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On a chain-like connectivity of three-dimensional digital pictures*

by

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0. Introduction

In recent years, there have been many interesting papers on the digital geometry. These works develop as the theoretical foundation of digital image processing. Among them, the problem of connectedness is one of the most interesting topics.

It is well-known that there are 4- and 8-connectedness of 2D digital pictures and also the 6- and 26-connectedness of 3D digital pictures. In this paper, we propose a new definition of connectivity of 3D digital pictures. Although the exact definition is given in next section, it means the following topological properties: Let S_1 and S_2 be two subsets the same 3D digital picture. S_1 and S_2 are connected when they satisfy the

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following conditions (1)-(3):

(1) S_1 and S_2 are toruses,

(2) S_1 goes through a hole of S_2,

(3) S_2 goes through a hole of S_1.

The above connectedness of S_1 and S_2 is illustrated in Figure 1. This connectedness relation is called a chain-like connectivity. After giving the exact definition, we consider the decision problem whether or not S_1 and S_2 of an arbitrary 3D digital picture is connected in a chain-like style. We prove that any one-pebble acceptor cannot determine whether an arbitrary 3D picture is connected in this style. This result seems to be interesting in contrast with the following problems of the usual definition of connectedness.

(1) It is still open whether a finite state automaton (FSA) can determine connectedness of a 2D picture (see, e.g., [1]).

(2) A one-pebble acceptor can determine whether an arbitrary 2D picture is connected (see [2]).

(3) A one-way 3D parallel/sequential acceptor cannot determine whether an arbitrary 3D picture is connected (see [3]).

An idea of proof in this paper comes from the paper [2]. That is, the number of equivalence classes by a one-pebble acceptor are larger than a number of diferent digital pictures 's of 0's and 1's.

It is assumed that the readers are familiar with some basic notions and definitions of 3D digital topology (see [4]).

1. Preliminaries and definitions

In this paper, we exclusively treat a three-dimensional (3D) digital image Σ in the usual way.

Consider the 3D Euclidean space E and its subset D consisting of all lattice points in E. Let Σ be a connected subset of D such as $\Sigma = \{(i,j,k) \mid 1 \leq i \leq \ell,\ 1 \leq j \leq m,\ 1 \leq k \leq n\}$. A 3D digital picture f is a mapping from Σ to $\{0, 1\}$, i.e., $f: \Sigma \longrightarrow \{0, 1\}$. Each point (i,j,k) is called a voxel. To avoid special case, we assume that f(i,j,k)=0 if at least one of these i, j, k is equal to either 1 or ℓ, 1 or m, 1 or n, respectively. The set of such points, $\{(i,j,k) \mid i=1 \vee i=\ell \vee j=1 \vee j=m \vee k=1 \vee k=n\}$, is called the <u>border</u> of Σ. We also assume that each voxel which is not in Σ has a value #, i.e., a 3D digital picture is a rectangular parallelepiped sinking in the sea of #'s. Usually, the subset of Σ, $\{(i,j,k) \mid f(i,j,k)=1\}$, is called S and its complement is called $\bar{S}$. For every pair of distinct points $X=(x_1,x_2,x_3)$, $Y=(y_1,y_2,y_3)$, X and Y are <u>6-adjacent</u> if

$$|x_1-y_1| + |x_2-y_2| + |x_3-y_3| = 1;$$

X and Y are <u>26-adjacent</u> if

$$\max(|x_1-y_1|, |x_2-y_2|, |x_3-y_3|)=1.$$

If points X and Y are 6-adjacent (26-adjacent), the X is called 6-neighbor (26-neighbor) of Y. To avoid ambigious situations, we assume that the opposite type of adjacency is used for S and $\bar{S}$, i.e., if 6-adjacency (26-adjacency) is used for S, 26-adjacency (6-adjacency) is used for $\bar{S}$. A <u>6-path</u> (<u>26-path</u>) is a sequence of lattice points, $\pi=P_0,P_1,\ldots,P_m$, where P_i is a 6-neighbor (26-neighbor) of P_{i-1}, for all i such that $1 \leq i \leq m$. Any two points P, Q are called <u>6-connected</u> <u>(26-connected)</u> <u>in</u> <u>S</u> if there exists a 6-path (26-path) $P=P_0,\ldots,P_m=Q$ from P to Q, where $P_i \in S$. The relation, "connected", is an equivalence relation. It partitions S into equivalence classes. These classes are colled the <u>connected components</u> of S. In the same way, we can define connected components of $\bar{S}$. Clearly, exactly one component of $\bar{S}$ contains the border of Σ. This component is called the <u>background</u> of S; all other components of $\bar{S}$ are called <u>cavities</u> of S.

