ON A QUESTION ARISING FROM COMPLEX MULTIPLICATION THEORY

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§0. Introduction

An abelian variety A defined over ℂ, equipped with complex multiplication and level structure, is described up to isomorphism by some invariants that are "analytic" in nature. (The details of this description are reviewed in §1.) Let s be an arbitrary automorphism of ℂ. Tate conjectured [8] and Deligne proved [1] a formula for the analytic invariants of A^s, the conjugate of A under s, in terms of classfield theory. (See also Lang [4], in which summaries of the contents of [1,8] can be found.) Tate's formula generalizes the classical reciprocity law of Shimura-Taniyama [6,7] to the case in which s does not necessarily fix the CM type of A. (Tate's formula is reviewed in §1.)

Now figuring prominently in Tate's formula is a certain cocycle. The task we set for ourselves in this paper is to abstract the construction of the cocycle figuring in Tate's formula making possible the subsequent specialization of that construction to the function field case. This task is carried out in §2.

The eponymous question of the paper is not the question answered by the investigation of §2, but rather the question raised by it: What interpretation can be given to the new cocycle which we have constructed in the function field case? This is an open problem; the author expects the solution

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to be found in an as-of-yet-undeveloped theory of higher-

dimensional Drinfeld modules with complex multiplication

in which, in particular, an analogue of Tate's formula is valid.

§1. The Basic Problem of Complex Multiplication Theory

1.0. Notation: We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Complex conjugation is denoted by ρ. By the term numberfield,

we understand an extension of $\mathbb{Q}$ of finite degree embedded in $\mathbb{C}$.

Given a numberfield K, let

$$r_K : (\text{idèle group of } K) \longrightarrow \text{Gal}(K^{ab}/K)$$

denote the reciprocity law of classfield theory, where K^{ab} denotes

the maximal abelian extension of K in $\overline{\mathbb{Q}}$. Let $\hat{\mathbb{Z}}$ denote the

profinite completion of $\mathbb{Z}$, and let $\chi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \hat{\mathbb{Z}}^{\times}$ denote

the cyclotomic character. Let K^{abs} denote the largest subfield

of K^{ab} in which every archimedean place of K splits completely,

and let

$$\overline{r}_K : (K \otimes \hat{\mathbb{Z}})^{\times} \longrightarrow \text{Gal}(K^{abs}/K)$$

denote the unique homomorphism rendering the diagram

$$
\begin{array}{ccc}
(\text{idèle group of } K) & \xrightarrow{\ r_K\ } & \text{Gal}(K^{ab}/K) \\
\big\downarrow{\text{forget components at } \infty} & & \big\downarrow{\sigma \mapsto \sigma^{-1}\big|_{K^{abs}}} \\
(K \otimes \hat{\mathbb{Z}})^{\times} & \xrightarrow[\ \overline{r}_K\ ]{} & \text{Gal}(K^{abs}/K)
\end{array}
$$

commutative. Note that, in particular,

$$\bar{r}_{\mathbb{Q}}(\chi(\sigma)) = \text{the restriction of } \sigma \text{ to } \mathbb{Q}^{abs}.$$

1.1. Let K be a <u>CM numberfield</u>, i. e. a numberfield K such that for all $x \in K$ and $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\sigma \rho x = \rho \sigma x \in \sigma K \not\subseteq \mathbb{R}.$$

Let $\Phi \subseteq \text{Hom}(K,\mathbb{C})$ be a <u>CM type</u>, i. e. a subset Φ such that

$$\Phi \cap \rho\Phi = \emptyset, \qquad \Phi \cup \rho\Phi = \text{Hom}(K,\mathbb{C}).$$

Let $\mathcal{O}_K$ denote the ring of integers of K, and $H_1(A)$ the first singular homology group of the complex manifold underlying A. A homomorphism $\theta: \mathcal{O}_K \longrightarrow \text{End}_{\mathbb{C}}(A)$ relative to which $H_1(A)$ becomes a rank one projective $\mathcal{O}_K$-module is termed (a structure of) <u>complex multiplication</u> by $\mathcal{O}_K$. The complex multiplication θ is said to be <u>of type</u> Φ if

$$\Phi = \left\{ \tau \in \text{Hom}(K,\mathbb{C}) \mid \text{Lie}(A)_\tau \neq 0 \right\},$$

where for each $\tau \in \text{Hom}(K,\mathbb{C})$,

$$\text{Lie}(A) \overset{\text{def}}{=} \left\{ v \in \text{Lie}(A) \mid \forall x \in \mathcal{O}_K, \; \theta(x)_* v = \tau(x)v \right\}.$$

An <u>continuous parameterization</u> of an abelian variety A defined over $\mathbb{C}$ endowed with complex multiplication θ by $\mathcal{O}_K$ is by definition an $\mathcal{O}_K$-equivariant isomorphism of real Lie groups

$$\lambda: (\mathfrak{a} \otimes \mathbb{R})/\mathfrak{a} \overset{\sim}{\longrightarrow} A(\mathbb{C})$$

where $\mathfrak{a}$ is a suitably chosen fractional ideal of $\mathcal{O}_K$.

