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The statement AN is equivalent to the statement $n(\beta\omega\setminus\omega) > c$

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1. Introduction and results. A filter $\mathfrak{F}$ on ω is said to be ample, if there is an infinite subset a of ω such that, whenever $x \in \mathfrak{F}$, $a \setminus x$ is finite. A filter $\mathfrak{F}$ on ω is said to be weakly ample, if for each free ultrafilter (uf) $\mathcal{U}$ on ω, there is a function f on ω such that $f(\mathcal{U}) \supset \mathfrak{F}$. Let us denote by AN the statement: "every free weakly ample filter on ω is ample." In [2], we showed

PROPOSITION 1.

(i) AN implies the existence of c^+ Ramsey ufs on ω, where c denotes the cardinality of 2^ω.

(ii) The existence of c^+ Ramsey ufs on ω does not imply AN.

(iii) Ⓟ implies AN, where Ⓟ denotes the statement: "every free filter on ω generated by a set cardinality less than c is ample."

It seems to be interesting to consider how strong the statement AN is. As to this, we first show

THEOREM 1. AN is equivalent to the statement that $\beta\omega\setminus\omega$ can not be covered by a family of c nowhere dense sets, where $\beta\omega$ denotes the Čech-Stone compactification of ω.

Let us denote by $n(\beta\omega\setminus\omega)$ the Baire number of $\beta\omega\setminus\omega$ (i.e. the minimal cardinal of a family of nowhere dense sets covering $\beta\omega\setminus\omega$). As to the Baire number of $\beta\omega\setminus\omega$, the systematical estimation was given and several consistencies were shown in [1]. In [1], it is shown

PROPOSITION 2 (5.2.V in [1]) The statement $n(\beta\omega\setminus\omega) > c$ does not imply that $\forall\kappa < c\ (\ |2^{\kappa}| \leq c\ )$.

Since Ⓟ implies that $\forall\kappa < c\ (\ |2^{\kappa}| \leq c\ )$ ([3]), by Theorem 1 and Proposition 2, AN does not imply Ⓟ.

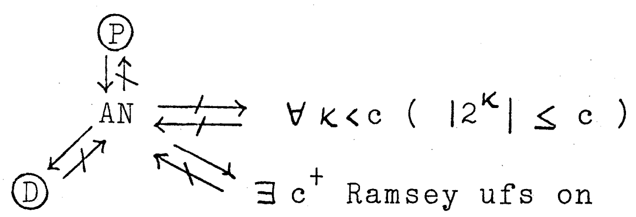
Define the pseudo-ordering $<^*$ on ω^{ω} by $f <^* g$ iff $\lim_{n\to\infty}(\ g(n) - f(n)\ ) = \infty$. A family F of subsets of ω^{ω} is said to be unbounded, if there does not exist $g \in \omega^{\omega}$ such that, whenever $f \in F$, $f <^* g$. Then, it holds

PROPOSITION 3 (4.6 and 4.7 in [1])

(i) The statement $n(\beta\omega\setminus\omega) > c$ implies the statement Ⓓ : "every unbounded family of subsets of ω^{ω} has the cardinality c."

(ii) Ⓓ does not imply that $n(\beta\omega\setminus\omega) > c$.

By Propositions 1∼3 and Theorem 1, the following diagram holds.



The only interesting in the above diagram which is not mentioned is whether AN + $\forall\kappa < c$ ($|2^\kappa| \leq c$) implies Ⓟ or not. As to this, we show

THEOREM 2. AN + $\forall\kappa < c$ ($|2^\kappa| \leq c$) does not imply Ⓟ.

We shall prove Theorem 1 in the following section and Theorem 2 in section 3.

2. Proof of Theorem 1. We first show that $n(\beta\omega\setminus\omega) > c$ implies AN. So, assume that $n(\beta\omega\setminus\omega) > c$. Let $\mathfrak{F}$ be any weakly ample filter on ω. For each $f \in \omega^\omega$, set

$$D_f = \{ \mathcal{U} \in \beta\omega\setminus\omega \ ; \ f(\mathcal{U}) \supset \mathfrak{F} \}.$$

Since $\mathfrak{F}$ is weakly ample, it holds that

$$\bigcup\{ D_f \ ; \ f \in \omega^\omega \} = \beta\omega\setminus\omega.$$

So, there is some $f \in \omega^\omega$ such that D_f is not nowhere dense in $\beta\omega\setminus\omega$. Take an infinite subset a_0 of ω such that

(*) $\quad \forall x \subset a_0 \ (\ |x| = \omega \Rightarrow \exists \mathcal{U} \in D_f \ (\ x \in \mathcal{U} \ ) \ ).$

Set $a_1 = f''a_0$. Then, by (*), it holds that a_1 is infinite and $\forall x \in \mathfrak{F}$ ($a_1 \setminus x$ is finite). Hence, $\mathfrak{F}$ is ample.

