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ONE-STEP RECURRENT TERMS IN λ-β-CALCULUS

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Abstract

A one-step recurrent term is a term in λ-β-calculus whose one-step reductums are all reducible to the term. It is a weakened notion of minimal form or recurrent term in the λ-β-calculus. In this paper, a one-step recurrent term which is not recurrent is shown. That term becomes a counter example for a conjecture presented by J.W.Klop. By analysis of the reduction cycles of one-step recurrent terms, a neccessary and sufficient condition for a one-step recurrent term to be recurrent is given.

0. Introduction

The reduction graph of a lambda term [1] is a directed graph which has lambda terms at each node. Each arc represents a one-step reduction from a term to another term. Thus, all the terms in the graph are reducible from the term. The structure of the reduction graph of lambda terms or terms in combinatory reduction systems has been studied in [1,4,5,9]. When we use the lambda-calculus as a model of computation, given a term, we have to select an appropriate reduction path to reach the terminal node representing its computation result. The reduction strategies tells us which branch we should follow. Some useful strategies and non-existence of some strategies with special properties is show in [1, 2, 8].

As an attempt to solve a well-know open problem [1,2] concerning the reduction strategy, Klop [4] defined some notion and gave a conjecture. First, we review his definitions. Two terms are said to be *cyclically equivalent* when they are reducible to each other. An equivalence class by the relation is called a *plane*. A term in a plane is called an *exit* when the term is reducible to another term which is not reducible to any term in the plane. Klop presented the following conjecture:

If a plane has an exit, then every point in the plane is an exit.

One of the authors of this paper introduced a notion of *one-step recurrent term* and gave a reformulation of the above conjecture [6]. A term is said to be recurrent if the result of any reduction of the term can be reducible to the term [7]. A term is called one-step recurrent if the result of any one-step reduction of the term is reducible to the term. One-step recurrent term is a weakened notion of recurrent term. Recurrent terms are called minimal forms in [1,3]. Using the notion of recurrence, we can reformulate Klop's conjecture as follows:

Every one-step recurrent term is recurrent.

In [6], the conjecture was proved for the one-step recurrent term with at most two redexes, and some properties of one-step recurrent terms are studied.

In this paper, we solve the problem in negative form, i.e., we give a one-step recurrent term which is not recurrent. And we examine the difference of one-step recurrent term and recurrent term. As a result of the analysis, we obtain a necessary and sufficient condition for a one-step recurrent term to be recurrent.

1. One – step recurrent terms and recurrent terms

In this section, we define the notions of one-step recurrent terms and recurrent terms. And we prove that the set of all recurrent terms is a proper subset of the set of all one-step recurrent terms. This is one of the main theorem of this paper. We states two fundamental lemmas which we use through the discussion.

First, we begin by explaining the notations and terminology, almost of which are usual ones.

We use the letter M, M_1, $M_2, \cdots$, M_i, N, $\cdots$ for λ-terms. The upper case greek letters Δ, Δ_1, $\cdots$ stand for redexes, $\mathcal{F}$ stand for set of redexes of a term. We use the symbol $\equiv$ for identity (up to α-conversion) of terms. The set of all redexes in a term M is denoted by $redex(M)$. We use $\rightarrow$ for one step reduction, and $\twoheadrightarrow$ for the reflexive transitive closure of $\twoheadrightarrow$. When there is a reduction $M \twoheadrightarrow N$ we say that M is reducible to N or that N is reducible from M. If the reduction is one-step reduction, we say that M is one-step reducible to N or that N is one-step reducible from M. When $M \equiv N$ the reduction is called a *cyclic reduction of* M. The lower case greek letters σ, τ, $\sigma_1, \cdots$ stand for reductions. Given a sequence of reductions $\sigma_i : N_i \twoheadrightarrow N_{i+1}$ $(i = 0, 1, \cdots, k)$, the successive composition of σ_i's is denoted by $\sigma_1\sigma_2\cdots\sigma_k : N_1 \twoheadrightarrow N_2 \twoheadrightarrow \cdots \twoheadrightarrow N_k \twoheadrightarrow N_{k+1}$.

If all N_i's are the same term and all σ_i's are identical to σ, then $\sigma_1\ \sigma_2\ \cdots\ \sigma_k$ is denoted by σ^k. Given a reduction $\sigma : M \twoheadrightarrow M'$, a redex Δ in M, a set $\mathcal{F}$ of redexes in M and a redex Δ' in M', Δ/σ stands for the set of all residuals of Δ by the reduction σ. $\mathcal{F}/\sigma$ stands for the union of the set Δ_i/σ for all $\Delta_i \in \mathcal{F}$. If Δ' is a residual of Δ by σ, i.e., $\Delta' \in \Delta'/\sigma$, we rite $\Delta \stackrel{\sigma}{\longmapsto} \Delta'$ or $(\Delta, M) \stackrel{\sigma}{\longmapsto} (\Delta', M')$. When Δ' is not a residual of any redex in M, we say that σ creates Δ' and write $\stackrel{\sigma}{\longmapsto} \Delta'$. When there is no residual of Δ in M', we say that σ erases Δ.