1.2. Let K be a CM numberfield. Let A be an abelian variety defined over $\mathbb{C}$ endowed with complex multiplication θ by $\mathcal{O}_K$ of type Φ. Let s be an automorphism of $\mathbb{C}$. Let $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denote the restriction of s to $\overline{\mathbb{Q}}$. Let $s\theta$ denote the composition of θ with the "transport of structure" isomorphism $\text{End}_{\mathbb{C}}(A) \overset{\sim}{\longrightarrow} \text{End}_{\mathbb{C}}(sA)$, where sA denotes the conjugate of A relative to s.

Then sA is endowed with complex multiplication $s\theta$ of type $s\Phi$.
Select continuous parameterizations $\lambda:(\mathcal{a}\otimes\mathbb{R})/\mathcal{a}\xrightarrow{\sim}A(\mathbb{C})$ and
$\mu:(\mathcal{b}\otimes\mathbb{R})/\mathcal{b}\xrightarrow{\sim}sA(\mathbb{C})$. Let A_{tor} and sA_{tor} denote the torsion
subgroups of $A(\mathbb{C})$ and $sA(\mathbb{C})$, respectively. Then there exists
unique $g\in(K\otimes\hat{\mathbb{Z}})^{\times}$ rendering the diagram

$$
\begin{array}{ccc}
K/\mathcal{a} & \xrightarrow[\sim]{\lambda} & A_{tor} \\
{\scriptstyle x\mapsto gx}\downarrow & {\scriptstyle \mu} & \downarrow{\scriptstyle a\mapsto sa} \\
K/\mathcal{b} & \xrightarrow[\sim]{} & sA_{tor}
\end{array}
$$

commutative. It can be shown without great difficulty that
modulo $K^{\times}\subseteq(K\otimes\hat{\mathbb{Z}})^{\times}$, g depends only upon K, Φ and σ. Set

$$g_{K}(\sigma,\Phi)\overset{def}{=}gK^{\times}\in(K\otimes\hat{\mathbb{Z}})^{\times}/K^{\times}.$$

One of the most basic problems of complex multiplication
theory is the determination of $g_{K}(\sigma,\Phi)$. The reciprocity
law of Shimura-Taniyama [6,7] is, in effect, a formula for
$g_{K}(\sigma,\Phi)$ in the case $s\Phi=\Phi$. Tate conjectured [8] a
formula in the general case more or less equivalent to the
"0-dimensional case" of Langlands' conjecture [5] on the
conjugation of Shimura varieties. Both Tate's conjectural
formula and 0-dimensional Langlands' conjecture were later
proven by Deligne [1,2]. The remainder of §1 is devoted to
a presentation of Tate's formula.

1.3. The cocycle $g_{?}(?,?)$ constructed in the preceding paragraph
satisfies many functional equations. We note here several of
the most important functional equations. Let K be a CM field,

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Φ a CM type of K, and σ, τ elements of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let L be a CM numberfield containing K, and let Ψ be the unique CM type of L such that

$$\Phi = \left\{ \psi \big|_K \in \mathrm{Hom}(K,\mathbb{C}) \; \middle| \; \psi \in \Psi \right\}.$$

Then the following relations hold:

$$g_K(\sigma, \tau\Phi) g_K(\tau, \Phi) = g_K(\sigma\tau, \Phi). \qquad (1.3.1)$$

$$(\tau \otimes 1) g_K(\sigma, \Phi) = g_{\tau K}(\sigma, \Phi\tau^{-1}). \qquad (1.3.2)$$

$$g_K(\sigma, \Phi) g_K(\sigma, \rho\Phi) \equiv \chi(\sigma) \quad \mod K^{\times}. \qquad (1.3.3)$$

$$g_L(\sigma, \Psi) \equiv g_K(\sigma, \Phi) \quad \mod L^{\times}. \qquad (1.3.4)$$

Of these relations, only (1.3.3) is not a purely formal consequence of the definition. In order to prove (1.3.3) one considers polarizations. (See Tate [8] or Lang [4].)

1.4. Deligne uncovered a much deeper functional equation for $g_?(?,?)$, a consequence of his theory of absolute Hodge cycles on abelian varieties [2]. Let K be a CM field. Given a CM type Φ of K, one attaches the characteristic function

$$[\Phi]_K \overset{\mathrm{def}}{=} \left(\sigma \longmapsto \begin{array}{l} 1 \text{ if } \sigma\big|_K \in \Phi \\ 0 \text{ if } \sigma\big|_K \notin \Phi \end{array} \right) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \{0,1\}.$$

Deligne observed [1] that given CM types $\Phi_1, \ldots, \Phi_n$ and $m_1, \ldots, m_n \in \mathbb{Z}$ such that

$$\sum m_i [\Phi_i]_K = 0,$$

one has

$$\prod g_K(\sigma, \Phi_i)^{m_i} \equiv 1 \quad \mod K^{\times} \qquad (1.4.1)$$

for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

1.5. We consider Tate's <u>half-transfer</u> construction. Let K be a CM numberfield. For each embedding $\tau: K \to \overline{\mathbb{Q}}$, select a lifting $w_\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ subject to the constraint

$$w_{\rho\tau} = \rho\, w_\tau \ . \qquad\qquad (1.5.1)$$

Then, according to Tate [8], for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and CM type Φ of K, there exists unique $F_K(\sigma, \Phi) \in \mathrm{Gal}(K^{ab}/K)$ such that

