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The Greatest Fixed-Points and Rational Omega-Tree Languages
(Extended abstract)

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Rational omega-tree languages are defined by extending the notion of rational (or regular) expressions for languages, and their properties are investigated. They are characterized by means of greatest fixed-points of certain functions, and are shown to be precisely the omega-tree languages recognized by Rabin's "special automata".

Kleene's theorem in classical theory of automata can be stated without referring to automata, as follows; L is a rational language if and only if L is a component of the least fixed-point of a function $f(\mathbb{X}) = \mathbb{L}\mathbb{X} : P(A^*)^n \rightarrow P(A^*)^n$ where $n \geq 1$ and $\mathbb{L}$ is an n x n matrix of finite languages. Our motivation of this work is to see whether such characterizations are possible for sets of infinite words and infinite trees, and if so, for what class(es). It turns out that the class mentioned in the abstract and some others are characterized by using the greatest fixed-points, rather than the least fixed-points, of certain (linear and nonlinear) functions.

For an alphabet A, let A^* (A^ω, respectively) be the set of finite words (ω-words) over A, and let $A^\infty = A^* \cup A^\omega$. Subsets of A^* (A^ω, A^∞, resp.) are called languages (ω-languages, ∞-languages). By extending the notion of rational (or regular) expressions for languages, we define the class $Rat(A^\infty)$ of rational ∞-languages; $L \in Rat(A^\infty)$ if and only if L is an ∞-language obtained from subsets of A by a finite number of union, concatenation, $*$-operation, and ω-operation (where for $L \subseteq A^{\infty}$, $L^{\omega} = \{ \lim_{k\to\infty} w_0 w_1 \ldots w_k \in A^{\infty} \mid w_0, w_1, \ldots \in L \}$). Then we define $Rat(A^{\omega}) = Rat(A^{\infty}) \cap P(A^{\omega})$ and $Rat(A^*) = Rat(A^{\omega}) \cap P(A^*)$.

Our first result is the following characterization of $Rat(A^{\omega})$.

Theorem 1

The following conditions for an ω-language L are equivalent.

(i) L belongs to $Rat(A^{\omega})$.

(ii) There exist $n \geq 1$ and L_{ij} in $Rat(A^*)$ $(i,j=1,\ldots,n)$ such that

$$L = \bigcup \{ L_{ij} L_{jk} L_{kl} \ldots \mid j,k,l,\ldots \text{ in } \{1,\ldots,n\} \}.$$

(iii) There exist $n \geq 1$ and $\mathbb{L}$ in $Rat(A^*)^{n\times n}$ such that L is a component of the greatest fixed-point of the function $f(\mathbb{X}) = \mathbb{L}\mathbb{X} : P(A^{\omega})^{n} \to P(A^{\omega})^{n}$.

The essential part of the proof of the theorem can be stated as

Lemma

For a given matrix $\mathbb{L} = (L_{ij}) \in P(A^{\infty})^{n\times n}$ let $\mathbb{K} = (K_{ij}) \in P(A^{\infty})^{n\times n}$ be defined by $\mathbb{K} = \bigcup_{k=1}^{\infty} \mathbb{L}^{k}$. Then $(K'_1,\ldots,K'_n)$ with $K'_i = \bigcup_{j=1}^{n} K_{ij} K_{jj}^{\omega} A^{\omega}$ $(i=1,\ldots,n)$ is the greatest fixed-point of the function $f(\mathbb{X}) = \mathbb{L}\mathbb{X} : P(A^{\omega})^{n} \to P(A^{\omega})^{n}$.

When we consider A^{ω} as the product space of the discrete space A, the class of closed subsets of A^{ω} which are rational can be characterized as follows.

Theorem 2

The following conditions for an ω-language L are equivalent.

(i) L belongs to $Rat(A^{\omega})$ and L is closed.

(ii) There exist $n \geq 1$ and finite languages L_{ij} ($\subseteq A^*$) $(i,j=1,\ldots,n)$ such that

$$L = \bigcup \{ L_{ij} L_{jk} L_{kl} \ldots \mid j,k,l,\ldots \text{ in } \{1,\ldots,n\} \}.$$

(iii) There exist $n \geq 1$ and an $n \times n$ matrix $\mathbb{L}$ of finite languages ($\subseteq A^*$) such that L is a component of the greatest fixed-point of the function $f(\mathbb{X}) = \mathbb{L}\mathbb{X} : P(A^{\omega})^n \to P(A^{\omega})^n$.

The argument leading to these results on ω-languages can be extended to sets of infinite trees, and we can characterize two (proper) sebclasses of the class of recognizable sets of infinite trees (in the sense of Rabin[1]); one is the class of sets recognized by "special automata" in the sense of Rabin[2], and the other is its subclass consisting of closed sets.

