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Mixed Motives

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I would like to report on the concept of "mixed motive" for absolute Hodge cycles, introduced by U. Jannsen (unpublished). Jannsen associates such motives to arbitrary quasi-projective smooth varieties over a field k of characteristic zero and thus extends and at the same time simplifies the definition of motives for projective smooth varieties as it was given by Deligne [2].

Before we give the definition of mixed motive we study the cohomology of a smooth quasi projective variety X/k. There are three cohomology theories, Betti cohomology, deRham cohomology and l-adic cohomology, each of which is endowed with an additional structure, the first two ones with a mixed Hodge structure and the last one with the structure of a galois module.

I. Betti cohomology: For an imbedding $\sigma: k \to \mathbb{C}$ we denote by

$$H^n_\sigma(X) = H^n(\sigma X, \mathbb{Q})$$

the singular cohomology with rational coefficient of the analytic space $\sigma X = (X_k \times_\sigma \mathbb{C})(\mathbb{C})$. This $\mathbb{Q}$-vectorspace has a mixed $\mathbb{Q}$-Hodge structure, i.e. there is an increasing filtration $(W_m)_{m\in\mathbb{Z}}$ of $H^n_\sigma(X)$ (the weight filtration) and decreasing filtration $(F^n)_{n\in\mathbb{Z}}$ on $H^n_\sigma(X) \otimes_{\mathbb{Q}} \mathbb{C}$ (the Hodge filtration) which induces a $\mathbb{Q}$-Hodge structure of weight m on $G^W_m H^n_\sigma(X) = W_m H^n_\sigma(X)/W_{m-1}H^n_\sigma(X)$, i.e. a decomposition

$$Gr_m^W H_\sigma^n(X) \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{pq}$$

with $H_\sigma^{pq} = \bar{H}_\sigma^{qp}$ and $F^p Gr_m^W H_\sigma^n(X) = \bigoplus_{p' \geq p} H_\sigma^{p'q'}$.

The weight filtration is trivial if X is projective (i.e. $W_m H_\sigma^n(X) = H_\sigma^n(X)$ for $m \geq 2n$ and $=0$ for $m < 2n$) .

<u>II deRham cohomology.</u> We define the deRham cohomology $H_{DR}^n(X)$ as the Zariski hypercohomology of the algebraic deRham complex $\Omega^{\bullet}_{X/k}$,

$$H_{DR}^n(X) := \mathbb{H}^n (X_{Zar}, \Omega^{\bullet}_{X/k}) \ .$$

This k-vector space is endowed with a decreasing filtration $(F^n)_{n \in \mathbb{Z}}$ (the Hodge filtration) and an increasing filtration $(W_m)_{m \in \mathbb{Z}}$ (the weight filtration). The weight filtration is trivial when X is projective.

<u>III l-adic cohomology.</u> Let $\bar{k}$ be a fixed algebraic closure of k , $G_k = Gal(\bar{k}|k)$ and let l be a prime number. We define the l-adic cohomology of X by

$$H_l^n(X) := H_{et}^n(X \times_k \bar{k}, \mathbb{Q}_l) \ .$$

This is a $\mathbb{Q}_l$-vectorspace acted on by G_k via the G_k-action on $X \times_k \bar{k}$. Moreover the G_k-module $H_l^n(X)$ has a weight filtration $(W_m)_{m \in \mathbb{Z}}$ which is the filtration induced by the Leray spectral sequence

$$H_{et}^p(X \times_k \bar{k}, R^q j_* \mathbb{Q}_l) \Rightarrow H_{et}^{p+q}(X \times_k \bar{k}, \mathbb{Q}_l)$$

associated to an open immersion $j: X \hookrightarrow \bar{X}$ into a smooth projective variety. (The weight filtration of $H_\sigma^n(X)$ and $H_{DR}^n(X)$ can be obtained in quite a similar way).