Recurrent terms are called minimal forms in [1]. The notion of one-step recurrent terms is defined in [6].

Definition 1.1 A term M is *recurrent iff* every term reducible from M is reducible to M. M is *one-step recurrent iff* every term one-step reducible from M is reducible to M.

The set of all recurrent terms and the set of all one-step recurrent terms are denoted by Λ_∞ and Λ_1 respectively. Since any one-step reduction is a reduction, any recurrent term is a one-step recurrent term. However the converse is not true in general.

Theorem 1.2 There exists a one-step recurrent term which is not recurrent.

Proof Let $M \equiv XXYZ$ where $X \equiv \lambda xyz.xxy(yz)$, $Y \equiv \lambda xz.x(xz)$, $Z \equiv \lambda z.II(IIz)$ and $I \equiv \lambda x.x$. The term M has three redexes

Δ_0 : the leftmost redex,

Δ_1 : the subterm II in the left position in Z, and

Δ_2 : the subterm II in the right position in Z.

By reducing Δ_0 and Δ_1, we have a reduction

$$M \equiv XXYZ \twoheadrightarrow XXY(YZ) \twoheadrightarrow XXY(Y(\lambda z.IIz)).$$

Since

$$Y(\lambda z.IIz) \twoheadrightarrow \lambda w.(\lambda z.IIz)((\lambda z.IIz)w) \twoheadrightarrow \lambda w.II(IIw) \equiv Z,$$

we have a reduction $XXYZ \twoheadrightarrow XXY(Y(\lambda z.IIz)) \twoheadrightarrow XXYZ$ which erases Δ_0 and Δ_1. Even if we reduce Δ_0 and Δ_2, we have the (syntactically) same reduction. Thus, there are reductions which erase each redex in M. Therefore, M is one-step recurrent.

Next, consider the term $XXY(\lambda z.Iz)$ which is obtained by reducing all the redexes Δ_0, Δ_1 and Δ_2. Any reduction of the term does not produce a term which has the subterm II. Therefore, it is not reducible to M. Thus, M is not recurrent. □

Given a set $\mathcal{F}$ of redexes in M, a reduction $M \stackrel{\sigma}{\twoheadrightarrow} N$ is called a complete development of $(M, \mathcal{F})$ iff it erases all the residuals of $\mathcal{F}$ and all the redexes contracted through σ are residuals of some redexes in $\mathcal{F}$. The resulting term by any complete developments of $(M, \mathcal{F})$ is unique, so we denote it by $G_{\mathcal{F}}(M)$. When $\mathcal{F}$ is the set of all redexes in M, we write it as $G(M)$. The following lemma says that if a reduction erases some redexes of a term then the resulting term can be obtained by reducting the redexes first and followed by some reduction.

<u>*Lemma* 1.3</u> Let $\mathcal{F}$ be a set of redexes in M and $\sigma : M \twoheadrightarrow N$ be a reduction. If N has no residual of $\mathcal{F}$, then $G_{\mathcal{F}}(M)$ is reducible to N.

<u>*Proof*</u> By induction on the length of σ. Suppose that σ is of the form $\sigma : M \stackrel{\sigma_0}{\longrightarrow} M' \stackrel{\sigma_1}{\twoheadrightarrow} N$ and that σ_0 reduces a redex Δ in M. Then by induction hypothesis for σ_1 and $\mathcal{F}/\sigma_0$, $G_{\mathcal{F}/\sigma_0}(M')$ is reducible to N. Let τ_0 be a complete development of $(M, \mathcal{F})$, τ_1 be a complete development of $(M', \mathcal{F}/\sigma_0)$ and σ_2 be a complete development of $(G_{\mathcal{F}}(M), \Delta/\tau_0)$. Since both $\sigma_0\tau_1$ and $\tau_0\sigma_2$ are the complete development of $(M, \mathcal{F} \cup \{\Delta\})$, they produce the same term $G_{\mathcal{F}/\sigma_0}(M')$.

Thus, we have $G_{\mathcal{F}}(M) \xrightarrow{\sigma_2}\!\!\!\!\rightarrow G_{\mathcal{F}/\sigma_0}(M')$. Therefore, $G_{\mathcal{F}}(M)$ is reducible to N. (See Figure 1.) □

Figure 1

Lemma 1.4 The following three conditions are equivalent.

(1) M is recurrent.

(2) G(M) is reducible to M.

(3) There is a cyclic reduction of M which erases all the redexes in M.

Proof The equivalence of (1) and (2) is proved in [3]. (2) ⇒ (3) is trivial. (3) ⇒ (2) is an easy consequence of Lemma 1.3. □

2. Compatibility of redexes

In this section, the notion of compatibility of redexes in a term is defined, and a sufficient condition is given for a one-step recurrent term to be recurrent.

