数理解析研究所講究録
第 588 巻 1986 年 193-195

Hilbert irreducibility sequences and nonstandard arithmetic

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Let $\mathbf{Q}^*$ and $\mathbf{Z}^*$ be enlargements of $\mathbf{Q}$ and $\mathbf{Z}$ respectively. Our aim of this paper is to give a sufficient condition for $x \in \mathbf{Z}^* - \mathbf{Z}$ that $\mathbf{Q}(x)$ has no algebraic extension of degree not more than m within $\mathbf{Q}^*$. As its application to number theory, we give irreducibility sequences explicitly.

By an arithmetical prime divisor, we mean a prime number or the archimedean prime p_∞. For each arithmetical prime p, we define p-adic absolute value of a rational number x,

$$|x|_p = p^{-n}$$

$$|x|_{p_\infty} = |x|$$

where $x = rp^n$ and r has no p factor. For each finite set S, of arithmetcal primes, we define

$$H_S(x) = \prod_{p \in S} \max(1, |x|_p)$$

$$H(x) = \prod_{p} \max(1, |x|_p) = \max(|m|, |n|)$$

where $x = m/n$ and g.c.d.$(m,n)=1$.

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THEOREM. *Let x be a nonstandard rational number. Assume (1) there is a finite set S of standard prime divisors such that*

$$\frac{\log(H_S(x)H_S(x^{-1}))}{\log H(x)} > 2-\frac{1}{m}+\varepsilon$$

for some standard positive real ε,

(2) for any nonzero standard rational number r and any natural number n with $2\leqq n\leqq m$, there is no nonstandard rational $y\in\mathbf{Q}^*-\mathbf{Q}$ such that $rx=y^n$.

Then $\mathbf{Q}(x)$ has no algebraic extension of degree not more than m within $\mathbf{Q}^*$.

Let us give an application of the theorem to standard number theory. A sequence of integers $a_1,a_2,\ldots,a_n,\ldots$ is called a m-irreducibility sequence if for any polynomial $f(X,Y)\in\mathbf{Z}[X,Y]$ with X-deg$(f)\leqq m$, there are only finitely many a_n such that $f(X,a_n)$ is reducible. A sequence of integers is called a Hilbert irreducibility sequence (H.i.seq.) if it is a m-irreducibility sequence for all natural number m. In his papers [3] and [4], V.G.Sprindzuk proved that

$$a_n=[\exp\sqrt{\log\log n}]+n!2^{n^2}$$

is a H.i.seq.. Oer theorem can give a different type of H.i. seq. from those given by Sprindzuk. For example, we will show that $2^n p_n$, $2^n(n^3+1)$ and $n!2^{n^2}$ are H.i.seq.s.

In nonstandard arithmetic, we have a beautiful charactorization of a H.i.seq. due to Gilmor and Robinson.

PROPOSITION 1. *a_n is a* H.i.seq. *if and only if for any nonstandard natural number $\omega\in\mathbf{N}^*-\mathbf{N}$, $\mathbf{Q}(a_\omega)$ is relatively* *algebraically closed in* $\mathbf{Q}^*$.

As for m-irreducibility we have the following sufficient condition for a sequence to be an m-irreducibility sequence

PROPOSITION 2. *If for any nonstandard natural number* ω, $\mathbf{Q}(a_\omega)$ *has no proper algebraic extension of degree not more than* $m!$ *within* $\mathbf{Q}^*$, *then* a_n *is an m-irreducibility sequence.*

Unfortunately the converse of Proposition 2 is not true but if $m!$ is replaced by m, then its converse holds.

PROPOSITION 3. *If* a_n *is an m-irreducibility sequence, then for any nonstandard natural number* ω, $\mathbf{Q}(a_\omega)$ *has no algebraic extension of degree not more than m within* $\mathbf{Q}^*$.

It is easily shown that Proposition 1 is a easy consequence of Proposition 2 and 3.

For the proofs of Theorem, Proposition 2 and 3, please refer to [5].

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