数理解析研究所講究録
第 587 巻 1986 年 80-89

Critically $(k,\bar{k})$-connected graphs

Kiyoshi Ando
Nippon Ika University
Kosugi Nakahara-ku Kawasaki Japan

Yoko Usami
Ochanomizu University
Otsuka Bunkyo-ku Tokyo Japan

Abstract

A fragment of a connected simple graph G is a subset A of V(G) consisted of components of G–S such that $V(G)-A-S \neq \emptyset$ where S is a minimum cut of G. A minimal fragment of G is said to be an end of G. The complement of G is denoted by $\bar{G}$. A simple graph G is said to be critically $(k,\bar{k})$-connected if $\kappa(G-x) = \kappa(G)-1$ or $\kappa(\bar{G}-x) = \kappa(\bar{G})-1$ for each x of V(G) where $\kappa(G)$ means the vertex connectivity of G. We proved the followings:

Let G be a critically $(k,\bar{k})$-connected graph $(k \geq \bar{k} \geq 2)$. We denote by η and a the number of ends of G and the minimum order of the ends of G, respectively. Similarly $\bar{\eta}$ and $\bar{a}$ denote those of $\bar{G}$. Suppose $2a > k$ and $2\bar{a} > \bar{k}$. Then

(A) If there is no minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta = 2,\ 3 \text{ or } 4 \quad \text{and} \quad \frac{\eta}{\eta-1}\left\lfloor \frac{k}{\bar{a}} \right\rfloor \geq \bar{\eta} \geq \frac{\eta(k+1)}{2(2\bar{a}-1)}.$$

Furthermore, if $\eta = 2$, then $|G| \leq 2k+\bar{\eta}\bar{k}$, if $\eta = 3$, then $|G| \leq \frac{9}{4}k+\bar{\eta}\bar{k}-\frac{11}{4}$ and if $\eta = 4$, then $|G| \leq 2k+\bar{\eta}\bar{k}-6$.

(B) If there is a minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta(G) = 2 \text{ or } 3 \quad \text{and} \quad \left\lceil \frac{\eta a}{k} \right\rceil \leq \bar{\eta} \leq \left\lfloor \frac{k}{\bar{a}} \right\rfloor.$$

§1 Introduction and main results

In this paper we consider only finite simple graphs. We denote by V(G) the vertex set of a graph G. Let G be a connected graph. We call a set S of V(G) a *cut* of G if G–S is disconnected, and S is said to be a *minimum cut* of G if $|S| \leq |S'|$ for any cut S' of G. The order of a minimum cut of G is called the *vertex connectivity* of G and denoted by $\kappa(G)$. The minimum degree of the vertices of G is denoted by $\delta(G)$. The complement of a graph G is denoted by $\overline{G}$. As usual for a real number r we denote by $\lceil r \rceil$ and $\lfloor r \rfloor$ the integers such that $r \leq \lceil r \rceil < r+1$ and $r-1 < \lfloor r \rfloor \leq r$.

A non-empty subset A of V(G) is called a *fragment* of G if A is consisted of components of G−S and $V(G)-A-S \neq \emptyset$ for some minimum cut S of G. A minimal fragment of G is called an *end* of G. We denote by $\eta(G)$ the number of ends of G.

A graph G is said to be *critically k-connected* if $\kappa(G) = k$ and $\kappa(G-x) = k-1$ for each vertex x of V(G). G. Chartrand, A. Kaugars and D. R. Lick [2] have shown that if G is a critically k-connected graph, $k \geq 2$, then $\delta(G) \geq \frac{3k-1}{2}$ and this bound is sharp.

In [1] we introduced critically (k,k)-connectedness of graphs. More generally we define here that a graph is said to be *critically* $(k,\overline{k})$*-connected* if $\kappa(G) = k$, $\kappa(\overline{G}) = \overline{k}$ and $\kappa(G-x) = k-1$ or $\kappa(\overline{G}-x) = \overline{k}-1$ for each vertex x of V(G). In [1] we proved the following theorem concerning critically (k,k)-connected graphs.