Now, let $N_{27}(P)$ denote the 26 voxels in 3x3x3 neighborhood of P excluding P itself.

<u>Definition 1.1</u>. Let P be a voxel of S. P is called a removable voxel if the following (i)-(iv) are satisfied:

(i) $S \cap N_{27}(P)$ has the same number of components (in the S sense) as $(S \cap N_{27}(P)) \cup \{P\}$.

(ii) $\bar{S} \cap N_{27}(P)$ has the same number of components (in the $\bar{S}$ sense) as $(\bar{S} \cap N_{27}(P)) \cup \{P\}$.

(iii) $S \cap N_{27}(P)$ has the same number of holes as $(S \cap N_{27}(P)) \cup \{P\}$.
(iv) $\bar{S} \cap N_{27}(P)$ has the same number of holes as $(\bar{S} \cap N_{27}(P)) \cup \{P\}$.

In (iii) of above definition, the hole in $S \cap N_{27}(P)$, $(S \cap N_{27}(P)) \cup \{P\}$ mean an object which corresponds to a torus of the Euclidean space. (Figure 2 shows a torus in 2D case. The hole in the above definition means the 3D version of Figure 2.) For (iv), this situation is similar.

From this definition, we know that the removable voxel is the same as the simple point of 3D (see [5]). For this simple point, it has been shown that removal of a point preserves topology (in the 3D sense) — the number of objects and holes, and cavities — if and only if the point is simple.

Here, we consider characterization of holes for the general case. Even in ordinary topology it is difficult to do it. An object is defined to have no hole if every simple closed curve (in 3D) in the object is continuously deformable (remove and addition of simple points) within the object to a single point.

Now, we define a new connectivity relation among objects with a hole. Then, we can build an object without hole by adding some P_i's. This part which consists of added voxels mentioned above is called the inside of hole. Obviously, the inside of hole is not unique.

<u>Definition 1.2</u>. Let S_1 and S_2 be two objects with a hole of the same 3D digital picture Σ . S_1 and S_2 is called separate if and only if there exist insides of S_1 and S_2 which do not meet. Otherwise, S_1 and S_2 are connected in a chain-like style.

The above definition is for two objects S_1 and S_2. We extend this definition to the general case.

<u>Definition 1.3</u>. Let S and R be two objects with hole. Then, S and R are connected in a chain-like style if and only if there

exists a sequence $S_0, S_1, \ldots, S_n$ satisfying the following conditions:

(i) $S_0=S$, $S_n=R$.

(ii) S_{i-1} is connected to S_i in a chain-like style for every $1 \leqq i \leqq n$.

In particular, a single object with hole is connected in a chain-like style.

The chain-like connectivity is obviously reflexive, symetric, and transitive.

In the next section, we consider a problem of recognition by an acceptor.

A 3D finite automaton (FSA) is defined as follows:

Definition 1.4. A 3D finite automaton is a triple (Q, V,), where

Q is the set of states,

V is the set of symbols,

$\delta : QxV \longrightarrow 2^{QxVx\Delta}$ (or $\longrightarrow QxVx\Delta$, in the deterministic case) is trasition function, and $\Delta = \{L,R,F,B,U,D,N\}$ is the set of movedirections ("left", "right", "front", "back", "up", "down", and "no move").

Definition 1.5. A 3D finite state acceptor is a triple $A=(M, q_0, Q_A)$, where M is a 3D finite state automaton, q_0 is M's initial state, and Q_A is a set of accepting states.

A configuration of acceptor and acceptance of a 3D input tape are defined in the usual way.

A 3D tape-bounded automaton M is defined as follows: M "bounce off" #'s, so that it is essentially confined to the non-# portion of its input tape. Especially, we require that if M has just moved in direction d and read the symbol #, it must not rewrite the # as anything else, and must move in the opposite direction.