The notion is come from the analysis of the construction of cyclic reductions from some simple cycles. First we explain the intuitive idea of the analysis.

If we want to show that a one-step recurrent term M is a recurrent, we only have to construct a reduction $\sigma : M \twoheadrightarrow M$ which erases all the redexes in M by Lemma 1.4. Since M is one-step recurrent, we have reductions $\sigma_1, \sigma_2, \cdots, \sigma_k : M \twoheadrightarrow M$ each of which erases a redex Δ_i in M. So it would be natural to try to construct the reduction σ from σ_i's. The reduction σ_i erases the redex Δ_i, however, the residuals of another redex Δ_j would

(1) disappear, or

(2) appear in one position, or

(3) appear in more than two places.

If (1) or (2) is true for all σ_i's, each reduction would decrease the number of residuals to be erased. Therefore, all redexes could be erased. However, if (3) is true for some reductions, an essential difficulty arises for the case in which $\Delta_i, \Delta_j \in \Delta_j/\sigma_i$ and $\Delta_j, \Delta_i \in \Delta_i/\sigma_j$. In this case any times of reductions of σ_i and σ_j leaves the residuals of σ_i or σ_j in the both positions Δ_i and Δ_j in the resulting term. (See Figure 2.)

Figure 2

Thus we can not erase both Δ_i and Δ_j at the same time by this way. (In fact the term given in Theorem 1.2 is such a term.) If such case does not happen for the term, we can construct the desired reduction. That is the main theorem in this section.

Definition 2.1 Let Δ_1 and Δ_2 be distinct redexes in M. We write $\Delta_1 \succ \Delta_2$ *iff* there is a cyclic reduction $\sigma : M \twoheadrightarrow M$ such that

(a) $\Delta_2/\sigma = \emptyset$,

(b) $\Delta_1, \Delta_2 \in \Delta_1/\sigma$.

Δ_1 and Δ_2 are *incompatible iff* $\Delta_1 \succ \Delta_2$ and $\Delta_2 \succ \Delta_1$. Δ_1 and Δ_2 are *compatible iff* they are not incompatible. M is *compatible iff* every two redexes in M are compatible.

Lemma 2.2 Let Δ be a redex in M, σ be a reduction $M \twoheadrightarrow M$ and k be the number of redexes in M. If $\Delta \notin \Delta/\sigma^i$ for all $i \leq k$, then $\Delta/\sigma^k = \emptyset$.

Proof Let M_i be the term M after the reduction σ^i. Since all M_i's are syntactically identical, M_i has the corresponding redex occurrence of Δ' in the same position in it for each redex Δ'. So let it be denoted by (Δ', M_i).

Suppose that $\Delta/\sigma^k \neq \emptyset$. Then M_k has a residual of (Δ, M_0). Therefore, each M_i $(i = 0, 1, \cdots, k)$ has residual (Δ_i, M_i) which is a redex of Δ, and $(\Delta_i, M_i) \overset{\sigma_i}{\longmapsto} (\Delta_{i+1}, M_{i+1})$ where $\Delta_0 = \Delta$. (See Figure 3.)

Figure 3

At first stage M_1, since $\Delta \notin \Delta/\sigma$, the redex (Δ_1, M_1) is distinct from (Δ_0, M_1). Now assume that the redexes $(\Delta_0, M_i), \cdots, (\Delta_i, M_i)$ are distinct in the i-th stage M_i. Then at $i+1$-th stage, M_{i+1} has $i+1$ redexes $(\Delta_1, M_{i+1}), \cdots, (\Delta_{i+1}, M_{i+1})$ each two of which are distinct, because they are the residuals of distinct redexes in the previous stage. Moreover, they are not identical to (Δ_0, M_{i+1}), because $\Delta \notin \Delta/\sigma^{i+2}$. Thus M_{i+1} has $i+1$ redexes. Therefore, M_{k+1} has k redexes. A contradiction. Therefore, $\Delta \in \Delta/\sigma^i$ for some $i \leq k$. $\square$

<u>*Remark* 2.3</u> In the definition of $\Delta_1 \succ \Delta_2$, the existence of a reduction $\sigma : M \twoheadrightarrow M$ is required such that

(a) $\Delta_2/\sigma = \emptyset$,

(b) $\Delta_1, \Delta_2 \in \Delta_1/\sigma$.

However, the requirement (a) can be removed as follows. Suppose that σ satisfies the condition (b). Since $\Delta_1, \Delta_2 \in \Delta_1/\sigma$, we have $\Delta_2 \in \Delta_1/\sigma^i$ for all i. Therefore $\Delta_2 \notin \Delta_2/\sigma^i$. Thus, we have $\Delta_2/\sigma^k = \emptyset$ by Lemma 2.2, where k is the number of redexes in M. So the reduction σ^k satisfies both (a) and (b). Therefore, all we have to show to prove $\Delta_1 \succ \Delta_2$ is an existence of a reduction $\sigma : M \twoheadrightarrow M$ which satisfies (b). $\square$

<u>*Lemma* 2.4</u> Let M be a compatible one-step recurrent term. Then for any set $\mathcal{F}$ of redexes in M, there is a reduction $\sigma^* : M \twoheadrightarrow M$ such that $\mathcal{F}/\sigma^* = \emptyset$.

