数理解析研究所講究録
第592巻 1986年 119-126

Gevrey Classでの Fuchs双曲型方程式

(Fuchsian Hyperbolic Equations in Gevrey Classes)

上智大理工 田原秀敏
(Sophia University, Hidetoshi TAHARA)

In this note, I want to report my recent results on Fuchsian hyperbolic equations in Gevrey classes. Fuchsian hyperbolic equations were introduced by the author in [5] and they have been investigated in various spaces; e.g., by [5] in analytic-hyperfunction spaces, by [6] in C^∞ spaces (or in Sobolev spaces), by [1] in distribution spaces, and [7,8,9] in Gevrey classes etc.

§1. Review of C^∞ Results.

First, let us consider the following operator:

$$L(t,x,t\partial_t,t^{\varkappa}\partial_x) \ (=L)$$
$$= (t\partial_t)^m + \sum_{\substack{j+|\alpha|\leq m \\ j<m}} a_{j,\alpha}(t,x)(t\partial_t)^j(t^{\varkappa}\partial_x)^{\alpha},$$

where $(t,x)=(t,x_1,\ldots,x_n)\in[0,T]\times\mathbb{R}^n$ $(T>0)$, $m\in\mathbb{N}$ $(=\{1,2,3,\ldots\})$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n$ $(=\{0,1,2,\ldots\}^n)$, $|\alpha|=\alpha_1+\ldots+\alpha_n$, $a_{j,\alpha}(t,x)\in C^\infty([0,T]\times\mathbb{R}^n)$ $(j+|\alpha|\leq m$ and $j<m)$, $\varkappa=(\varkappa_1,\ldots,\varkappa_n)\in\mathbb{N}^n$, and

$$\partial_t = \frac{\partial}{\partial t}, \quad t^{\varkappa}\partial_x = (t^{\varkappa_1}\frac{\partial}{\partial x_1}, \ldots, t^{\varkappa_n}\frac{\partial}{\partial x_n}),$$

$$(t^{\varkappa}\partial_x)^{\alpha} = (t^{\varkappa_1}\frac{\partial}{\partial x_1})^{\alpha_1}\ldots(t^{\varkappa_n}\frac{\partial}{\partial x_n})^{\alpha_n}$$

$$(= t^{\varkappa_1\alpha_1+\ldots+\varkappa_n\alpha_n}(\frac{\partial}{\partial x_1})^{\alpha_1}\ldots(\frac{\partial}{\partial x_n})^{\alpha_n}).$$

In addition, we impose the following condition on L:

(A-1) All the roots $\lambda_i(t,x,\xi)$ $(1\leq i\leq m)$ of

$$\lambda^m + \sum_{\substack{j+|\alpha|=m \\ j<m}} a_{j,\alpha}(t,x)\lambda^j\xi^{\alpha} = 0$$

are <u>real</u>, <u>simple</u> and <u>bounded</u> on $\{(t,x,\xi);\ |\xi|=1\}$.

Then, L is one of the most elementary examples of Fuchsian hyperbolic operators (in [6]). Therefore, we can define the characteristic exponents $\rho_1(x),\ldots,\rho_n(x)$ by the roots of

$$L(0,x,\rho,0) = 0.$$

Let $\mathcal{E}(\mathbf{R}^n)$ be the Schwartz space on $\mathbf{R}^n$, and let $C^{\infty}([0,T],\mathcal{E}(\mathbf{R}^n))$ [resp. $C^{\infty}((0,T),\mathcal{E}(\mathbf{R}^n))$] be the space of all infinitely differentiable functions on [0,T] [resp. (0,T)] with values in $\mathcal{E}(\mathbf{R}^n)$. Then, we have the following results.

<u>Unique Solvability (Tahara [6-III]).</u>

<u>Assume that</u> $\rho_i(x)\notin \mathbf{Z}_+$ <u>holds for any</u> $x\in\mathbf{R}^n$ <u>and</u> $1\leq i\leq m$. <u>Then, the equation</u>

$$L(t,x,t\partial_t,t^{\varkappa}\partial_x)u = f$$

<u>is uniquely solvable in</u> $C^{\infty}([0,T],\mathcal{E}(\mathbf{R}^n))$.