Definition 1.6. A 3D one-pebble acceptor $A^{(1)}$ is a 3D tape-bounded finite state acceptor whose state set Q and vocabulary V are of the special forms

$Q=Q' \times \{0,1\}$,

$V=(V' \times \{0,1\}) \cup \{\#\}$,

and whose transition function δ satisfies the following restrictions: If state (q_1,u_1) and symbol (w_1, v_1) give rise to state (q_2, u_2) and symbol (w_2, v_2), where every u and v is 0 or 1, then $w_2=w_1$ and only the following combinations of (u_1, v_1, u_2, v_2) are possible: (0,0,0,0), (0,1,0,1), (0,1,1,0), (1,0,1,0), and (1,0,0,1).

2. Unacceptability of a chain-like connectivity by a 3D one-pebble acceptor.

In this section, we show that any 3D one-pebble acceptor cannot determine a chain-like connectivity.

First of all, we consider a 3D picture Σ with 7 units in thickness. So, for some m,

$$\Sigma = \{(i,j,k) \mid 1 \leq i,j \leq m+2,\ 1 \leq k \leq 7\}$$

Figure 3 represents Σ. Here we define two defferent 5x5x5 patterns as shown in Figure 4a-b. Then, consider arbitrary n by n matrix of those 5x5x5 patterns (see Figure 5). Adjacent local patterns are 6-connected as shown in Figure 4c-d. Note here that each local pattern consists of two disconnected (in normal definitions) objects.

Now, suppose that 3D picture Σ contains the patterns of Figure 5 having 7 units in their thickness (i.e., the top and bottom plane form some parts of Σ's border). Then, we check the behaviors of 3D deterministic finite-state acceptor on 3D picture Σ.

Lemma 2.1. Let S be the set of all rectangular parallelopipeds represented in Figure 5. Then, any 3D deterministic finite-state acceptor A partition the set S into at most $(140n|Q|+1)^{140|Q|}$ equivalence classes, where $|Q|$ is the number of states of acceptor A.

Proof: Since 3D acceptor A is deterministic finite-state, the number of the equivalence class generated depends on the enterning, leaving positions and the states of A only. There are $4 \times 7 \times 5n \times |Q|$ possible enterning configulations (i.e., the surface area of an element of S by the number of states of A). There also exist $4 \times 7 \times 5n \times |Q|$ plus one possible leaving configulations (including the case where A does not leave). Thus, the number of the possible combinations of enterning and leaving configulations are $(140n|Q|+1)^{140n|Q|}$. Therefore, any 3D deterministic finite-state acceptor can partition the set S into at most $(140n|Q|+1)^{140n|Q|}$ equivalence classes.

Now, we prove the main theorem of this paper.

Theorem 2.2. Any 3D deterministic one-pebble acceptor cannot determine the chain-like connectivity of arbitrary given digital picture.

Proof: Consider the set T of the patterns represented in Figure 6. Let the hatched parts of its elements are the elements of S defined in Lemma 2.1. On the other hand, the outer parts have a fixed pattern. Suppose here that there exists a 3D deterministic one-pebble acceptor $A^{(1)}$ which accepts an element t of T if and only if t has connected objects in chain-like style. Without loss of generality, we can assume that $A^{(1)}$ has a physical marker, and $A^{(1)}$ starts its accepting behavior at the upper-north-west corner of the elements of T. It is obvious that the accepting behaviors of $A^{(1)}$ depend on the patterns of central parts. Now, let $A^{(1)'}$ be the acceptor which is get from the acceptor $A^{(1)}$ by taking away the maker from $A^{(1)}$. Obviously, $A^{(1)'}$ is a 3D deterministic finite-state acceptor. So, from the result of Lemma 2.1, it partitions the set T into at most $(140n|Q|+1)^{140n|Q|}$ equivalence classes. We call these equivalence classes $A^{(1)'}$-equivalence classes. On the other hand, the set T has 2^{n^2} elements since there are 2^{n^2} patterns of the central part. Since $2^{n^2} > (140n|Q|+1)^{140n|Q|}$ for sufficiently large n, the largest equivalence class of $A^{(1)'}$ has at least

$$2^{n^2} / (140n|Q|+1)^{140n|Q|}$$

different rectangular parallelopipeds. $A^{(1)}$ starts at upper-north-west corner of these two different rectangular parallelopipeds with its maker. Then, $A^{(1)}$ carries its maker into central part in one of $140n|Q|$ ways. Now, let us define any two $A^{(1)}_{0}$-equivalent 3D pictures of T are $A^{(1)}_{1}$-equivalent provided that $A^{(1)}$ carries its maker into central parts and brings it back in the same way for both pictures. Then, at least one of these $A^{(1)}_{1}$-equivalence classes has