$$F_K(\sigma, \Phi) \overset{def}{\equiv} \prod_{\tau \in \Phi} (w_{\sigma\tau}^{-1} \sigma\, w_\tau) \qquad \mathrm{mod}\ G(K^{ab}),$$

independent of the choice of a lifting $w_\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for each embedding $\tau: K \to \overline{\mathbb{Q}}$ subject to condition (1.5.1) and the choice of an ordering of the product. (A proof of this independence in a more general context will be given in §2.) Tate termed this construction the <u>half-transfer</u> because

$$F_K(\sigma, \Phi) F_K(\sigma, \rho\Phi) = \mathrm{Ver}_{K/\mathbb{Q}}(\sigma), \qquad (1.5.2)$$

where $\mathrm{Ver}_{K/\mathbb{Q}}: \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \to \mathrm{Gal}(K^{ab}/K)$ denotes the transfer homomorphism. Tate went on to show that there exists unique $f_K(\sigma, \Phi) \in \hat{K}^\times/K^\times$ such that

$$((\rho \otimes 1) f_K(\sigma, \Phi)) f_K(\sigma, \Phi) \equiv \chi(\sigma) \qquad \mathrm{mod}\ K^\times, \qquad (1.5.3)$$

$$\overline{r}_K(f_K(\sigma, \Phi)) = F_K(\sigma, \Phi). \qquad (1.5.4)$$

(A proof of this uniqueness result in a more general context will be given in §2.) Since $g_K(\sigma, \Phi)$ possesses property (1.5.3) and, in the case $\sigma\Phi = \Phi$, possesses (1.5.4) by Shimura-Taniyama reciprocity, Tate was inspired to conjecture [8]

$$f_K(\sigma, \Phi) = g_K(\sigma, \Phi). \qquad (1.5.5)$$

According to Deligne [1], (1.5.5) is the consequence of (1.3.1,2,3,4), (1.4.1) and Shimura-Taniyama reciprocity.

1.6. In order to complete the task of motivating the abstract cocycle construction of §2, we prove the following

Proposition. Let $f: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}$ be a locally constant function. The following two conditions are equivalent:

(I)   The function f is an integral linear combination

of functions of the form $[\Phi]_K$, K an arbitrary CM field

and Φ an arbitrary CM type of K.

(II)   For all $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$f(\sigma\rho\tau) + f(\sigma\tau) = f(\rho) + f(1).$$

Proof. The implication (I) $\Rightarrow$ (II) is obvious. We turn to the proof of (II) $\Rightarrow$ (I). For all $\sigma, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$f(\sigma(\tau\rho\tau^{-1}\rho)) = f(\rho) + f(1) - f(\sigma\rho) = f(\sigma).$$

Therefore, for a suitable CM numberfield K galois over $\mathbb{Q}$, the function f factors through $\mathrm{Gal}(K/\mathbb{Q})$. Select a CM type Φ of K arbitrarily and identify Φ with a subset of $\mathrm{Gal}(K/\mathbb{Q})$ in the evident fashion. Set

$$w \stackrel{\mathrm{def}}{=} f(\rho) + f(1).$$

Replacing f by $f - w[\Phi]_K$, we may assume that $w = 0$. For each $\tau \in \Phi$, set

$$g_\tau \stackrel{\mathrm{def}}{=} [\Phi]_K - [(\Phi \cup \{\rho\tau\}) \sim \{\tau\}]_K.$$

Then

$$f = \sum_{\tau \in \Phi} f(\tau) g_\tau. \quad \blacksquare$$

§2. The Abstract Cocycle Construction

2.0. Notation: Let k be a global field and let ∞ be a place of k. We assume that, in case char(k) = 0, the place ∞ is the unique archimedean place of k. Let k_∞ denote the completion of k at ∞, k_∞^s a fixed separable algebraic closure of k, and k^s the separable algebraic closure of k in k^s. Note that k^s is a separable algebraic closure of k. Let us agree to restrict the use of the term global field henceforth to the designation of subfields of k^s containing k and of finite degree over k. Set

$$G \stackrel{def}{=} Gal(k^s/k), \quad D \stackrel{def}{=} Gal(k_\infty^s/k_\infty),$$

identifying D with a closed subgroup of G in the evident fashion. Given any subfield K of k^s containing k, we write

$$G(K) \stackrel{def}{=} Gal(k^s/K) \subseteq G.$$

Given a global field K and a place v of K, we say that v is infinite if v lies above ∞, and finite otherwise. We write

$$K^{ab} \stackrel{def}{=} \text{the maximal abelian extension of K in } k^s,$$
$$K^{abs} \stackrel{def}{=} \text{the largest subfield of } K^{ab} \text{ in which every}$$
$$\text{infinite place of K splits completely.}$$

Set

$$A \stackrel{def}{=} \left\{ x \in k \;\middle|\; |x|_v \geq 1 \text{ for all finite } v \right\},$$
$$\hat{A} \stackrel{def}{=} \varprojlim A/\mathcal{O},$$

where the inverse limit is extended over all nonzero ideals $\mathcal{O}$ of A. Given a global field K, we write