The three cohomology theories $H^n_\sigma(X)$, $H^n_{DR}(X)$, $H^n_l(X)$ are related by canonical "comparison isomorphisms"

$$I_{\infty,\sigma}: H^n_\sigma(X) \underset{\mathbb{Q}}{\otimes} \mathbb{C} \xrightarrow{\sim} H^n_{DR}(X) \underset{k,\sigma}{\otimes} \mathbb{C} ,$$

$$I_{l,\bar\sigma}: H^n_\sigma(X) \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_l \xrightarrow{\sim} H^n_l(X)$$

where $\bar\sigma: \bar k \to \mathbb{C}$ is an extension of the imbedding $\sigma: k \to \mathbb{C}$. Jannsen proves that the isomorphism $I_{\infty,\sigma}$ respects the Hodge filtration and the weight filtration on the cohomology groups H^n_σ and H^n_{DR} and that the isomorphisms $I_{l,\bar\sigma}$ respect the weight filtrations and is compatible with G_k-action on $H^n_l(X)$ and on the imbeddings $\bar\sigma: \bar k \to \mathbb{C}$.

Having established these compatibilities he defines quite abstractly the <u>category of mixed realizations MR_k</u> as the category of all systems

$$H = (H_\sigma, H_{DR}, H_l;\ I_{\infty,\sigma}, I_{l,\bar\sigma})_{\sigma,\bar\sigma,l}$$

where

1). H_σ is a $\mathbb{Q}$-vector space endowed with a mixed $\mathbb{Q}$-Hodge structure.

2). H_{DR} is a k-vector space endowed with a decreasing filtration $(F^n)_{n\in\mathbb{Z}}$ and an increasing filtration $(W_m)_{m\in\mathbb{Z}}$.

3). H_l is a $\mathbb{Q}_l$-vector space endowed with a G_k-action and a filtration $(W_m)_{m\in\mathbb{Z}}$.

4). $I_{\infty,\sigma}: H_\sigma \underset{Q}{\otimes} \mathbb{C} \xrightarrow{\sim} H_{DR} \underset{k,\sigma}{\otimes} \mathbb{C}$ is an isomorphism respecting the filtrations $(F^n)_{n\in\mathbb{Z}}$ and $(W_m)_{m\in\mathbb{Z}}$ on both sides.

5). $I_{\infty,\bar\sigma} : H_\sigma \otimes \mathbb{Q}_l \xrightarrow{\sim} H_l$ is an isomorphism respecting the filtration $(W_m)_{m\in\mathbb{Z}}$ on both sides and is compatible with the G_k-action on H_l and on the imbeddings $\bar\sigma: \bar k \to \mathbb{C}$.

The morphisms $f: H \to H'$ in $\underline{MR}_k$ are systems of linear mappings $f_\alpha: H_\alpha \to H'_\alpha$, $\alpha = \sigma, DR, l$, respecting filtrations and galois actions.

The category $\underline{MR}_k$ of mixed realizations is endowed with an internal tensor product given by

$$H \otimes H' = (H_\sigma \otimes H'_\sigma, H_{DR} \otimes H'_{DR}, H_l \otimes H'_l; I_{\infty,\sigma} \otimes I'_{\infty,\sigma}, I_{l,\bar\sigma} \otimes I'_{l,\bar\sigma})$$

where the filtrations are defined by

$$W_m(H_\alpha \otimes H'_\alpha) = \sum_{r+s=m} W_r H_\alpha \otimes W_s H'_\alpha$$

$$(\alpha \in \{\sigma, DR, l\}).$$

$$F^n(H_\alpha \otimes H'_\alpha) = \sum_{p+q=m} F^p H_\alpha \otimes F^q H_\alpha$$

$\underline{MR}_k$ contains the identity object

$$1 = (\mathbb{Q}, k, \mathbb{Q}_l; id_{\infty,\sigma}, id_{l,\sigma})$$

and for each object $M \in \underline{MR}_k$ the dual object $\check{M} = \underline{Hom}(M, 1)$.
It turns out that $\underline{MR}_k$ is a Tannakian category over $\mathbb{Q}$ with fiber functor $\underline{MR}_k \to Vec_{\mathbb{Q}}$, $H \mapsto H_\sigma$.

Now let $\overset{o}{V}_k$ (rsp. V_k) be the category of smooth quasi-projective (resp. projective) varieties over k . By what we have said above we have for each n a functor

$$H^n: \overset{o}{V}_k \to \underline{MR}_k \ , \ X \mapsto (H^n_\sigma(X), H^n_{DR}(X), H^n_l(X); I_{\infty,\sigma}, I_{l,\bar\sigma})$$

and we define the functor

$$H: \overset{o}{V}_k \to \underline{MR}_k \quad \text{by} \quad H(X) = \bigoplus_n H^n(X)$$

Definition (U. Jannsen). The category $\underline{MM}_k$ of mixed motives (resp. motives) for absolute Hodge cycles is the Tannakian subcategory of $\underline{MR}_k$ generated by the image of $\overset{o}{V}_k$ (resp. V_k) under H .