Proof By induction on the number n of redexes in $\mathcal{F}$.

Base step $n = 1$. Since M is one-step recurrent, the redex in $\mathcal{F}$ is erased by some reduction $\sigma^* : M \twoheadrightarrow M$.

Induction step Let $\mathcal{F} = \mathcal{F}_0 \cup \{\Delta\}$ where $\Delta \notin \mathcal{F}$. By induction hypothesis, there is a reduction $\sigma : M \twoheadrightarrow M$ such that $\mathcal{F}_0/\sigma = \emptyset$. Let k be the number of redexes in M.

Case 1 $\Delta/\sigma^k = \emptyset$.

Since $\mathcal{F}_0/\sigma = \emptyset$ we have $\mathcal{F}_0/\sigma^k = \emptyset$. Therefore $\mathcal{F} \cup \{\Delta\}/\sigma^k = \emptyset$. Then put $\sigma^* = \sigma^k$.

Case 2 $\Delta/\sigma^k \neq \emptyset$.

Then by Lemma 2.2 $\Delta \in \Delta/\sigma^i$ for some $i \leq k$. Since M is one-step recurrent, there is a reduction $\tau : M \twoheadrightarrow M$ such that $\Delta/\tau = \emptyset$. Let $\theta = \sigma^i\tau : M \xrightarrow{\sigma_i}\!\!\!\!\!\rightarrow M \xrightarrow{\tau}\!\!\!\!\!\rightarrow M$.

Now suppose that $\Delta/\theta^k \neq \emptyset$. Then by Lemma 2.2, we have $\Delta \in \Delta/\theta^j$ for some $j \leq k+1$. Let $M_0 \equiv M$, and M_1,M_2,M_3,M_4, be the terms after reduction M_1 be the term M after σ^i,θ, θ^j and $\theta^j\sigma^i$ respectively. (See Figure 4.)

Figure 4

Since M_0, M_1, M_2, M_3 and M_4 are syntactically identical to M, each M_i has a redex occurrence of Δ at the corresponding position. Let it be (Δ, M_i). Since $\Delta \in \Delta/\theta^j$, M_1 has a residual (Δ', M_1) of (Δ, M_0) such that $(\Delta, M_0) \overset{\sigma_i}{\longmapsto} (\Delta', M_1)$, $(\Delta', M_1) \overset{\tau\theta^{j-1}}{\longmapsto} (\Delta, M_3)$. The reduction τ erases (Δ, M_1), so that (Δ', M_1) is distinct from (Δ, M_1). Therefore Δ and Δ' are distinct. Thus we have two redexes Δ, Δ' and two reductions $\sigma^i, \tau\theta^{j-1}\sigma^i$ such that $\Delta, \Delta' \in \Delta/\sigma_i$

and $\Delta, \Delta' \in \Delta'/\tau\theta^{j-1}\sigma^i$. This contradicts the assumption that M is compatible. Thus we have $\Delta/\theta^k = \emptyset$.

On the other hand, we have $\mathcal{F}_0/\theta^k = \emptyset$ by the definition of $\theta = \sigma^i$ and the assumption $\mathcal{F}_0/\sigma = \emptyset$. Therefore, we have $\mathcal{F} \cup \{\Delta\}/\theta^k = \emptyset$. Then we can put $\sigma^* = \theta^k$. □

Theorem 2.5 Every compatible one-step recurrent term is recurrent.

Proof Let $\mathcal{F}$ be the set of all redexes in a compatible one-step recurrent term M. By Lemma 2.4, there is a reduction $M \xrightarrow{\sigma^*}\!\!\!\!\rightarrow M$ such that $\mathcal{F}/\sigma^* = \emptyset$. Then M is recurrent by Lemma 1.4. □

Remark 2.6 The converse of Theorem 2.5 does not holds in general, i.e., every recurrent term is not always compatible. For example consider the term $N \equiv VVXYWI(XXYZ)$ where $V \equiv \lambda vxyzwiz.vvxywi(xxy(wi))$, $X \equiv \lambda xyz.xxy(yz), Y \equiv \lambda xz.x(xz)$, $Z \equiv \lambda z.II(IIz), I \equiv \lambda x.x$ *and* $W \equiv \lambda iu.ii(iiu)$. Recall that we constructed the term $M \equiv XXYZ$ in Theorem 1.2. The term M has three redexes

Δ_0 : the leftmost redex,

Δ_1 : the left redex II in Z, and

Δ_2 : the right redex II in Z.

