数理解析研究所講究録
第 588 巻 1986 年 175-189

Hyper-principle and the functional structure of ordinal diagrams

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The theory of ordinal diagrams (which will simply be called the diagrams) has been developed over the years, and has been applied to consistency proofs of subsystems of analysis. The importance of the accessibility proof of the diagrams has been discussed in various literatures (see [1], [3], [5]), and hence let us confine ourselves to a brief remark on this matter. The essence of the diagrams can be characterized by its functional structure; that is, by determining the universe of the functionals which produce "moduli of finiteness" for various decreasing sequences of the diagrams.

I have recently reached a "concrete" version of the accessibility proof; the rough idea of this approach is to be explained subsequently. (See [5]~[7] for details.)

Let $\mathbf{I}$ and $\mathbf{A}$ be two accessible sets and let $O \equiv O(\mathbf{I}, \mathbf{A})$ be the system of diagrams based on $\mathbf{I}$ and $\mathbf{A}$. First define F_i and G_i for each $i \in \mathbf{I} \cup \{\infty\}$.

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Definition. Suppose $i \in I \cup \{\infty\}$.

$$F_i = \{\alpha \in O;\ \forall j < i \text{ (every } j\text{-section of } \alpha \text{ is } j\text{-accessible in } F_j)\}$$

$$G_i = \{\alpha \in F_i;\ \alpha \text{ is } i\text{-accessible in } F_i\}$$

An element of F_i is said to be i-fit, and an element of G_i is said to be i-grounded.

Then the theorem to be proved is this.

(Thm) $\forall i \forall \alpha(\alpha \in O \leftrightarrow \alpha \in G_i \leftrightarrow \alpha \in F_i)$

From this follows i-accessibility of every diagram for every i. The theorem is proved via the key proposition:

(KP) If $\alpha \in G_i$, then $\forall j < i(\alpha \in G_j)$.

That is, the groundedness property is inherited downward. The proof of the key proposition requires a certain manipulation which heavily relies on the theory of fundamental sequences of the diagrams, latter having been developed in [2].

Now the notions of i-fitness and i-groundedness should be interpreted "constructively." Let $(C, \prec)$ be a concretely given linearly ordered structure, and let c be in C. acc$(C, \prec, c, M)$ will express that M be a method which, for every f a $\prec$-decreasing sequence from C led by c, gives a modulus of finiteness of f; that is,

$$\forall n \geq M(c, f)(f(n) = \text{empty}).$$

$\alpha \in F_i$ and $\alpha \in G_i$ should then be read as follows.

$$\alpha \in F_i : \exists M_{i,\alpha}(\text{a method depending on } i \text{ and } \alpha)$$
$$\forall j < i\ \forall \sigma \subset_j \alpha(M_{i,\alpha}(j, \sigma) \text{ is a method}$$
$$\text{to concretely demonstrate that } \sigma \in G_j)$$

$$\alpha \in G_i : \alpha \in F_i \wedge \exists N_{i,\alpha}(\text{a method})\, acc(F_i, <_i, \alpha, N_{i,\alpha})$$

So, we are in the world where, in claiming any property about any objects, one has to have a method to establish that relation. We can symbolically express this as follows. Put $M_i = \lambda\alpha M_{i,\alpha}$ and $N_i = \lambda\alpha N_{i,\alpha}$. Then

$$G_i \leftrightarrow \mathcal{M}_i = (M_i, N_i),$$
$$F_i \leftrightarrow \{\mathcal{M}_j^*\}_{j<i} = M_i.$$

Here N_i represents a modulus of finiteness for $(F_i, <_i, \alpha)$ for some α, M_i is the method for (determining the elements of) F_i, $\mathcal{M}_i$ is the method for G_i and $\mathcal{M}_j^*$ is a slight modification of $\mathcal{M}_j$.