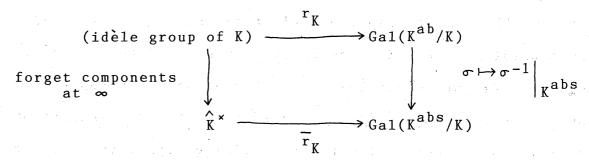$$\hat{K} \stackrel{def}{=} K \underset{A}{\otimes} \hat{A}.$$

We denote by

$$r_K : (\text{idèle group of K}) \longrightarrow Gal(K^{ab}/K)$$

the reciprocity law homomorphism of global classfield theory,

and define

$$\overline{r}_K : \hat{K}^\times \longrightarrow Gal(K^{abs}/K)$$

to be the unique homomorphism rendering the diagram

$$\begin{array}{ccc}
(\text{idèle group of } K) & \xrightarrow{\quad r_K \quad} & Gal(K^{ab}/K) \\
\text{forget components} \Big\downarrow & & \Big\downarrow \quad \sigma \mapsto \sigma^{-1}\Big|_{K^{abs}} \\
\text{at } \infty & & \\
\hat{K}^\times & \xrightarrow[\quad \overline{r}_K \quad]{} & Gal(K^{abs}/K)
\end{array}$$

commutative.

2.1. Let X denote the set of locally constant functions $\Phi : G \longrightarrow \mathbb{Z}$ such that for all $\sigma, \tau \in G$,

$$\int_D \Phi(\sigma\rho\tau)\,d\rho = \int_D \Phi(\rho)\,d\rho,$$

where

$$d\rho \overset{\text{def}}{=} \text{normalized Haar measure on D.}$$

Given $\sigma \in G$ and $\Phi \in X$, we define $\sigma\Phi \in X$ and $\Phi\sigma \in X$ by the rules

$$(\sigma\Phi)(\tau) \overset{\text{def}}{=} \Phi(\sigma^{-1}\tau), \quad (\Phi\sigma)(\tau) \overset{\text{def}}{=} \Phi(\tau\sigma^{-1}).$$

For each global field K, set

$$X(K) \overset{\text{def}}{=} \left\{ \Phi \in X \mid \forall_{\sigma \in G(K)}, \ \Phi\sigma = \Phi \right\}.$$

__Lemma__. Let K be a global field, B an abelian group (written additively) and $f : G \longrightarrow B$ a function factoring through the double coset space $D\backslash G/G(K)$. Then for all $\Phi \in X(K)$, the function

$$\sigma \longmapsto \sum_{\tau \in G/G(K)} f(\sigma\tau)\,\Phi(\tau) : G \longrightarrow B$$

is constant.

<u>Proof</u>. We may assume without loss of generality that B = $\mathbb{Z}$ and that f is the characteristic function of a subset S of G such that DSG(K) = S. Set

$$d\gamma \overset{\text{def}}{=} \text{ normalized Haar measure on G.}$$

Then

$$[G:G(K)]^{-1} \sum_{\tau \in G/G(K)} f(\sigma\tau)\, \mathcal{Z}(\tau) = \int_G f(\sigma\gamma)\, \mathcal{Z}(\gamma)\, d\gamma$$

$$= \int_G \left(\int_D f(\rho^{-1}\sigma\gamma)\, d\rho \right) \mathcal{Z}(\gamma)\, d\gamma = \int_G f(\gamma) \left(\int_D \mathcal{Z}(\sigma^{-1}\rho\gamma)\, d\rho \right) d\gamma$$

$$= \left(\int_G f(\gamma)\, d\gamma \right) \left(\int_D \mathcal{Z}(\rho)\, d\rho \right) . \quad \blacksquare$$

2.2 For each global field K, let W(K) denote the set of functions w:G $\longrightarrow$ G such that w factors through G/G(K) and such that for all $\sigma \in$ G and $\tau \in$ D,

$$w(\sigma) \in \sigma G(K), \quad w(\tau\sigma)w(\sigma)^{-1} \in D.$$

Given any $\tau \in$ G(K), let τ_K denote the image of τ in Gal(K^{abs}/K) under restriction.

<u>Lemma</u>/<u>Definition</u>. There exists for all global fields K, $\sigma \in$ G, and $\mathcal{Z} \in$ X(K) a unique element $F_K(\sigma, \mathcal{Z})$ of Gal(K^{abs}/K) such that for all w $\in$ W(K),

$$F_K(\sigma, \mathcal{Z}) = \overline{\prod_{\tau \in G/G(K)}} (w(\sigma\tau)^{-1}\sigma\, w(\tau))_K^{\mathcal{Z}(\tau)} .$$

<u>Proof</u>. Provisionally, let us denote the product on the right by $F_K(\sigma, \mathcal{Z}, w)$. At issue is the dependence of $F_K(\sigma, \mathcal{Z}, w)$ upon w.