Let σ_1 be the reduction which reduces Δ_0 and Δ_1. Then we have $XXYZ \xrightarrow{\sigma_1}\!\!\!\!\rightarrow XXY(Y(\lambda z.(IIz)))$ where the subterm II in the result of σ_1 is a residual of Δ_2. Since there is a reduction τ : $Y(\lambda x.IIz) \longrightarrow \lambda w.(\lambda z.IIz)((\lambda z.IIz)w) \longrightarrow\!\!\!\!\rightarrow \lambda w.II(IIw) \equiv Z$, we have $XXYZ \xrightarrow{\sigma_1\tau}\!\!\!\!\rightarrow XXYZ$. Since the subterm II's in the result M are the residual of the redex II in $Y(\lambda z.IIz)$, Δ_1 and Δ_2 are the residuals of Δ_2 by $\sigma_1\tau$. Thus $\Delta_1, \Delta_2 \in \Delta_2/\sigma_1\tau$. We can apply the similar argument for the reduction σ_2 which erases Δ_0 and Δ_1. Therefore we have $\Delta_1, \Delta_2 \in \Delta_1/\sigma_2\tau$. Therefore $XXYZ$ is incompatible. So N is not compatible. □

3. Admissible class of redexes

In Theorem 1.2 of section 1, we gave a term M which has two incompatible redexes Δ_1 and Δ_2, i.e., $\Delta_1 \succ \Delta_2$ and $\Delta_2 \succ \Delta_1$. For that term, we have shown the impossibility of erasing both redexes by any cyclic reduction. In Theorem 2.5 of section 2, we proved that all redexes of a recurrent term can be erased by some cyclic reduction, if any two redexes in the term are compatible. However, as we have shown in Remark 2.6, the compatibility is not always a necessary condition for the redexes to be erased by some cyclic reduction. In fact, even if a term has incompatible redexes in it, all redexes can be erased by some cyclic reduction — recall Lemma 1.3.

In this section we examine the reason why incompatible redexes can be erased by some cyclic reduction when the term is recurrent. And we give a necessary and sufficient condition for a one-step recurrent term to be recurrent.

Definition 3.1 We define the equivalence relation $\sim$ of redexes in a term, inductively by

(1) $\Delta \sim \Delta$,

(2) $\Delta_1 \succ \Delta_2, \Delta_2 \succ \Delta_1 \Longrightarrow \Delta_1 \sim \Delta_2$,

(3) $\Delta_1 \sim \Delta_2, \Delta_2 \sim \Delta_3 \Longrightarrow \Delta_1 \sim \Delta_3$.

"$\sim$" is the equivalence relation generated by "incompatibility". We call an equivalence class module "$\sim$" simply an equivalence class or a class.

Proposition 3.2 Let Δ_1 and Δ_2 be redexes in a term M. If $\Delta_1 \sim \Delta_2$, then there is a reduction $\sigma : M \twoheadrightarrow M$ such that $\Delta_2 \in \Delta_1/\sigma$.

Proof By induction on the definition of "$\sim$".

Base step (1) Take an empty reduction as σ, then we have $\Delta \overset{\sigma}{\longmapsto} \Delta$.

Base step (2) Suppose that $\Delta_1 \succ \Delta_2$ and $\Delta_2 \succ \Delta_1$. Then there is a reduction such that $\Delta_1 \overset{\sigma}{\longmapsto} \Delta_2$ by the definition of $\Delta_1 \succ \Delta_2$.

Induction step (3) Suppose that $\Delta_1 \sim \Delta_2$ and $\Delta_2 \sim \Delta_1$. By induction hypothesis, we have reductions σ_1 and σ_2 such that $\Delta_1 \overset{\sigma_1}{\longmapsto} \Delta_2, \Delta_2 \overset{\sigma_2}{\longmapsto} \Delta_1$. Therefore $\Delta_1 \overset{\sigma_1\sigma_2}{\longmapsto} \Delta_3$. Put $\sigma = \sigma_1\sigma_2$. □

Recall that given a reduction $\tau : N_1 \twoheadrightarrow N_2$ and a redex Δ_2 in N_2, we say that τ creates Δ_2 *iff* $\Delta_2 \notin \Delta_1/\tau$ for all redex $\Delta_1 \in N_1$, and we write $\overset{\tau}{\longmapsto} \Delta_2$.

Definition 3.3 An equivalence class $\mathcal{F}$ of redexes in a term M is *admissible iff* there is a redex $\Delta \in \mathcal{F}$ and a reduction $\sigma : M \twoheadrightarrow M$ such that σ creates Δ.

Proposition 3.4 For each admissible equivalence class $\mathcal{F}$, there exist some reductions $\sigma_0, \sigma_1, \cdots, \sigma_n : M \twoheadrightarrow M$, and the elements of $\mathcal{F}$ can be numbered such that

(1) $\mathcal{F} = \{\Delta_0, \Delta_1, \cdots, \Delta_n\}$,

(2) σ_0 creates Δ_0,

(3) $\Delta_{i+1} \in \Delta_i/\sigma_i \qquad$ for $i = 0, 1, \cdots, n-1$.