Asymptotic Expansions (Tahara [6-V]).

Assume that $\rho_i(x)-\rho_j(x)\notin \mathbb{Z}$ holds for any $x\in \mathbb{R}^n$ and $1\leq i\neq j\leq m$. Then, we have the following results on

(E) $\qquad L(t,x,t\partial_t,t^{\varkappa}\partial_x)u = 0 \quad$ in $C^\infty((0,T),\mathcal{E}(\mathbb{R}^n))$.

(1) Any solution $u(t,x)\in C^\infty((0,T),\mathcal{E}(\mathbb{R}^n))$ of (E) can be expanded asymptotically into the form

$$u(t,x) \sim \sum_{i=1}^{m}\left(\varphi_i(x)t^{\rho_i(x)} + \sum_{k=1}^{}\sum_{h=0}^{mk}\varphi_{i,k,h}(x)t^{\rho_i(x)+k}(\log t)^h\right) \qquad (*)$$

(as $t\longrightarrow +0$) for some unique $\varphi_i(x)$, $\varphi_{i,k,h}(x)\in\mathcal{E}(\mathbb{R}^n)$.

(2) Conversely, for any $\varphi_i(x),\ldots,\varphi_n(x)\in\mathcal{E}(\mathbb{R}^n)$ there exist a unique solution $u(t,x)\in C^\infty((0,T),\mathcal{E}(\mathbb{R}^n))$ of (E) and unique coefficients $\varphi_{i,k,h}(x)\in\mathcal{E}(\mathbb{R}^n)$ such that the asymptotic relation in (1) holds.

Example. Let P be of the form

$$P = (t\partial_t)^2 - t^{2\varkappa_1}\partial_{x_1}^2 - t^{2\varkappa_2}\partial_{x_2}^2$$
$$+t^{p_1}a_1(t,x)\partial_{x_1} + t^{p_2}a_2(t,x)\partial_{x_2}$$
$$+b(t,x)(t\partial_t)+c(t,x),$$

where $\varkappa_1,\varkappa_2,p_1,p_2\in\mathbb{N}$. Then, if $p_1\geq\varkappa_1$ and $p_2\geq\varkappa_2$ hold , we can apply our C^∞ results to P.

In this example, the condition "$p_1\geq\varkappa_1$ & $p_2\geq\varkappa_2$" seems to be essential to the C^∞ results quoted above. Therefore, if we want to consider the case without "$p_1\geq\varkappa_1$ & $p_2\geq\varkappa_2$", we must treat

P in suitable subclasses in $C^\infty([0,T],\mathcal{E}(\mathbf{R}^n))$ or $C^\infty((0,T),\mathcal{E}(\mathbf{R}^n))$.

§2. Results in Gevrey Classes.

In order to treat P in Example for the general case, let us consider here the following operator:

$$P(t,x,t\partial_t,\partial_x) = L(t,x,t\partial_t,t^{\varkappa}\partial_x)$$

$$+ \sum_{j+|\alpha|<m} t^{p(j,\alpha)} b_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha,$$

where $L(t,x,t\partial_t,t^{\varkappa}\partial_x)$ is the same as in §1, $p(j,\alpha)\in\mathbf{N}$ $(j+|\alpha|<m)$ and $b_{j,\alpha}(t,x)\in C^\infty([0,T]\times\mathbf{R}^n)$ $(j+|\alpha|<m)$.