Given also $w' \in W(K)$, we have

$$F_K(\sigma, \Phi, w') F_K(\sigma, \Phi, w)^{-1} = \prod_{\tau \in G/G(K)} (h(\sigma\tau)^{-1} h(\tau))^{\Phi(\tau)},$$

where $h: G \longrightarrow \text{Gal}(K^{abs}/K)$ is given by the rule

$$h(\sigma) \overset{\text{def}}{=} (w'(\sigma) w(\sigma)^{-1})_K.$$

Now the function h factors through the double coset space $D\backslash G/G(K)$

by the lemma of ¶2.1,

$$F_K(\sigma, \Phi, w) = F_K(\sigma, \Phi, w'). \quad \blacksquare$$

__Theorem__/__Definition__. There exists one and only one way to assign

to each triple (K, σ, Φ) consisting of a global field K, $\sigma \in G$,

and $\Phi \in X(K)$ an element $f_K(\sigma, \Phi)$ of $\hat{K}^{\times}/K^{\times}$ such that for all

global fields $L \supseteq K$, Φ, $\Phi' \in X(K)$, $\sigma, \tau \in G$, the following

relations hold:

(I) $\qquad\qquad f_K(\sigma, \tau\Phi) f_K(\tau, \Phi) = f_K(\sigma\tau, \Phi).$

(II) $\qquad\qquad (\tau \underset{A}{\otimes} 1) f_K(\sigma, \Phi) = f_{\tau K}(\sigma, \Phi\tau^{-1}).$

(III) $\qquad\qquad f_K(\sigma, \Phi) \equiv f_L(\sigma, \Phi) \mod L^{\times}.$

(IV) $\qquad\qquad f_K(\sigma, \Phi) f_K(\sigma, \Phi') = f_K(\sigma, \Phi + \Phi').$

(V) $\qquad\qquad \overline{r}_K(f_K(\sigma, \Phi)) = F_K(\sigma, \Phi).$

The proof is deferred, pending some discussion of the theorem

and the proofs of some lemmas.

2.3. Let us consider the theorem in the case $(k, k_{\infty}^s, k^s) =$

$(\mathbb{Q}, \mathbb{C}, \overline{\mathbb{Q}})$. Then for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, CM numberfields K and

CM types Φ of K, the value of the symbol $f_K(\sigma, \Phi)$ defined in §1,

following Tate, coincides with the value of the symbol

$f_K(\sigma, [\Phi]_K)$ as defined by the theorem above.

2.4. Next, let us consider the theorem in the case that the characteristic of k is <u>nonzero</u>. Then the cocycle $f_?(?,?)$ does not figure in a reciprocity law analogous to (1.5.5), as far as the author knows. But inspired by Drinfeld's paper [3], we pose the eponymous question of the paper: Does there exist a theory of higher-dimensional Drinfeld modules with complex multiplication in which a cocycle $g_?(?,?)$ can be defined <u>without</u> <u>the use of classfield theory</u>, such that a significant reciprocity law of the form "f=g" is valid? The author is convinced that this is indeed possible; a few steps toward this goal were taken in the author's paper [ ].

2.5. Let K be a global field and let U(K) denote the set of nonzero elements u of K such that for all finite places v of K, $|u|_v = 1$. For each finite set S of finite places of K and $0<\varepsilon<1$, let $U(K;S,\varepsilon)$ denote the set of elements u of U(K) such that for all places v of S, $|u-1|_v \leq \varepsilon$.

<u>Lemma 2.5.1</u>. For all positive integers n and global fields K, there exists a finite set S of finite places of K and $0<\varepsilon<1$ such that $U(K)^n \supseteq U(K;S,\varepsilon)$.

<u>Proof</u>. This is due to Chevalley, but the author cannot find a suitable reference; therefore the following sketch of a proof is offered. We may assume at the outset that n is a power of a prime number p. We then distinguish four cases:

(I) $p = char(K) = n$.

(II) $p = char(K) < n$.

(III) $p \neq char(K)$ and $K \supseteq \mu_n(k^s)$.

(IV) $p \neq char(K)$ and $K \not\supseteq \mu_n(k^s)$.

In case (I), we consider the space $V(K)$ consisting differentials of the form $\frac{du}{u}$, u an arbitrary element of $U(K)$. The sequence

$$1 \to U(K)^P \longrightarrow U(K) \xrightarrow{\;u \mapsto \frac{du}{u}\;} V(K) \longrightarrow 0$$

is well known to be exact. Therefore, in order to insure that $U(K)^P$ contain $U(K;S,\varepsilon)$, it is necessary only that S be large enough so that any differential ω belonging to $V(K)$ such that ω vanishes at v for all $v \in S$ vanishes <u>identically</u>. In case (II) note that

$$U(K)^n \supseteq U(K;S,\varepsilon) \quad \Rightarrow \quad U(K)^{nP} \supseteq U(K;S,\varepsilon^P).$$

Thus case (II) is disposed of by an induction the base step of which is case (I). In case (III), one considers the finite Kummer extension L of K in k^S obtained by adjoining every $n^{\underline{th}}$ root of every element of $U(K)$ to K. One insures that $U(K)^n$ contain $U(K;S,\varepsilon)$ by taking S to be any finite set of finite places of K unramified in L such that

$$\{(v,L/K) \in \mathrm{Gal}(L/K) \mid v \in S\}$$

generates $\mathrm{Gal}(L/K)$, where $(v,L/K)$ is the Artin symbol. Finally in case (IV), it is enough, in view of case (III), to prove that the index

$$I_n \overset{\mathrm{def}}{=} [U(K_n)^n \cap U(K) : U(K)^n]$$

is bounded, where K_n denotes the field obtained by adjoining the $n^{\underline{th}}$ roots of unity in k^S to K. Now the map