The foundations of ordinal diagrams means, therefore, to determine the universe of the "methods" which yield the "moduli of finiteness" for i-decreasing sequences of the diagrams uniform in i. It should be expected that such universe is of peculiar feature as may be guessed from the outline given so far. The properties such as fitness and groundedness are not classical, since the existence of methods are incorporated in the definientia multifoldly. We are therefore naturally led to an intuitionistic system in the attempt to formalize the accessibility proof, and the nature of accessibility can be embodied by the "functional" interpretation of the existential quantifiers. They occur in the form $\forall f \exists n P(f, n)$, where $P(f, n)$ is of first order and $\exists$-free. A functional X such that $\forall f P(f, X(f))$ will represent a modulus of finiteness (of f). The implication $\vdash$ should be interpreted so as to be compatible with the idea of the "methods." That is, $\exists X A(X) \vdash \exists Y B(Y)$ should be interpreted as $\exists Z \forall X (A(X) \vdash B(Z(X)))$. We are thus led to the "modified realizability" (mr-) translation of the formulae of our system. The mr-translation of G is defined so as to incorporate the idea of methods and the mr-interpretation of connectives. Through these procedures, the desired universe of methods can be determined.

In Part I of the work under the presented title, (Thm) is proved in a system called ASOD (analytic system especially designed for ordinal diagrams). The completely "positive" proof of this theorem naturally determines the system ASOD. The crucial step, (KP), is proved via the construction of a (bar) tree Λ of diagrams.

In Part II the theory of "methods", which will be called the "hyper-principle (HP)" is developed. It consists of generalized notions of types, functionals and formulae (called type-forms, term-forms and formula-forms respectively), and serves as the basis of the functional interpretation of the diagrams. The term-forms without parameters are called hyper-functionals, and they are interpreted with "methods", where the continuity principle of the functionals of the lowest type is assumed. The formula-forms admit only three logical connectives $\wedge$, $\vdash$ and $\forall$. There is a special predicate symbol Δ, which is to be the mr-translation of G. Some axioms on the predicates and the hyper-functionals are introduced so as to represent their intrinsic meanings, and the truth-values and the validity are defined so that the axioms are made valid by the "methods". A formula-form which is valid in this sense is said to be HP-valid. The entire content of this part is called HP. The bar induction and the continuity principle are informally used here.

In Part III, the mr-translations are applied to the formulae of ASOD according to the idea described above. An mr-translation is of the form

$$(*)\quad \exists X_1 \ldots \exists X_n A(X_1, \ldots, X_n),$$

where $A(X_1, \ldots, X_n)$ is a formula-form. We now attain our objective.

Theorem. If B is a closed formula of ASOD and the mr-translation of B is as in (*), then hyper-terms $\Phi_1^*, \ldots, \Phi_n^*$ can be found so that $A(\Phi_1^*, \ldots, \Phi_n^*)$ be HP-valid.

This yields the Conclusion that the functional structure of ordinal diagrams is represented by HP.

Before going into technical details, let us make some remarks.

Firstly, although it appears that we have carried out our project indirectly through the general system ASOD, the entire argument realizes the intrinsic idea of "methods", and hence the concreteness is not lost.

Secondly, the informal use of the bar induction as well as the continuity principle does not affect the spirit of the "foundations" (of the diagrams), since, as was explained previously, our intention is to construct the "methods" which interpret the groundedness property, and that has been done. If one clears all the intermediate treatments in the whole course, only the concrete methods will remain.

Now some of the technicalities. First let us quote some definitions and arguments from [5] and [6].

Definition. $H_i = \{(k, b, \beta);\ k < i \text{ and } \beta \in G_k\}$

$\gamma[i]$: a new symbol corresponding to γ if $\gamma \in H_i$

$J_i = \{\gamma[i];\ \gamma \in H_i\}$

$\prec_i$: the order of J_i induced from $<_i$

$I\langle i\rangle = \{j \in I;\ j \geq i\}$

$O(i) = O(I\langle i\rangle, A, J_i)$

$D_i = \{\kappa \in O(i);\ \kappa \text{ is i-accessible in } O(i)\}$

For an $\alpha \in F_i$, $\alpha[i]$, the "i-projection" of α, is defined to be the figure obtained from α by replacing in it each i-active element of H_i by its corresponding symbol in J_i. For a κ in $O(i)$, the unique $\alpha \in F_i$ such that $\alpha[i] = \kappa$ is called the "i-elevation" of κ, and is denoted by $\kappa\{i\}$. (Such an α exists.)