Proof Since $\mathcal{F}$ is admissible, there is a redex $\Delta_0 \in \mathcal{F}$ and a reduction $\sigma_0 : M \twoheadrightarrow M$ such that σ_0 creates Δ_0. Let $\{\Delta_1, \Delta_1, \cdots, \Delta_n\}$ be other redexes in $\mathcal{F}$. Since $\mathcal{F}$ is an equivalence class, we have $\Delta_i \succ \Delta_{i+1}$ for $i = 0, 1, \cdots, n-1$. Then by Proposition 3.2, there is a reduction $\sigma_i : M \twoheadrightarrow M$ such that $\Delta_{i+1} \in \Delta_i/\sigma_i$. □

We denote the condition (2) and (3) of Proposition 3.4 by

$$\overset{\sigma_0}{\longmapsto} \Delta_0 \overset{\sigma_1}{\longmapsto} \Delta_1 \overset{\sigma_2}{\longmapsto} \cdots \overset{\sigma_{n-1}}{\longmapsto} \Delta_n.$$

Lemma 3.5 Let $\mathcal{F}$ be a set of redexes in M, Δ be a redex in M, k be the number of redexes in M, and σ, τ be reductions $M \twoheadrightarrow M$. If $\mathcal{F}/\sigma = \emptyset$ and τ creates Δ, then $(\sigma\tau)^{k+1}$ creates Δ and $\mathcal{F} \cup \{\Delta\}/(\sigma\tau)^{k+1} = \emptyset$.

Proof Since $\mathcal{F}/\sigma = \emptyset$, we have $\mathcal{F}/(\sigma\tau)^i = \emptyset$ for all i. Since τ creates Δ, it follows that $(\sigma\tau)^i$ creates Δ for all i. So it suffices to show that $\Delta/(\sigma\tau)^{k+1} = \emptyset$. Since $(\sigma\tau)^i$ creates Δ, Δ can not be a residual of any redex by the reduction $(\sigma\tau)^i$. Therefore, $\Delta \notin \Delta/(\sigma\tau)^i$. Therefore, by Lemma 2.2, we have $\Delta/(\sigma\tau)^{k+1} = \emptyset$. □

Lemma 3.6 Let $\mathcal{F}$ be a set of redexes in M, $\Delta_0, \Delta_1, \cdots, \Delta_n$ be redexes in M and $\sigma, \tau_0, \tau_1, \cdots, \tau_n : M \twoheadrightarrow M$. If

(1) $\mathcal{F}/\sigma = \emptyset$,

(2) τ_0 creates Δ_0,

(3) $\Delta_{i+1} \in \Delta_i/\tau_i$ for $i = 0, 1, \cdots, n+1$,

then $\mathcal{F} \cup \{\Delta_0, \Delta_1, \cdots, \Delta_n\}/\theta_n = \emptyset$. where k is the number of redexes in M and $\theta_0 = (\sigma\tau_0)^{k+1}$, $\theta_{i+1} = (\theta_i\tau_0\tau_1\cdots\tau_{i+1})^{k+1}$ for $i = 0, 1, \cdots, n-1$,

Proof By induction on n.

Base step $n = 0$. Lemma 3.6 is identical to Lemma 3.5 for this case.

Induction step By induction hypothesis $\mathcal{F} \cup \{\Delta_0, \Delta_1, \cdots, \Delta_i\}/\theta_i = \emptyset$. Since $\xmapsto{\tau_0} \Delta_0 \xmapsto{\tau_1} \Delta_1 \longmapsto \cdots \xmapsto{\tau_i} \Delta_i \xmapsto{\tau_{i+1}} \Delta_{i+1}$, it follows that $\tau_0\tau_1\cdots\tau_i\tau_{i+1}$ creates Δ_{i+1}. Therefore, by Lemma 3.5, we have $\mathcal{F}\cup \{\Delta_0, \Delta_1, \cdots, \Delta_i, \Delta_{i+1}\}$ $/(\theta_i\tau_0\tau_1\cdots\tau_i\tau_{i+1})^{k+1} = \emptyset$. Thus $\mathcal{F} \cup \{\Delta_0, \cdots, \Delta_{i+1}\}$ $/\theta_{i+1}^{k+1} = \emptyset$. □

Lemma 3.7 For any admissible equivalence class $\mathcal{F}$ of redexes in M, there is a reduction $\sigma : M \twoheadrightarrow M$ such that $\mathcal{F}/\sigma = \emptyset$.

Proof By Proposition 3.4, there exist some reductions $\tau_0, \tau_1, \cdots, \tau_n$: $M \twoheadrightarrow M$ and the elements of $\mathcal{F}$ are numbered such that

(1) $\mathcal{F} = \{\Delta_0, \Delta_1, \cdots, \Delta_n\}$,

(2) τ_0 creates Δ_0,

(3) $\Delta_{i+1} \in \Delta_i/\tau_i$ for $i = 0, 1, \cdots, n-1$.