Theorem A ([1]) If G is a critically (k,k)-connected graph, $k \geq 2$, $\delta(G) \geq \frac{3k-1}{2}$ and $\delta(\overline{G}) \geq \frac{3k-1}{2}$, then $|G| \leq 4k$.

In fact in [1] we proved the following stronger assertion:

Theorem ([1]) Let G be a critically (k,k)-connected graph, $k \geq 2$. Let a (resp.$\overline{a}$) be the order of minimum end of G (resp $\overline{G}$). If $2a > k$ and $2\overline{a} > k$, then $|G| \leq 4k$ and $(\eta(G), \eta(\overline{G})) = (2,2)$.

In this paper we will study critically $(k,\overline{k})$-connected graphs and we will show more general results descrived as follows:

<u>Main Theorem.</u> Let G be a critically $(k,\bar{k})$-connected graph $(k \geq \bar{k} \geq 2)$. We denote by η and a the number of ends of G and the minimum order of the ends of G, respectively. Similarly $\bar{\eta}$ and $\bar{a}$ denote those of $\bar{G}$. Suppose $2a > k$ and $2\bar{a} > \bar{k}$. Then

(1) (A) If there is no minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta = 2, 3 \text{ or } 4 \quad \text{and} \quad \frac{\eta}{\eta-1}\left\lfloor\frac{k}{\bar{a}}\right\rfloor \geq \bar{\eta} \geq \frac{\eta(k+1)}{2(2\bar{a}-1)}.$$

In particular $\eta \leq \bar{\eta}$.

(B) If there is a minimum cut of G containing all the ends of $\bar{G}$ then

$$\eta = 2 \text{ or } 3 \quad \text{and} \quad \left\lceil\frac{\eta a}{\bar{k}}\right\rceil \leq \bar{\eta} \leq \left\lfloor\frac{k}{\bar{a}}\right\rfloor.$$

(2) In case (A) in (1) we have

If $\eta = 2$ then $|G| \leq 2k+\bar{\eta}\bar{k}$.

If $\eta = 3$ then $|G| \leq \frac{9}{4}k+\bar{\eta}\bar{k}-\frac{9}{4}$.

If $\eta = 4$ then $|G| \leq 2k+\bar{\eta}\bar{k}-6$.

In case (B) in (1) there is no upper bound of the order of G for each k and $\bar{k}$.

§2 Preliminaries

In this section we introduce some more notation and present preliminary lemmas which we will use in the following two sections to prove our main results. Let G be a connected graph. We denote by $\mathscr{C}(G)$ the family of all minimum cuts of a graph G, and set $C(G) = \bigcup_{S \in \mathscr{C}(G)} S$. We denote by G[A] the subgraph of G induced by $A \subset V(G)$. Let $N_G(x)$ be the set of the vertices adjacent to x in G. For $A \subset V(G)$, we put $N_G(A) = \bigcup_{x \in A} N_G(x) - A$ and $N_G[A] = N_G(A) \cup A$. Recall a fragment of G is a non-empty subset A of V(G) such that (i) $N_G(A)$ is a minimum cut of G and (ii) $G-N_G[A]$ is non-empty, and that an end of G is a minimal fragment of G. We call a minimum fragment of G an *atom* and we denote by a_G the order of an atom of G. If there is no danger of ambiguity, we abbreviate $\eta(G)$, $\eta(\overline{G})$, a_G and $a_{\overline{G}}$ to η, $\overline{\eta}$, a and $\overline{a}$, respectively.

The following lemma expresses the essential relation between a graph G and its complement $\overline{G}$, so that we call it "Complement lemma".

Lemma (Complement Lemma) *Let G be a graph and let A, B be subsets of V(G). If B is not contained in* $N_G[A]$, *then* $N_{\overline{G}}[B]$ *contains A.*

Proof. Let x be a vertex of B not contained in $N_G[A]$. It is immediate that $N_{\overline{G}}(x) \supset A$, since $x \notin N_G[A]$. ■

In the above lemma, if $A \cap B = \emptyset$, then we can replace the closed neighbourhoods $N_G[A]$ and $N_{\overline{G}}[B]$ with the open neighbourhoods $N_G(A)$ and $N_{\overline{G}}(B)$, respectively. Therefore the next lemma (we also call it "Complement lemma") is an immediate consequence of the above lemma. This lemma will play a fundamental roll in our argument through this paper.