A function which produces the fundamental sequences for a system **C** will be called a fundamental method for **C**.

These constitute the design of the accessibility proof as mentioned. Namely, we can show that the i-groundedness property is reduced to the $<_i$-accessibility in $O(i)$. That is, $\alpha \in G_i$ if and only if $\alpha[i] \in D_i$. Through certain constructions induced by the fundamental sequences in $O(j)$ for some $j < i$, we can demonstrate that every diagram is i-grounded, and is hence $<_i$-accessible (for every i). Notice that j is the least element of $I \cup \{\infty\}$ in O(j) even if it is not in the original system O; this simple fact has a decisive effect on the entire argument.

For Part I, the language $\mathcal{L}$ of ASOD consists of the language of intuitionistic arithmetic with constants for elementary theory of the diagrams, $\forall$ on function type, $\forall f$ (but <u>without</u> $\exists f$), augmented by the following constants.

$\mathcal{I}$, $\mathcal{A}$, $\mathcal{J}$: symbols which correspond to the fundamental methods for **I**, **A** and J respectively.

G, Ord, lss: predicates which correspond to G, O(i) and the orders of O(i) respectively.

An $\mathcal{L}$-formula is said to be $\mathcal{L}$-recursive if it is free of G, Ord and the quantifiers ($\forall$, $\exists$). (Note: lss is allowed.)

The base of the system ASOD is the intuitionistic arithmetic applied to $\mathcal{L}$-formulae; added are the axioms on projection and elevation, the axioms on the constants listed above, TI(**I**)(transfinite induction along the order of **I**), TI(**A**) and the bar induction BI(R, A)(applied to R and A), where R is $\mathcal{L}$-recursive and A is an arbitrary $\mathcal{L}$-formula. The axioms on G, Ord and lss are definitions by transfinite induction along the order of **I**, which are formal expressions of respective definitions.

The essence of the proof of (KP) in this system is to construct a sequence from O(i), say $\{\lambda_m\}$, for any κ in O(i) from the fundamental sequence for (i, κ) in O(i), say $\{\kappa_m\}_m$, so that $\lambda_m \leq_i \kappa_m$ and that the i-accessibility of λ_m implies that of κ_m. Repeating this process, we define a tree Λ of the diagrams in O(i) with κ at the bottom and then apply (formal)BI to Λ to obtain (KP). Here Λ satisfies the finiteness property (that is, Λ is a bar-tree) and the heredity with respect to the i-accessibility downward.

Part II is the main theme of the present report. It consists of the theories of type-forms, term-forms and formula-forms.

Definition. 1) The language $\mathcal{L}_{tp}$ for type-forms is the language of $\mathcal{L}$-terms and $\mathcal{L}$-recursive formulae in Part I augmented by

$$N_0, \text{ ept}, \ \Lambda, \ \{ \ \}, \to, \ \mathcal{C}, [\ ], \Rightarrow, \ \Pi, \ T, \langle \ \rangle, \ \mathcal{R}.$$

2) Definitions of type-forms, the variables in them (free and bound) and the reduction rules (RR)(where necessary) are given below.

(1) N_0 is the atomic type-form.

(2) $\Lambda x(t)$ (x is a variable not bound in t and t is a type-form.)

(3) $S \to t$ (S and t are type-forms.)