Let k be the number of redexes in M, $\theta_0 = \tau_0^{k+1}$, $\theta_{i+1} = (\theta_i \tau_0 \tau_1 \cdots \tau_i \tau_{i+1})^{k+1}$ for $i = 0, 1, \cdots, n-1$ and $\sigma = \theta_n$. Then we have $\mathcal{F}/\sigma = \emptyset$ by Lemma 3.6. □

Theorem 3.8 A one-step recurrent term is recurrent *iff* all the equivalence class of the redexes of the term are admissible.

Proof Only − if − part: Let M be a recurrent term. Then there is a reduction $\sigma : M \twoheadrightarrow M' \equiv M$ which erases all the redexes in M. Therefore M' has no residual of the original term M. Thus, every redex in M' is created by σ. Therefore, every equivalence class is admissible.

If − part: Let $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_m$ be all the equivalence classes. By induction on $i = 1, 2, \cdots, m$, we prove the existence of a reduction $\sigma_i : M \twoheadrightarrow M$ such that $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_i/\sigma_i = \emptyset$.

Base step: Since $\mathcal{F}_1$ is admissible there is a reduction σ_1 such that $\mathcal{F}_1/\sigma_1 = \emptyset$ by Lemma 3.7.

Induction step: By induction hypothesis there is a reduction $\sigma_i : M \twoheadrightarrow M$ such that $\mathcal{F}_1 \cup \cdots \mathcal{F}_i/\sigma_i = \emptyset$. Since $\mathcal{F}_{i+1}$ is admissible, by Proposition 3.4 there are some reductions $\tau_0, \tau_1, \cdots, \tau_l : M \twoheadrightarrow M$ and the elements of $\mathcal{F}_i$ are numbered such that

(1) $\mathcal{F}_{i+1} = \{\Delta_0, \Delta_1, \cdots, \Delta_l\}$,

(2) τ_0 creates Δ_0,

(3) $\Delta_{j+1} \in \Delta_j/\tau_j$ for $j = 0, 1, \cdots, l-1$.

Then we can apply Lemma 3.6 for $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_i$, $\Delta_0, \Delta_1, \cdots, \Delta_l, \sigma_i, \tau_0, \tau_1, \cdots, \tau_l$, obtaining a reduction $\sigma_{i+1} : M \twoheadrightarrow M$ such that $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_i \cup \{\Delta_0, \Delta_1, \cdots, \Delta_l\}$ $/\sigma_{i+1} = \emptyset$. Thus $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_i \cup \mathcal{F}_{i+1} /\sigma_{i+1} = \emptyset$. □

Remark 3.9 In Theorem 1.2, as an example of one-step recurrent term which is no recurrent we constructed the following term $M = XXYZ$ where $X \equiv \lambda xyz.xxy(yz), Y \equiv \lambda xz.x(xz)$, $Z \equiv \lambda z.II(IIz)$ and $I \equiv \lambda x.x$. M has three redexes $\Delta_0, \Delta_1, \Delta_2$. Δ_0 is the leftmost redex. Δ_1 and Δ_2 are in the subterm Z. The equivalence classes of the redexes of the term are $\{\Delta_0\}$ and $\{\Delta_1, \Delta_2\}$. Since neither Δ_1 or Δ_2 can not be created by any cyclic reduction of M, the class $\{\Delta_1, \Delta_2\}$ is not admissible. That is the reason why the term is not recurrent. □

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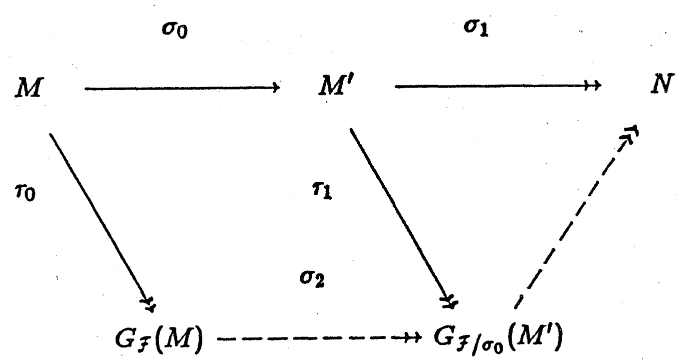

Figure. 1.

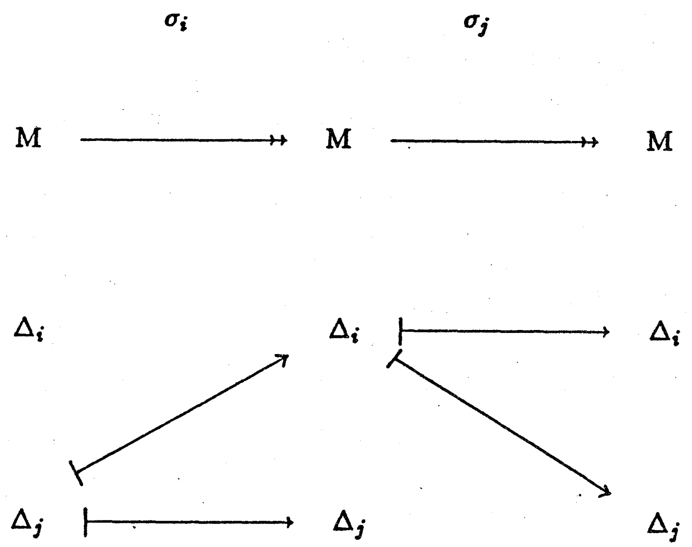

Figure. 2.