$$2^{n^2} / (140n|Q|)(140n|Q|+1)^{140n|Q|}$$

elements. Now, consider any different 3D pictures x_1, y_1 of an $A^{(1)}_{1}$-equivalence class. To distinguish these pictures, $A^{(1)}$ must bring its maker into central part at least two times in one of $(140n|Q|)(140n|Q|-1)$ ways since if $A^{(1)}$ carries its maker in the same way for both times, it falls into cycling. By the same argument as above, at least

$$2^{n^2} / (140n|Q|)(140n|Q|-1)(140n|Q|+1)^{140n|Q|}$$

elements are $A^{(1)}_{2}$-equivalent.

Continue in this way, there are

$$2^{n^2} / (140n|Q|)!(140n|Q|+1)^{140n|Q|}$$

3D pictures of T which are $A^{(1)}_{(140n|Q|)!}$-equivalent. For a sufficiently large n, we have $2^{n^2} > (140n|Q|)!(140n|Q|+1)^{(140n|Q|)}$. Then, $2^{n^2}/(140n|Q|)!(140n|Q|+1)^{(140n|Q|)} > 1$ for such n. However, $A^{(1)}$ cannot carry its maker into the central part more than $(140n|Q|)!$ times without cycling. Here, let us consider two different elements t_1, t_2 in a $A^{(1)}_{(140n|Q|)!}$-equivalence class. Note that there exists at least one pair of corresponding 5x5x5 local patterns of their central parts which are different each other (see, Figure 7) For any pair of such pictures, we can construct a new outer pattern such that t_1 has connected objects in the chain-like style, but t_2 has no such objects. We call these new 3D pictures t'_1 and t'_2 (see Figure 8). Note here that y'_1 and y'_2 are still $A^{(1)}_{(140n|Q|)!}$-equivalent. Therefore, if $A^{(1)}$ accepts t'_1, it also accepts t'_2. It is a contradiction.

Thus, we have this theorem.

From Theorem 2.1, we have easily the following corollary:

Corollary 2.2. A 3D FSA cannot determine the chain-like connectivity.

3. Remarks

It has been shown that a 3D one-pebble acceptor cannot recognize the chain-like connectivity. By the similar argument, it is easily shown that one-pass parallel/sequential acceptor given in [3] cannot recognize this connectivity. It is shown without difficulty that a 3D tape-bounded acceptor can accept this set. How is a situation for a 3D two-pebble acceptor ? This question seems to be interesting. We will treat this problem at further papers.

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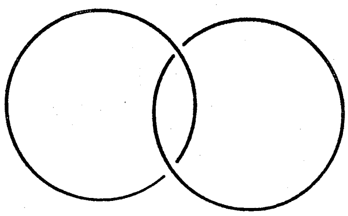


Figure 1.

1 1 1 1 1
1 0 0 0 1
1 0 0 0 1
1 0 0 0 1
1 1 1 1 1

Figure 2.

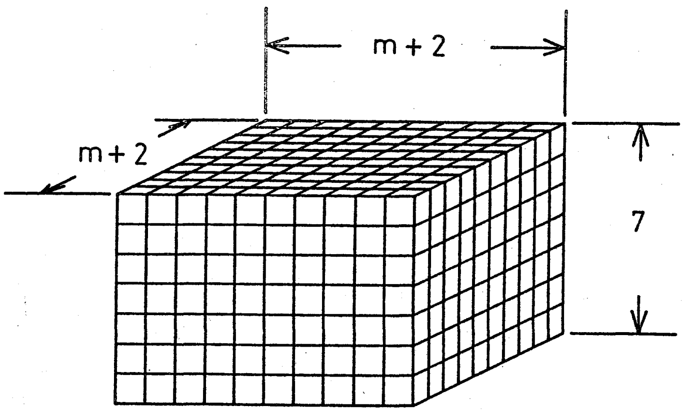


Figure 3.