$$u \mapsto \sqrt[n]{u} : U(K_n)^n \bigcap U(K) \longrightarrow (U(K_n)/\!/\mu_n(K_n))^{G_n}$$

is an isomorphism, where

$$G_n \stackrel{\text{def}}{=} \text{Gal}(K_n/K).$$

Thus by considering the long exact sequence in G_n-cohomology

for the short exact sequence

$$1 \longrightarrow \mu_n(K_n) \longrightarrow U(K_n) \longrightarrow U(K_n)/\!/\mu_n(K_n) \longrightarrow 1,$$

one arrives at the estimate

$$I_n \leq \#H^1(G_n, \mu_n(K_n)).$$

The quantity on the right is easily shown to be bounded for n

ranging over the powers of the prime p. ∥

Given a global field K, let $\mathcal{O}_K$ denote the ring of elements of K

regular at all finite places of K, and let $\hat{U}(K)$ denote the

closure of the group U(K) in the profinite group $(\mathcal{O}_K \otimes_A \hat{A})^\times$.

By Lemma 2.5.1, $\hat{U}(K)$ is precisely the profinite completion of

U(K). For any global field K galois over k, and left G-module

M, let $\text{Hom}_G(X(K), M)$ denote the group of homomorphisms $f : X(K) \longrightarrow M$

of abelian groups such that for all $\sigma \in G$ and $\mathcal{P} \in X(K)$,

$$f(\mathcal{P}\sigma^{-1}) = \sigma f(\mathcal{P}).$$

Lemma 2.5.2. For all global fields K galois over k, the

homomorphism $\text{Hom}_G(X(K), \hat{K}^\times/K^\times) \longrightarrow \text{Hom}_G(X(K), \text{Gal}(K^{\text{abs}}/K))$ induced

by $\bar{r}_K$ is an isomorphism.

Proof. The sequence of Gal(K/k)-modules

$$1 \longrightarrow \hat{U}(K)/U(K) \longrightarrow \hat{K}^\times/K^\times \xrightarrow{\ \bar{r}_K\ } \text{Gal}(K^{\text{abs}}/K) \longrightarrow 1$$

is exact. It therefore suffices to prove the following two

statements:

$\hat{U}(K)/U(K)$ is an injective $Gal(K/k)$-module. (2.5.3)

$Hom_G(X(K),\hat{U}(K)/U(K)) = 0.$ (2.5.4)

Now $\hat{\mathbb{Z}}/\mathbb{Z}$ is infinitely and uniquely divisible, i. e. a vector space over $\mathbb{Q}$. The statement (2.5.3) now follows from the observation that there exists a $Gal(K/k)$-equivariant isomorphism

$$\hat{U}(K)/U(K) \xrightarrow{\sim} U(K) \otimes (\hat{\mathbb{Z}}/\mathbb{Z}),$$

and, further, (2.5.4) now reduces to

$$Hom_G(X(K),U(K) \otimes \mathbb{Q}) = 0. \qquad (2.5.5)$$

Let $|?|$ denote an extension to k_∞^s of the absolute value $|?|_\infty$ of k_∞. Then for all $u \in U(K)$ such that there exists $\mathcal{D} \in X(K)$ and $f \in Hom_G(X(K),U(K) \otimes \mathbb{Q})$ such that $f(\mathcal{D}) = u \otimes 1_\mathbb{Q}$, and for all $\sigma \in G$,

$$\log |\sigma u| = \int_D \log |\rho \sigma u| \, d\rho = \log |u|, \qquad (2.5.6)$$

by appeal to the definition of X. But (2.5.6) implies that u is a root of unity, hence $u \otimes 1_\mathbb{Q} = 0$. This establishes (2.5.5) and completes the proof of the lemma. ▌

Lemma 2.5.7. For all global fields $L \supseteq K$, where L is galois over K, the sequence

$$1 \longrightarrow K^\times \longrightarrow \hat{K}^\times \longrightarrow (\hat{L}^\times/L^\times)^{Gal(L/K)} \longrightarrow 1$$
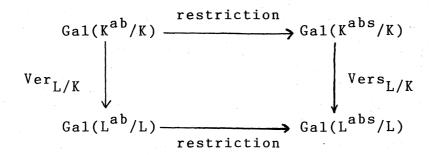
is exact.

Proof. Hilbert's Theorem 90. ▌

2.6. For each pair $L \supseteq K$ of global fields, let

$$Ver_{L/K}: Gal(K^{ab}/K) \longrightarrow Gal(L^{ab}/L)$$

denote the **transfer homomorphism**. Let

$$Vers_{L/K}: Gal(K^{abs}/K) \longrightarrow Gal(L^{abs}/L)$$

denote the unique continuous homomorphism rendering the diagram

$$\begin{array}{ccc}
\text{Gal}(K^{ab}/K) & \xrightarrow{\ \text{restriction}\ } & \text{Gal}(K^{abs}/K) \\[2mm]
{\scriptstyle \text{Ver}_{L/K}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{Vers}_{L/K}} \\[2mm]
\text{Gal}(L^{ab}/L) & \xrightarrow[\ \text{restriction}\ ]{} & \text{Gal}(L^{abs}/L)
\end{array}$$

commutative; in order that $\text{Vers}_{L/K}$ exist, it is necessary and
sufficient that

$$\text{Ver}_{L/K}(\text{Gal}(K^{ab}/K^{abs})) \subseteq \text{Gal}(L^{ab}/L^{abs}). \qquad (2.6.2)$$