Lemma (Complement Lemma) *Let G be a graph and let B be a subset of V(G). If* $\mathcal{A}$ *is a family of subsets of V(G) such that* $B \cap A = \emptyset$ *for each A in* $\mathcal{A}$ *and* $N_{\overline{G}}(B) \not\supset \bigcup_{A \in \mathcal{A}} A$, *then there is a subset A in* $\mathcal{A}$ *such that* $N_G(A) \supset B$. ■

Lemma 1 *Let G be a connected graph and let W be a subset of V(G) such* *that (i) $|W| > \kappa(G)$, (ii) for any minimum cut S of G, $W-S$ is contained in a component of $G-S$. We denote by $\mathcal{A}$ the family of the maximal fragments of G each of which has no intersection with W. Then*

(1) *$A \cap B = \emptyset$ for any two distinct elements A, B in $\mathcal{A}$,*

(2) *$\mathcal{A} \neq \emptyset$ and any minimum cut of G is contained in $\bigcup_{A \in \mathcal{A}} N_G[A]$.*

proof. To prove (1) suppose not, i.e. suppose that there are two distinct fragments A_1 and A_2 in $\mathcal{A}$ such that $A_1 \cap A_2 \neq \emptyset$. Let $\widetilde{A}_i = V(G)-N_G[A]$ for $i = 1, 2$. Then since $A_1 \cap A_2 \neq \emptyset$ $|N_G(A_1 \cap A_2)| \geq \kappa(G)$. Consequently $\widetilde{A}_1 \cap \widetilde{A}_2 \neq \emptyset$, since $|N_G[\widetilde{A}_1] \cap N_G[\widetilde{A}_2]| \geq |W| > \kappa(G)$. Therefore $|N_G(\widetilde{A}_1 \cap \widetilde{A}_2)| = \kappa(G)$, which implies $A_1 \cup A_2$ is also a fragment of G, cotradicting the maximality of A_1 and A_2.

To prove (2) let S be any minimum cut of G and let H_S be the component of G-S containing W-S. Then the fragment $A = V(G)-N_G[V(H)]$ has no intersection with W , so there is an element A' in $\mathcal{A}$ containing A such that $S = N_G(A) \subset N_G[A']$.

■

We remark that in the above Lemma 1 if $W \subset C(G)$, then $W \subset \bigcup_{A \in \mathcal{A}} N_G(A)$, in particular, $|W| \leq \eta(G)\kappa(G)$.

As a slight extension of a result of Mader[4], we can easily show the followings which will be the firm bases of our arguments. (cf. Theorem 1 and Lemma 1 in [1])

Lemma 2 *Let G be a critically* $(k,\bar{k})$*-connected graph and let* $\{X_1, X_2, \ldots, X_\eta\}$ *and* $\{Y_1, Y_2, \ldots, Y_{\bar{\eta}}\}$ *be the set of all the ends of G and that of* $\bar{G}$*, respectively. Set* $X = \bigcup_{i=1}^{\eta} X_i$ *and* $Y = \bigcup_{j=1}^{\bar{\eta}} Y_j$*. Suppose* $2a > k$ *and* $2\bar{a} > \bar{k}$*. Then*

(*i*) $X \cap C(G) = \emptyset$ *and* $Y \cap (\bar{G}) = \emptyset$.