(4) $\mathcal{C}[(A); (t)]$ ($(A) \equiv A_1, \ldots, A_m$: $\mathcal{L}$-recursive formulae, mutually exclusive. $(t) \equiv t_1, \ldots, t_m$, t_{m+1}: type-forms)

(RR)

$$\mathcal{C}[(A); (t)] \Rightarrow \begin{cases} t_l \text{ if } A_l, & 1 \le l \le m, \\ t_{m+1} \text{ otherwise.} \end{cases}$$

(5) $T\langle i\rangle$ (i: an $\mathcal{L}$-term of atomic-type)

(RR)

$$T\langle i\rangle \Rightarrow \Lambda l\, \mathcal{C}[l = 0, \ l = 1; \ T_1\langle i\rangle, \ T_2\langle i\rangle, \text{ ept}],$$

where

$T_1\langle i\rangle \equiv \Lambda j \mathcal{C}[j < i; N_0 \to T\langle j\rangle, \mathrm{ept}]$,

$T_2\langle i\rangle \equiv S_q \to ((N_0 \to T_1\langle i\rangle) \to N_0)$,

$S_q = N_0 \to N_0$

(6) $\Pi(\mathcal{S};\ \phi)$ ($\mathcal{S} \equiv \Lambda x \mathcal{t}(x)$, ϕ: $\mathcal{L}$-term)

(RR)

$\Pi(\mathcal{S};\ \phi) \Rightarrow \mathcal{t}(\phi)$

(7) $\Pi(\mathcal{S};\ \phi)$ ($\mathcal{S} \equiv T\langle i\rangle$, ϕ: $\mathcal{L}$-term)

(8) Let $M(\mathcal{s}, \mathcal{t}, \nu, x; \mathcal{l})$ be an expression in $\mathcal{L}_{tp}$ without $\mathcal{R}$ which is determined from the indicated parameters. If it is a type-form for every numeral m, then $\mathcal{R}[\mathcal{s}, \mathcal{t}, \nu, x; \mathcal{l}]$ is a type-form for any $\mathcal{L}$-term $\mathcal{l}$.

(RR)

$\mathcal{R}[\mathcal{s}, \mathcal{t}, \nu, x; \mathcal{l}] \Rightarrow M(\mathcal{s}, \mathcal{t}, \nu, x; \mathcal{l})$

3) A term-form without free variables is called a hyper-type.

As a consequence of the definitions, we have that every hyper-type is "reducible" to a "normal (terminal)" form.

Let us explain what these type-forms represent by the objects of respective hyper-types (which are called the methods of the corresponding hyper-types).

Examples. 1. An object of hyper-type $\Lambda x(\mathcal{t})$ is a method to associate with each x an object of hyper-type $\mathcal{t}(x)$.

2. $\mathcal{s} \to \mathcal{t}$ represents maps in the usual sense.

3. The objects of $T\langle i\rangle$ are defined by transfinite induction on i according to the (RR) for $T\langle i\rangle$, so that $\mathcal{M}_i$ (the method for G_i) becomes an object of hyper-type $T\langle i\rangle$, M_i of $T_1\langle i\rangle$ and N_i of $T_2\langle i\rangle$.

$$G_i \leftrightarrow \mathcal{M}_i = (M_i, N_i) \leftrightarrow T\langle i\rangle = (T_1\langle i\rangle, T_2\langle i\rangle)$$

$$F_i \leftrightarrow \{\mathcal{M}_j^*\}_{j<i} = M_i \leftrightarrow T_1\langle i\rangle = \Lambda_j \mathcal{C}[j < i; N_0 \to T\langle j\rangle, \text{ept}]$$

$$\mathcal{M}_j^* \leftrightarrow (N_0 \to T\langle j\rangle)$$

These correspondences explain the reduction rule for $T\langle i\rangle$.

Definition. 1) The language $\mathcal{L}_{tm}$ for term-forms is $\mathcal{L}_{tp}$ augmented by the following.

The variable-forms $X_n^{\mathcal{S}}$ (n = 1, 2,..., and $\mathcal{S}$ is an arbitrary type-form), $\mathcal{J}_0$, η_0, ζ_0, μ_0, $\mathcal{B}$, λ, Π, $\mathcal{C}$, [ ]

where $\mathcal{S}$ is an arbitrary type-form.

2) The term-form of a certain type-form, its variables and variable-forms, and the associated variables (which are the variables occurring in type-forms) are defined simultaneously.

(1) The atomic constants: $\mathcal{J}_0$, η_0, ζ_0, μ_0, $\mathcal{B}^p$ (for some type-forms p)

(2) Variable-forms

(3) $\mathcal{L}$-terms

(4) $\Pi(\phi; \Psi)$, where the type-form of ϕ is either $\mathcal{S} \to t$, $\Lambda x t(x)$ or $T\langle i\rangle$.