Now, we shall prove that the inverse implication holds. The following fact which we shall use in the proof is well-known and easy.

FACT 1. There is a family W of subsets of ω such that

(1) $|W| = c$,

(2) $\forall x \in W \ (\ |x| = \omega \ )$,

(3) $\forall x, y \in W \ (\ x \neq y \Rightarrow x \wedge y$ is finite).

Assume that AN holds. Let $\mathcal{D} = \{ D_\alpha ; \alpha < c \}$ be any family of nowhere dense subsets of $\beta\omega\setminus\omega$. Let $\langle a_\alpha \mid \alpha < c \rangle$ be a monotone enumeration of a family W of subsets of ω which satisfies (1) $\sim$ (3) in Fact 1. For each $\alpha < c$, take $f_\alpha \in \omega^\omega$ such that $f_\alpha : \omega \longrightarrow a_\alpha$ one-to-one and onto. Define the filter $\mathfrak{F}$ on ω by

$$x \in \mathfrak{F} \quad \text{iff} \quad \forall \alpha < c \, \forall \mathcal{U} \in D_\alpha \, (x \in f_\alpha(\mathcal{U})).$$

Then, it is easy to see that $\mathfrak{F}$ is free and not ample. So, by AN, there is $\mathcal{U} \in \beta\omega\setminus\omega$ such that

$$\forall g \in \omega^\omega (g(\mathcal{U}) \not\supset \mathfrak{F}).$$

Then, it holds that, for any $\alpha < c$, $\mathcal{U} \notin D_\alpha$, since $f_\alpha(\mathcal{U}) \not\supset \mathfrak{F}$. Hence, $\mathcal{U} \notin \bigcup\mathcal{D}$. ∎

3. Proof of Theorem 2. Let $\mathfrak{M}$ be a countable transitive model of ZFC + GCH. We shall show that a generic extension of $\mathfrak{M}$ on the poset $P \times Q$ which will be defined below satisfies that AN + $\forall \kappa < c \, (|2^\kappa| < c)$ + ¬Ⓟ. The poset $P \times Q$ is alike the poset used in 5.V of [1]. Let P be the Solovey-Tennenbaum's poset used for the consistency of MA + $|2^\omega| = \omega_2$. Define the poset $Q \in \mathfrak{M}$ by, in $\mathfrak{M}$,

$$Q = \{ q \; ; \; \exists \alpha < \omega_1 \, (q : \alpha \longrightarrow 2) \}.$$

Let $G \times H$ be $\mathfrak{M}$-generic on $P \times Q$ and $\widetilde{\mathfrak{M}} = \mathfrak{M}[G \times H]$. Then, similar arguments in [1] show that

$$\widetilde{\mathfrak{M}} \models " \, |2^\omega| = |2^{\omega_1}| = \omega_2 + \text{AN} \, ".$$

We shall show that $\widetilde{\mathfrak{M}} \models$ ¬Ⓟ. Since CH holds in $\mathfrak{M}$, it holds that, in $\mathfrak{M}$, there is a dence embedding from Q to $P(\omega)$/finite.

So, we may assume that H is $\mathfrak{M}$-generic on $(P(\omega)/\text{finite})^{\mathfrak{M}}$.
Define $\mathfrak{F} \in \tilde{\mathfrak{M}}$ by

$$\tilde{\mathfrak{M}} \models \text{" } \mathfrak{F} = \{ x \subset \omega \ ; \ \exists\, a/\text{finite} \in H \ (a \setminus x \text{ is finite })\}.\text{"}$$

Since $\tilde{\mathfrak{M}} \models$ " $|H| = \omega_1$ ", it holds that

$$\tilde{\mathfrak{M}} \models \text{" } \mathfrak{F} \text{ is an } \omega_1\text{-generated free filter on } \omega.\text{"}$$

Moreover, since H is not in $\mathfrak{M}[G]$, we have that

$$\tilde{\mathfrak{M}} \models \text{" } \mathfrak{F} \text{ is not ample. "}$$

Hence, $\tilde{\mathfrak{M}} \models \neg$ Ⓟ. ▨

References

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