(*ii*) *Let A and B be any two distinct elements of* $\{X_1, X_2, \ldots, X_\eta, Y_1, \ldots Y_{\bar{\eta}}\}$. *Then* $A \cap B = \emptyset$. ■

§4 A proof of Main Theorem (1)

Throughout this section and the next section suppose G is a critically $(k,\bar{k})$-connected graph such that $2a > k$, $2\bar{a} > \bar{k}$ and $k \geq \bar{k}$. Let $\{X_1, X_2, \ldots X_\eta\}$ and $\{Y_1, Y_2, \ldots Y_{\bar{\eta}}\}$ be the set of all the ends of G and $\bar{G}$, respectively, and put $X = \bigcup_{i=1}^{\eta} X_i$ and $Y = \bigcup_{j=1}^{\bar{\eta}} Y_j$.

proof of (*A*) To prove the former part of (A) it suffices to show the following two inequalities: $\eta(k+1) \leq 2\bar{\eta}(2\bar{a}-1)$ and $\bar{\eta} + \left\lfloor \frac{k}{\bar{a}} \right\rfloor \eta \geq \eta\bar{\eta}$. If they hold, then $\bar{a}\eta + 2(2\bar{a}-1)\bar{\eta} - \eta \geq \bar{a}\bar{\eta}\eta$, which implies $\eta = 2$, 3 or 4. By the assumption of (A) $N_G(X_i) \not\supset Y$ for each i thus $X \subset N_{\bar{G}}(Y)$ and this implies the first inequality, since $\eta k \leq \eta(2a-1) \leq 2|X| - \eta \leq 2|N_{\bar{G}}(Y)| - \eta \leq 2\bar{\eta}\bar{k} - \eta \leq 2\bar{\eta}(2\bar{a}-1) - \eta$. To show the second inequality note that $N_G(X_i)$ contains at most $\left\lfloor \frac{k}{\bar{a}} \right\rfloor$ ends of $\bar{G}$ and $N_{\bar{G}}(Y_j)$ contains at most one end of G, since $2a > k \geq \bar{k}$. Furthermore, Complement lemma assures us that for any of pairs (i,j) either $N_G(X_i) \supset Y_j$ or $X_i \subset N_{\bar{G}}(Y_j)$. Thus $\bar{\eta} + \left\lfloor \frac{k}{\bar{a}} \right\rfloor \eta \geq \eta\bar{\eta}$. Next we prove the latter part of (A). For each end Y_j of $\bar{G}$, $N_{\bar{G}}(Y_j)$ can contain at most one end of G. On the other hand, the assumption of (A) implies each X_i contained in $N_{\overline{G}}(Y_j)$ for some j, so $\overline{\eta} \geqq \eta$.

proof of (B). To prove (B) it suffices to show the following three inequalities: $\eta \leqq 3$, $\eta a \leqq \overline{\eta}\overline{k}$ and $k \geqq \overline{\eta}\overline{a}$. The last one is immediate consequence of the assumption of (B). To show the former two inequalities put $H = \overline{G}[\bigcup_{i=1}^{\eta} X_i]$. Then as a consequence of Lemma 2 H has the complete η-partite graph with vertex clases $X_1, X_2, \ldots, X_\eta$ as its spanning subgraph. Therefore if $\eta \geqq 3$ then $\kappa(H) \geqq 2a > k \geqq \overline{k}$, so by the remark after Lemma 1 $\eta a \leqq |H| \leqq \overline{\eta}\overline{k}$. In particular, if $\eta \geqq 3$ then $a\overline{a}\eta \leqq \overline{a}\overline{\eta}\overline{k} \leqq k\overline{k}$ so $\eta \leqq \frac{k\overline{k}}{a\overline{a}} < 4$, thus the first inequality holds. In the case that $\eta = 2$, we may suppose $a > \overline{k}$, since otherwise $\eta a = 2a \leqq 2\overline{k} \leqq \overline{\eta}\overline{k}$. If $a > \overline{k}$ then $\kappa(H) \geqq a > \overline{k}$ and again by the same remark $\eta a \leqq \overline{\eta}\overline{k}$. ■

§4 A proof of Main Theorem (2)

At first we introduce two new families of subsets of V(G), $\mathcal{A}$ and $\mathcal{B}$, which will hold the key of our proof. Recall X is the union of all the ends of G and Y is that of $\overline{G}$. Let $\mathcal{A}$ be the family of the maximal fragments of G each of which has no intersection with Y. Similarly $\mathcal{B}$ stands for the family of the maximal fragments of $\overline{G}$ each of which has no intersection with X. To prove Main Theorem (2) we need the following two lemmas which express remarkable properties of $\mathcal{A}$ and $\mathcal{B}$. Throughout this section assume that there is no minimum cut of G containing Y. We remark that there is no minimum cut of $\overline{G}$ containing X, since $\eta a \geqq 2a > k \geqq \overline{k}$.