(5) $\lambda X\phi$

(6) $\mathcal{C}[(A); (\Phi)$

3) A term-form which does not have associated variables is called a hyper-term; a hyper-term without free variables or variable-forms is called a hyper-functional.

$\mathcal{B}^p$ is a bar constant of a type-form p, and the type-forms of constants such as $\mathcal{J}_0$, η_0,... are naturally determined from their intended attributes. $\Pi(\Phi;\ \Psi)$ represents the projection of Φ at Ψ, $\lambda X\Phi$ is the usual λ-notation and $C[(A);\ (\Phi)]$ represents the definition by cases.

Now we place some assumptions.

[Assumptions] 1. We shall henceforth work in the "universe of methods" (as explained previously).

2. We assume the continuity principle, CNPR (at the lowest type):

$$\forall S'(S'\restriction L(S) = S\restriction L(S) \vdash L(S') = L(S))$$

where $L(S) = \Pi(L;\ S)$.

Definition. 1) An assignment **a** of closed $\mathcal{L}$-terms to variables and an assignment **b** of "methods" to variable-forms of hyper-types are naturally defined.

2) If **ba** is an assignment to all the free variables and variable-forms in Φ, then it is said to be "complete" for Φ.

3) The interpretation of a term-form Φ by its complete assignment, $\mathbf{I}(\Phi,\ \mathbf{b},\ \mathbf{a})$, is a method of an appropriate hyper-type. It is defined according to the construction of Φ and its intended attributes. For example, a method $\mathcal{J}^*$ is assumed for $\mathcal{J}_0$ (which is supposed to interpret $\mathcal{J}$), and $\mathbf{I}(\Pi(\mathcal{B}^p;\ Z),\ \mathbf{b},\ \mathbf{a})$ is defined as follows, so that $\mathcal{B}$ satisfies the bar recursion at the lowest type (that is, the bar recursion is applied to trees of objects of atomic type).

Put $Z \equiv L,\ \Phi_1,\ \Phi_2,\ x,\ S$, where S is a sequence of objects of atomic type.

$$\mathbf{I}(\Pi(\mathcal{B}^p;\ \mathcal{Z}),\ \mathbf{b},\ \mathbf{a}) = \begin{cases} \mathbf{I}\Pi(\Phi_1;\ S\lceil x) \text{ if } \mathbf{I}L(S) \leq \mathbf{b}x, \\ \mathbf{I}\Pi(\Phi_2;\ S\lceil x,\ \lambda s\Pi(\mathcal{B};\ L, \Phi_1, \Phi_2, x+1, \\ (S\lceil x) * s)) \text{ if } \mathbf{b}x < \mathbf{I}L(S). \end{cases}$$

Proposition. The interpretation **I** is well-defined.

The proof is carried out according to the construction of Φ, and the case for $\mathcal{B}$ is dealt with by an informal application of the bar induction and CNPR above.

Definition. 1) The language $\mathcal{L}_0$ for formula-forms consists of the language of $\mathcal{L}$-recursive formulae (but $\mathcal{J}$ is replaced by $\mathcal{J}_0$ and lss is replaced by Σ), $\mathcal{L}_{tp}$, $\mathcal{L}_{tm}$, Δ, Θ, Σ, $\wedge$, $\vdash$, $\forall X$(X is a variable or a variable-form)

2) The formula-forms of $\mathcal{L}_0$ (and their variable-forms and the associated variables) are defined as usual from the $\mathcal{L}$-recursive formulae and the atomic formulae below:

$\Delta(i, \Phi, \alpha$, where i and α are atomic-type $\mathcal{L}$-terms and Φ is a term-form;

$\Theta(i, \Psi, \gamma)$

Σ i; j, γ, δ)

3) A formula-form which does not have associated free variables is called a hyper-formula.

There are seven axioms of $\mathcal{L}_0$-formula-forms, ($\mathcal{A}_0$ - 1) ~ ($\mathcal{A}_0$ - 7).