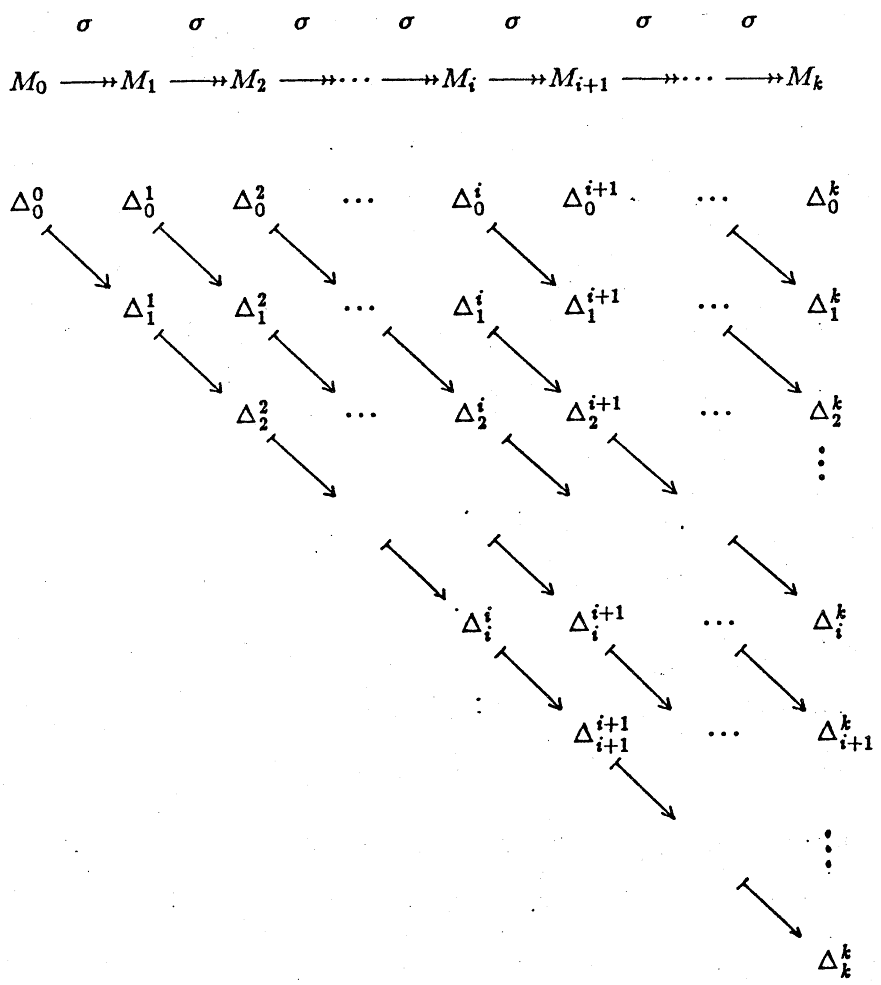

Figure. 3. $\Delta_j^i = (\Delta_i, M_j)$

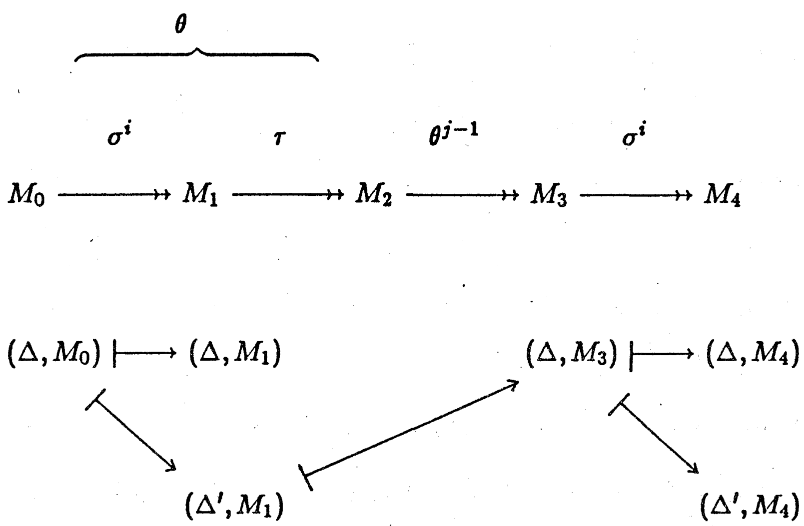

Figure. 4.