Lemma 3 *Suppose* $|G| > 2(k+\bar{k})$. *Then*

(1) *Each of* $\mathcal{A}$ *and* $\mathcal{B}$ *is a family of mutually disjoint subsets of* $V(G)$.

(2) $C(G) \subset \bigcup_{A\in\mathcal{A}} N_G[A]$ *and* $C(\bar{G}) \subset \bigcup_{B\in\mathcal{B}} N_{\bar{G}}[B]$.

(3) $|A| \le \bar{k}$ *for each* $A\in\mathcal{A}$, *and* $|B| \le k$ *for each* $B\in\mathcal{B}$.

proof. We give a proof for $\mathcal{A}$ (We can prove the result for $\mathcal{B}$ similarly). Let S be any minimum cut of G and let A_1 and $A_2 \in \mathcal{A}$ such that $A_1\cap A_2 \neq \emptyset$. For i = 1 and 2, let $\tilde{A}_i$ stand for $V(G)-N_G[A_i]$. Then according to the proof of Lemma 1 to prove (1) and (2) it suffices to show (i) Y−S is contained in a component of G−S and (ii) $\tilde{A}_1\cap\tilde{A}_2 \neq \emptyset$.

(i) By the assumption that $S \not\supset Y$, there is an end Y_S of $\bar{G}$ not contained in S. Also there is an end of G, say X_1, not contained in $N_{\bar{G}}(Y_S)$. By Complement lemma $N_G(X_1) \supset Y_S$, so $k \geq |Y_S|$ and $k+\bar{k} \geq |N_{\bar{G}}[Y_S]|$. Let $\tilde{Y}_S = V(G)-N_{\bar{G}}[Y_S]$. Then $|\tilde{Y}_S| > k$, since $|G| > 2(k+\bar{k})$. Hence $S \not\supset \tilde{Y}_S$. Consequently the subgraph $G[Y_S\cup\tilde{Y}_S-S]$ of G is connected, for $G[Y_S\cup\tilde{Y}_S]$ includes the complete bipartite graph with vertex classes Y_S and $\tilde{Y}_S$. Let H_S be the component of G−S containing $Y_S\cup\tilde{Y}_S-S$. Note that any other end of $\bar{G}$ which is disjoint from Y_S is contained in $\tilde{Y}_S$. Then $Y-S \subset V(H_S)$.

(ii) Assume $\tilde{A}_1\cap\tilde{A}_2 = \emptyset$. Then $|A_1\cup A_2| > 2\bar{k}$. Thus, without loss of generality, we may assume $|A_1| > \bar{k}$. On the other hand, by the assumption that $N_G(A_1) \not\supset Y$ there is an end of $\bar{G}$, say Y_1, not contained in $N_G(A_1)$, i.e. $Y_1 \not\subset N_G[A_1]$. However, since $Y_1\cap A_1 = \emptyset$, as an immediate consequence of Complement lemma $N_{\bar{G}}(Y_1) \supset A_1$ contradicting the assumption $|A_1| > \bar{k}$.

To prove (3), for each A in $\mathcal{A}$ put $S = N_G(A)$. Then, by the choice of the component H_S, $H_S \cap Y \neq \emptyset$ so $A \cap V(H_S) = \emptyset$. Thus $A \subset N_{\bar{G}}(Y_S) \cup S$, this implies $|A| \leq \bar{k}$, so (3) holds. ■

<u>Lemma 4</u> *Suppose* $|G| \geq 2(k+\bar{k})$. *Then*

(1) *For any* $A \in \mathcal{A}$ *and any* $B \in \mathcal{B}$ $\quad A \cap B = \emptyset$.