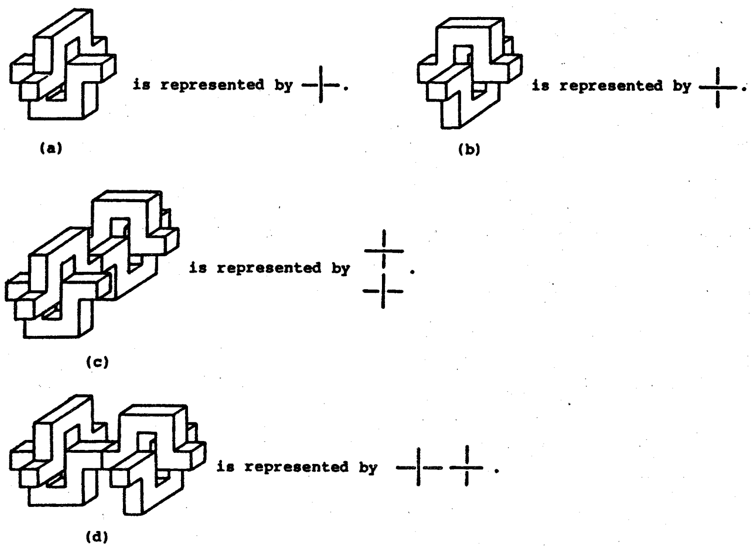


Figure 4.

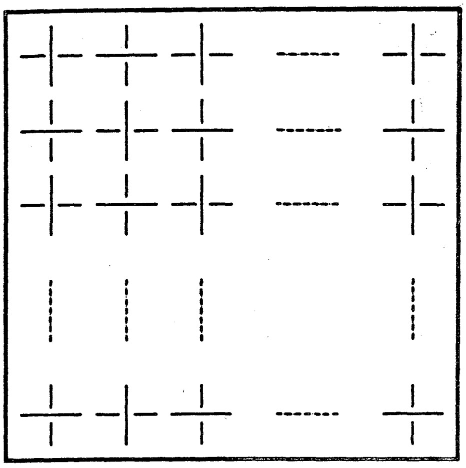


Figure 5.

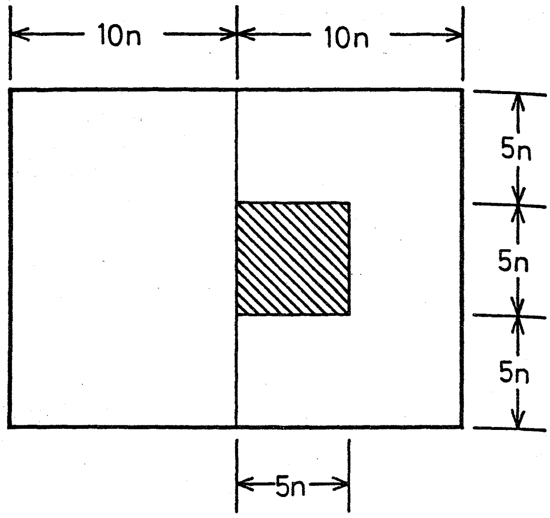
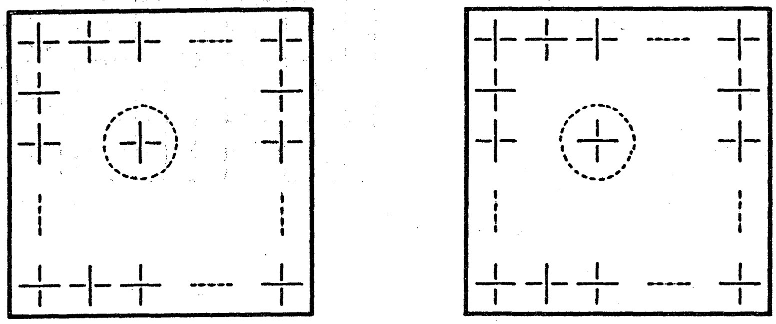
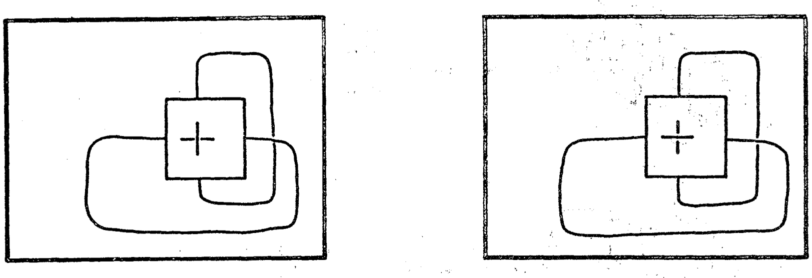


Figure 6.



(t_1) (t_2)

Figure 7.



(t_1') (t_2')

Figure 8.