We call a mapping $t:\{l,r\}^* \to A$ an ω-tree over A (where l and r are distinct symbols, and A is an alphabet), and denote their totality by T_A^{ω}. We also define the set $T_{A,B}^{\infty}$ of ∞-trees over alphabets A and B as the set of binary (finite or infinite) trees whose terminal (nonterminal, resp.) nodes are labeled by symbols in B (in A). The set of finite trees in $T_{A,B}^{\infty}$ is denoted by $T^*_{A,B}$.

We define the substitution, $*$-substitution, and ω-substitution of ∞-tree languages (i.e., subsets of $T_{A,B}^{\infty}$) as follows: For any $n \geq 0$, $T_0, T_1, \ldots, T_n \subseteq T_{A,B}^{\infty}$ and $b_1, \ldots, b_n \in B$,

the substitution $T_0[T_1/b_1, \ldots, T_n/b_n]$ is defined as the set of ∞-trees obtained from those in T_0 by substituting each occurrence of the symbols b_i with an ∞-tree in T_i, respectively;

the $*$-substitution is defined as $T_0[T_1/b_1, \ldots, T_n/b_n]^* = \bigcup_{k=0}^{\infty} T^{(k)}$ where $T^{(0)} = T_0$ and $T^{(k+1)}$ $(k \geqslant 0)$ is the set of ∞-trees obtained from those in $T^{(k)}$ by substituting an occurrence of b_i (for some i) with an ∞-tree in T_i;

the ω-substitution is defined as $T_0[T_1/b_1, \ldots, T_n/b_n]^{\omega} = \{ \lim_{k\to\infty} t_k \mid t_0 \in T_0,\ t_{k+1} \in t_k[T_1/b_1, \ldots, T_n/b_n]\ (k \geqslant 0)\}$.

Now we define $Rat(T^{\infty}_{A,B})$ as the class of sets $(\subseteq T^{\infty}_{A,B})$ obtained from finite sets of the form $\{ \overset{a}{b \quad b'} \mid a \in A$, b, b' are any symbols $\}$ by a finite number of union, substitution, $*$-substitution, and ω-substitution. The sets in $Rat(T^{\omega}_{A}) = Rat(T^{\infty}_{A,\phi})$ are called rational ω-tree languages, and those in $Rat(T*_{A,B}) = Rat(T^{\infty}_{A,B}) \cap P(T*_{A,B})$ are rational finite-tree languages.

These classes are related with finite automata as follows.

Theorem 3

$Rat(T^{\omega}_{A})$ is the class of recognized sets by Rabin's "special automata". $Rat(T*_{A,B})$ is the class of subsets of $T*_{A,B}$ which are recognized by tree automata for finite trees and consist of finite trees of height > 1.

Theorems 1 and 2 for ω-languages are now extended to ω-tree languages in the following way.

Theorem 4

For $T \subseteq T^{\omega}_{A}$, the following conditions are equivalent.

(i) T belongs to $Rat(T^{\omega}_{A})$.

(ii) There exist $n \geqslant 1$ and $T_i \in Rat(T*_{A,\{1,\cdots,n\}})$ $(i=0,1,\ldots,n)$ such that $T = T_0[T_1/1,\ldots,T_n/n]^\omega$.

(iii) There exist $n \geqslant 1$ and $T_i \in Rat(T*_{A,\{1,\cdots,n\}})$ $(i=1,\ldots,n)$ such that T is a component of the greatest fixed-point of the function $(T_1[X_1/1,\ldots,X_n/n], \ldots, T_n[X_1/1,\ldots,X_n/n]) : P(T_A^\omega)^n \to P(T_A^\omega)^n$.

Theorem 5

For $T \subseteq T_A^\omega$, the following conditions are equivalent.

(i) T belongs to $Rat(T_A^\omega)$ and is a closed set (with respect to the product topology on T_A^ω of the discrete topology on A).

(ii) There exist $n \geqslant 1$ and finite sets $T_i \subseteq T*_{A,\{1,\cdots,n\}}$ $(i=0,1,\ldots,n)$ such that $T = T_0[T_1/1,\ldots,T_n/n]^\omega$.

(iii) There exist $n \geqslant 1$ and finite sets $T_i \subseteq T*_{A,\{1,\cdots,n\}}$ $(i=1,\ldots,n)$ such that T is a component of the greatest fixed-point of the function $(T_1[X_1/1,\ldots,X_n/n], \ldots, T_n[X_1/1,\ldots,X_n/n]) : P(T_A^\omega)^n \to P(T_A^\omega)^n$.

The argument can also be extended to ∞-languages and ∞-tree languages.

References

[1] M.O.Rabin, Decidability of second-order theories and automata on infinite trees, Trans. Amer. Math. Soc. (1969) 1 - 35.

[2] M.O.Rabin, Weakly definable relations and special automata, in "Mathematical Logic and Foundations of Set Theory" (ed. Y. Bar-Hillel) North-Holland, 1970, 1 - 23.