(2) $\bigcup_{A \in \mathcal{A}} A \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$ *and* $\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{A \in \mathcal{A}} N_G(A)$.

proof. (1) Suppose not, i.e. there are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \cap B \neq \emptyset$. Let $\tilde{A} = V(G) - N_G[A]$ then $N_G[\tilde{A}] \not\supset B$, since $A \cap B \neq \emptyset$. As a consequence of Complement lemma $\tilde{A} \subset N_{\bar{G}}[B]$ thus $V(G) \subset N_G[A] \cup N_{\bar{G}}[B]$. According to the previous lemma $|A| \leq \bar{k}$ and $|B| \leq k$, so $|V(G)| < 2(k+\bar{k})$ contradicting the assumption.

(2) We show only $\bigcup_{A \in \mathcal{A}} A \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$. Recall Y is the union of all the ends of G. For each $A \in \mathcal{A}$ $N_G(A)$ can not contain whole Y and also $N_G(A) \not\supset \bigcup_{B \in \mathcal{B}} B$. According to (1) of this lemma it is an immediate consequence of Complement lemma that $\bigcup_{A \in \mathcal{A}} A \subset \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$. ■

By now we are all set to prove Main Theorem (2).

proof of Main Theorem (2) From the definition of critically $(k,\bar{k})$-connected graph $V(G) = C(G) \cup C(\bar{G})$. Therfore as a consequence of Lemma 3 (2) $V(G) = \bigcup_{A \in \mathcal{A}} N_G[A] \cup \bigcup_{B \in \mathcal{B}} N_{\bar{G}}[B]$. Finally, by Lemma 4 (2) $V(G) = \bigcup_{A \in \mathcal{A}} N_G(A) \cup \bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)$, so that $|G| \leq |\bigcup_{A \in \mathcal{A}} N_G(A)| + |\bigcup_{B \in \mathcal{B}} N_{\bar{G}}(B)|$. To complete the proof it remains to check the upper bound of $|\bigcup_{A\in\mathscr{A}} N_G(A)|$. We may suppose $\eta = 3$ or 4. We denote by $\#\mathscr{A}$ the number of fragments of $\mathscr{A}$. Because the family of ends of G is mutually disjoint the inequality $2a > \bar{k} \geq |A|$ implies each A of $\mathscr{A}$ contains exactly one end of G, so $\#\mathscr{A} = \eta$. For each B of $\mathscr{B}$, $N_{\bar{G}}(B)$ can contain at most one fragment A of $\mathscr{A}$, since $2a > \bar{k}$. Therefore by Complement lemma for each B there are $(\#\mathscr{A}-1)$ fragments of $\mathscr{A}$ such that $N_G(A)$ containes B. Consequently $|\bigcup_{A\in\mathscr{A}} N_G(A)| \leqq \#\mathscr{A}k-(\#\mathscr{A}-2)|\bigcup_{B\in\mathscr{B}} B| \leqq \eta k-\bar{\eta}\bar{a}(\eta-2)$. From the first inequality in the proof of Main Theorem (1) (A) it follows $\bar{\eta}\bar{a} \geqq \frac{1}{4}(\eta k+2\bar{\eta}+\eta)$, so finally $|\bigcup_{A\in\mathscr{A}} N_G(A)| \leqq \frac{1}{4}\{(6-\eta)\eta k-(\eta+2\bar{\eta})(\eta-2)\}$ and this completes the proof. ■

References

[1] K. Ando and Y. Usami, Critically (k,k)-connected graphs I, preprint

[2] G. Chartrand, A. Kaugars and D. R. Lick, Critically n-connected graphs, Proceedings of the American Math. Soc. Vol 32, no. 1 (1972) 63-68

[3] Y. O. Hamidoune, On critically h-connected simple graphs, Discrete Math. 32 (1980) 257-262

[4] W. Mader, Ein Eigenshaft der Atome endlicher Graphen, Arch. Math. 22 (1971) 333-336

[5] H. J. Veldman, Non-κ-critical vertices in graphs, Discrete Math. 44 (1983) 105-110