($\mathcal{A}_0$ - 1) (RR) of type-forms.

($\mathcal{A}_0$ - 2) The axiom on $\Delta(\equiv \Delta_1 \wedge \Delta_2)$. This claims that Δ be the constructive interpretation of G. That is, $\Delta(i, \Phi, \alpha)$

expresses the following: "it can be concretely demonstrated, by the method Φ, that α be in G_i."

Δ_1(one way implication of the inductive definition of G):

$$\forall i \forall \alpha \forall X \{ \Delta(i, X, \alpha) \rightarrow \forall j < i \forall \sigma (P_1(j, \sigma, \alpha) \vdash \Delta(j, \Pi(X; o, j, \sigma), \sigma)) \wedge \forall S \forall Y (P_2(i, \alpha, S) \wedge \forall n \forall j < i \forall \sigma (P_1(j, \sigma, \Pi(S; n)) \vdash \Delta(j, \Pi(Y; n, j, \sigma), \sigma) \vdash \forall n \geq \Pi(X; 1, S, Y)(\Pi(S; n) = \mathrm{ept}))\},$$

where $P_1(j, \sigma, \alpha)$ stands for σ is a j-section of α and $P_2(i, \alpha, S)$ stands for S is an i-decreasing sequence of diagrams led by α.

$$(\text{type-form of } X) = T\langle i \rangle$$

$$(\text{type-form of } Y) = T_2(i) = N_0 \rightarrow T_1\langle i \rangle$$

Δ_2 stands for the opposite implication.

($\mathcal{A}_0$ - 3) The axiom on Θ; $\Theta(i, \Psi, \gamma)$ expresses that Ψ guarantees

$$\gamma \in \mathrm{Ord}(i) (= O(i)).$$

A type form $\mathcal{R}[\mathcal{s}, \mathcal{t}, \nu, x; n]$ is used here.

($\mathcal{A}_0$ - 4) The axiom on Σ($\equiv$ lss, the orders of O(i)), which is $\mathcal{L}$-recursive.

($\mathcal{A}_0$ - 5) The axiom on $(\mathcal{J}_0, \eta_0, \zeta_0)$: $\mathcal{J}_0$ is the fundamental method for the diagrams which are guaranteed to be in J_i by η_0 and ζ_0.

($\mathcal{A}_0$ - 6) The axiom on μ_0 expresses that μ_0 serves as the "modulus of finiteness" function for the order $<_I$.

$$\forall S (S \text{ is } <_I\text{-decreasing} \vdash \Pi(\mu_0; S) \text{ gives the "modulus of finiteness" for } S)$$

$(\mathcal{A}_0 - 7)$ The formal expression of CNPR.

For the semantics of formula-forms, we place two more assumptions.

3. FSPR: Methods $\mathcal{J}^*$, η^*, ζ^* exist in correspondence to $\mathcal{J}_0$, η_0 and ζ_0 respectively so that they satisfy $(\mathcal{A}_0 - 5)$.

4. MFPR: A method μ^* is assumed for μ_0 to satisfy $(\mathcal{A}_0 - 6)$.

Definition. The truth-value of a formula-form relative to a complete assignment, inst(A, $\mathbf{b}$, $\mathbf{a}$), is defined naturally and "classically."

Examples. $\phi = \psi$ is true iff ϕ and ψ are equal as natural numbers.

$\Delta(i, X, \alpha)$ is false if (the assignment to) i, X, α do not make sense. Suppose $i \in \mathbf{I}$, $\alpha \in O$ and X is a method of hyper-type T<i>. Then the truth value of $\Delta(i, X, \alpha)$ is that of the succedent of $\rightarrow$ in Δ_1 of $(\mathcal{A}_0 - 2)$. It can be shown to be consistent with Δ_2 also.

The theories of type-forms, term-forms and formula-forms (including the semantics and the assumptions) is called the "hyper-principle", HP.

Now the story returns to Part III as was described earlier. The details will be omitted.

The manuscript has been typed by A. Shintani.

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