Now (2.6.2) follows from the well known fact that the diagram

$$\begin{array}{ccc}
(\text{idèle group of } K) & \xrightarrow{\ r_K\ } & \text{Gal}(K^{ab}/K) \\[2mm]
{\scriptstyle \text{inclusion}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{Ver}_{L/K}} \qquad (2.6.3) \\[2mm]
(\text{idèle group of } L) & \xrightarrow[\ r_L\ ]{} & \text{Gal}(L^{ab}/L)
\end{array}$$

commutes.   In turn, the commutativity of (2.6.1) and (2.6.3)
implies that the diagram

$$\begin{array}{ccc}
\widehat{K}^{\times} & \xrightarrow{\ \overline{r}_K\ } & \text{Gal}(K^{abs}/K) \\[2mm]
{\scriptstyle \text{inclusion}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{Vers}_{L/K}} \qquad (2.6.4) \\[2mm]
\widehat{L}^{\times} & \xrightarrow[\ \overline{r}_L\ ]{} & \text{Gal}(L^{abs}/L)
\end{array}$$

commutes.

<u>Proposition</u>. For all global fields $L \supseteq K$, $\underline{\Phi}, \underline{\Phi}' \in X(K)$ and $\sigma, \tau \in G$, the following relations hold:

(I) $\qquad\qquad F_K(\sigma, \tau\underline{\Phi})F_K(\tau, \underline{\Phi}) = F_K(\sigma\tau, \underline{\Phi})$.

(II) $\qquad\qquad \tau F_K(\sigma, \underline{\Phi})\tau^{-1} = F_{\tau K}(\sigma, \underline{\Phi}\tau^{-1})$.

(III) $\qquad\qquad F_L(\sigma, \underline{\Phi}) = \mathrm{Vers}_{L/K}(F_K(\sigma, \underline{\Phi}))$.

(IV) $\qquad\qquad F_K(\sigma, \underline{\Phi})F_K(\sigma, \underline{\Phi}') = F_K(\sigma, \underline{\Phi} + \underline{\Phi}')$.

<u>Proof</u>. The proofs of (I), (II), and (IV) are not particularly difficult, and so we omit them. The proof of (III) is not particularly easy; we give full details. Select $w \in W(L)$ and $v_0 \in W(K)$ arbitrarily. Set

$$v \overset{\mathrm{def}}{=} w \circ v_0.$$

Note that

$$w \circ v = v \in W(K). \qquad\qquad (2.6.5)$$

We define a function $u : G \longrightarrow G(K)$ by the formula

$$u(\gamma) \overset{\mathrm{def}}{=} v(\sigma\gamma)^{-1} \sigma v(\gamma).$$

Given $\gamma \in G(K)$ and $\delta \in G(L)$, we denote by γ_K and δ_L the images of γ and δ, respectively, in $\mathrm{Gal}(K^{\mathrm{abs}}/K)$ and $\mathrm{Gal}(L^{\mathrm{abs}}/L)$, respectively. For each $\gamma \in G$, we define a function $h_\gamma : G(K) \longrightarrow G(K)$ by the rule

$$h_\gamma(\delta) \overset{\mathrm{def}}{=} v(\gamma)^{-1} w(v(\gamma)\delta).$$

For each $\delta \in G(K)$, we define a function $p_\delta : G \longrightarrow \mathrm{Gal}(L^{\mathrm{abs}}/L)$ by the formula

$$p_\delta(\gamma) \overset{\mathrm{def}}{=} (h_1(\delta)^{-1} h_\gamma(\delta))_L.$$

By definition

$$F_K(\sigma, \underline{\Phi}) = \prod_{\gamma \in G/G(K)} u(\gamma)_K^{\underline{\Phi}(\gamma)} \qquad\qquad (2.6.6)$$

For each $\gamma \in G$, h_γ factors through $G(K)/G(L)$ and has the property that for all $\delta \in G(K)$,

$$h_\gamma(\delta) \in \delta G(L). \qquad (2.6.7)$$

Consequently, by definition, for all $\gamma \in G$,

$$\text{Vers}_{L/K}(u(\gamma)_K) = \prod_{\delta \in G(K)/G(L)} (h_\gamma(u(\gamma)\delta)^{-1} u(\gamma) h_\gamma(\delta))_L. \qquad (2.6.8)$$

For all $\gamma \in G$ and $\delta \in G(K)$, one verifies by direct calculation that

$$p_{u(\gamma)\delta}(\sigma\gamma)^{-1} p_{u(\gamma)\delta}(\gamma)(h_\gamma(u(\gamma)\delta)^{-1} u(\gamma) h_\gamma(\delta))_L$$

$$= (w(\sigma v(\gamma)\delta)^{-1} \sigma w(v(\gamma)\delta))_L. \qquad (2.6.9)$$