If $X \in \overset{o}{V}_k$ then automatically the mixed realization $H^n(X)$ is a mixed motive since it is a direct summand of $H(X)$, i.e. kernel of an idempotent in $\mathrm{End}(H(X))$, and since, by definition, $\underline{MM}_k$ is closed under forming kernels and cokernels. Moreover, it is easy to see, that the mixed motives are precisely the subquotients of $H(X) \otimes H(Y)^\vee = \underline{Hom}(H(Y),H(X))$, $X,Y \in \overset{o}{V}_k$. $\underline{MM}_k$ contains also the Tate motive

$$\mathbb{1}(1) = H^2(\mathbb{P}^1_k)^\vee = (\mathbb{Q}(1), k_l(1), \mathbb{Q}_l(1);\ id_{\infty,\sigma}(1), id_{l,\bar{\sigma}}(1))$$

where $\mathbb{Q}(1) = 2\pi i\mathbb{Q}$, $k(1) = k$, $\mathbb{Q}_l(1) = (\varprojlim \mu_{l^n}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, μ_{l^n} the group of l^n-th roots of unity. To each mixed motive we have also the Tate twists

$$M(r) = M \otimes \mathbb{1}(r) \in \underline{MM}_k$$

where $\mathbb{1}(0) = \mathbb{1}$, $\mathbb{1}(r) = \mathbb{1}(1)^{\otimes r}$ for $r > 0$ and $\mathbb{1}(r) = \mathrm{Hom}(\mathbb{1}(-r),\mathbb{1}$ for $r < 0$.

For each mixed motive M we call

$$\Gamma(M) = \mathrm{Hom}(\mathbb{1},M)$$

the group of absolute Hodge cycles. If M is the motive $H^{2p}(X)$ of a smooth projective variety X , then $\Gamma(M)$ is precisely the group of absolute Hodge cycles as defined by Deligne. Moreover for any two $M,N \in \underline{MM}_k$ we have

$$\mathrm{Hom}_{\underline{MM}_k}(M,N) = \Gamma(\underline{Hom}(M,N)) = \Gamma(N \otimes M^\vee) ,$$

showing in particular that the categories of motives defined by Deligne and by Jannsen are the same.

The mixed motives yield interesting aspects if we look at them from the Tannakian point of view. Consider the fiber functors

$$H_\sigma : \underline{MM}_k \to \mathrm{Vec}_{\mathbb{Q}} \quad \text{and} \quad H_\sigma : \underline{M}_k \to \mathrm{Vec}_{\mathbb{Q}}$$

of the Tannakian categories $\underline{MM}_k$ and $\underline{M}_k$, and let $MG(\sigma)$ and $G(\sigma)$ be the automorphism group of these fiber functors. These are pro-algebraic groups and the main theorem on Tannakian categories yields equivalences of tensor categories

$$\underline{M}_k \xrightarrow{\sim} \underline{\text{Rep}}\ G(\sigma)$$
$$\underline{MM}_k \xrightarrow{\sim} \underline{\text{Rep}}\ MG(\sigma) .$$

where $\underline{\text{Rep}}\ G$ denotes the category of finite dimensional representations of a pro-algebraic group $G/\mathbb{Q}$. The inclusion $\underline{M}_k \hookrightarrow \underline{MM}_k$ corresponds to a homomorphism $MG(\sigma) \xrightarrow{\psi} G(\sigma)$, and we obtain an exact sequence

$$0 \to U(\sigma) \to MG(\sigma) \xrightarrow{\psi} G(\sigma) \to 0$$

of pro-algebraic groups, in which $G(\sigma)$ is the maximal pro-reductive quotient of $MG(\sigma)$ and the kernel $U(\sigma)$ of ψ is connected and pro-unipotent.

Literature

[1] Deligne, P. Theorie de Hodge I, II, III, Actes ICM Nive (1970) and Publ. IHES 40, 44.

[2] Deligne, Milne, Ogus, Shih, Hodge cycles, Motives and Shimura Varieties. Springer Lecture Notes 900.

[3] Jannsen, U. Mixed Motives (to appear).