By definition,

$$F_L(\sigma, \mathcal{P}) = \prod_{\gamma \in G/G(K)} \prod_{\delta \in G(K)/G(L)} (w(\sigma v(\gamma)\delta)^{-1} \sigma w(v(\gamma)\delta))_L^{\mathcal{P}(\gamma)}. \qquad (2.6.10)$$

We claim that for all $\delta \in G(K)$, the function p_δ factors through the double coset space $D \backslash G / G(K)$. To verify the claim, let $\gamma \in G$, $\rho \in D$, and $\tau \in G(K)$ be selected arbitrarily. Then

$$p_\delta(\rho\gamma\tau) p_\delta(\gamma)^{-1} = (h_{\rho\gamma\tau}(\delta)^{-1} h_\gamma(\delta))_L$$

$$= (w(v(\rho\gamma\tau)\delta)^{-1} v(\rho\gamma) v(\gamma)^{-1} w(v(\gamma)\delta))_L$$

$$= (w(\rho_1 v(\gamma)\delta)^{-1} \rho_1 w(v(\gamma)\delta))_L = (w(v(\gamma)\delta)^{-1} \rho_2 \rho_1 w(v(\gamma)\delta))_L$$

$$= 1 \in \text{Gal}(L^{\text{abs}}/L),$$

where $\rho_1, \rho_2 \in D$ are given by the formulas

$$\rho_1 \overset{\text{def}}{=} v(\rho\gamma) v(\gamma)^{-1},$$

$$\rho_2 \overset{\text{def}}{=} w(v(\gamma)\delta) w(\rho_1 v(\gamma)\delta)^{-1}.$$

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By (2.6.8,9,10) and the lemma of ¶2.1,

$$\mathrm{Vers}_{L/K}(F_K(\sigma,\Phi))^{-1}F_L(\sigma,\Phi)$$

$$= \prod_{\gamma\in G/G(K)}\left(\prod_{\delta\in G(K)/G(L)}(p_{u(\gamma)\delta}(\sigma\gamma)^{-1}p_{u(\gamma)\delta}(\gamma))^{\Phi(\gamma)}\right)$$

$$= \prod_{\gamma\in G/G(K)}\left(\prod_{\delta\in G(K)/G(L)}(p_\delta(\sigma\gamma)^{-1}p_\delta(\gamma))^{\Phi(\gamma)}\right)$$

$$= \prod_{\delta\in G(K)/G(L)}\left(\prod_{\gamma\in G/G(K)}(p_\delta(\sigma\gamma)^{-1}p_\delta(\gamma))^{\Phi(\gamma)}\right)$$

$$= 1. \quad \blacksquare$$

2.7. **Proof of the theorem.** We claim that there exists a unique way to assign to each global field L galois over k, $\sigma\in G$, and $\Phi\in X(L)$ an element $\tilde{f}_L(\sigma,\Phi)$ of $\hat{L}^\times/L^\times$ such that for all $\sigma,\tau\in G$ and $\Phi,\Phi'\in X(L)$,

$$(\tau\otimes_A 1)\tilde{f}_L(\sigma,\Phi) = \tilde{f}_L(\sigma,\Phi\tau^{-1}), \qquad (2.7.1)$$

$$\tilde{f}_L(\sigma,\Phi)\tilde{f}_L(\sigma,\Phi') = \tilde{f}_L(\sigma,\Phi+\Phi'), \qquad (2.7.2)$$

$$\bar{r}_L(\tilde{f}_L(\sigma,\Phi)) = F_L(\sigma,\Phi). \qquad (2.7.3)$$

The claim is established by an appeal to Lemma 2.5.2 and to the proposition of ¶2.6. We claim also that for all global fields $M\supseteq L$, both M and L galois over k, $\sigma\in G$, and $\Phi\in X(L)$,

$$\tilde{f}_M(\sigma,\Phi)\equiv\tilde{f}_L(\sigma,\Phi) \quad \mathrm{mod}\ M^\times. \quad (2.7.4)$$

This claim follows from the definitions, Lemma 2.5.2, Lemma 2.5.7, and (III) of the proposition of ¶2.6. By Lemma 2.5.2, and (II) of the proposition of ¶2.6, for all global fields L galois over k, $\sigma,\tau\in G$ and $\Phi\in X(L)$,

$$\widetilde{f}_L(\sigma, \tau \Phi)\widetilde{f}_L(\tau, \Phi) = \widetilde{f}_L(\sigma\tau, \Phi). \qquad (2.7.5)$$

By Lemma 2.5.7, and relation (2.7.4) there exists unique $f_K(\sigma, \Phi) \in \hat{K}^\times/K^\times$ such that for all global fields $L \supseteq K$, with L galois over k,

$$f_K(\sigma, \Phi) \equiv \widetilde{f}_L(\sigma, \Phi) \quad \text{mod } L^\times. \qquad (2.7.6)$$

By Lemma 2.5.7, (III) of the proposition of ¶2.6, and (2.7.3), $f_K(\sigma, \Phi)$ possesses property (V) of the statement of the theorem. The remaining properties of $f_K(\sigma, \Phi)$ are deduced via Lemma 2.5.7 from (2.7.1,2,3,5). This settles the existence of $f_K(\sigma, \Phi)$. Relation (2.7.6) affirms the uniqueness of $f_K(\sigma, \Phi)$. ‖

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