A function $f(x)$ $(\in C^\infty(\mathbf{R}^n))$ is said to belong to the <u>Gevrey class</u> $\mathcal{E}^{\{s\}}(R^n)$, if $f(x)$ satisfies the following: for any compact subset K of $\mathbf{R}^n$ there are $C>0$ and $h>0$ such that

$$\sup_{x\in K}\left|\partial_x^\alpha f(x)\right| \leq C\, h^{|\alpha|}(|\alpha|!)^s \qquad \text{for any } \alpha\in\mathbf{Z}_+^n. \tag{2.1}$$

We denote by $C^\infty([0,T],\mathcal{E}^{\{s\}}(\mathbf{R}^n))$ [resp. $C^\infty((0,T),\mathcal{E}^{\{s\}}(\mathbf{R}^n))$] the space of all infinitely differentiable functions on [0,T] [resp. (0,T)] with values in $\mathcal{E}^{\{s\}}(\mathbf{R}^n)$ equipped with the locally convex topology in [4].

Let S_n denote the permutation group of n-numbers, put

$$M_{j,\alpha}(\tau,r) = \frac{\sum_{i=1}^{r}(\varkappa_{\tau(i)}-\varkappa_{\tau(r)})\alpha_{\tau(i)}+(m-j)\varkappa_{\tau(r)}-p(j,\alpha)}{(m-j-|\alpha|)\varkappa_{\tau(r)}}$$

and define the irregularity σ $(\geqq 1)$ by

$$\sigma = \max\left[1, \max_{\substack{j+|\alpha|<m \\ j<m}}\left\{\min_{\tau\in S_n}\left(\max_{1\leqq r\leqq n} M_{j,\alpha}(\tau,r)\right)\right\}\right].$$

Using this irregularity, we impose the following conditions:

(A-2) $1 < s < \sigma/(\sigma-1)$.

(A-3) $a_{j,\alpha}(t,x)$, $b_{j,\alpha}(t,x)\in C^{\infty}([0,T],\mathcal{E}^{\{s\}}(\mathbb{R}^n))$ for any (j,α).

When $\sigma=1$, (A-2) is read $1<s<\infty$. Then, we have the following results.

Theorem 1 (Unique Solvability).

Assume that $\rho_i(x)\notin \mathbb{Z}_+$ holds for any $x\in\mathbb{R}^n$ and $1\leqq i\leqq m$. Then, the equation

$$P(t,x,t\partial_t,\partial_x)u = f$$

is uniquely solvable in $C^{\infty}([0,T],\mathcal{E}^{\{s\}}(\mathbb{R}^n))$.

Theorem 2 (Asymptotic Expansions).

Assume that $\rho_i(x)-\rho_j(x)\notin \mathbb{Z}$ holds for any $x\in\mathbb{R}^n$ and $1\leqq i\neq j\leqq m$. Then, we have the following results on

(E_s) $\quad P(t,x,t\partial_t,\partial_x)u = 0$ in $C^{\infty}((0,T),\mathcal{E}^{\{s\}}(\mathbb{R}^n))$.

(1) Any solution $u(t,x)\in C^{\infty}((0,T),\mathcal{E}^{\{s\}}(\mathbb{R}^n))$ of (E_s) can be expanded asymptotically into the form

$$u(t,x) \sim \sum_{i=1}^{m}\left(\varphi_i(x)t^{\rho_i(x)} + \sum_{k=1}^{mk}\sum_{h=0}\varphi_{i,k,h}(x)t^{\rho_i(x)+k}(\log t)^h\right) \tag{**}$$

(as $t \longrightarrow +0$) for some unique $\varphi_i(x)$, $\varphi_{i,k,h}(x) \in \mathcal{E}^{\{s\}}(\mathbb{R}^n)$.

(2) Conversely, for any $\varphi_i(x),\ldots,\varphi_n(x) \in \mathcal{E}^{\{s\}}(\mathbb{R}^n)$ there exist a unique solution $u(t,x) \in C^\infty((0,T),\mathcal{E}^{\{s\}}(\mathbb{R}^n))$ of (E_s) and unique coefficients $\varphi_{i,k,h}(x) \in \mathcal{E}^{\{s\}}(\mathbb{R}^n)$ such that the asymptotic relation in (1) holds.

Here, the meaning of the asymptotic relation (**) [resp. (*)] is as follows: for any $a>0$ and any compact subset K of $\mathbb{R}^n$ there is an $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$

$$t^{-a}(t\partial_t)^j\Big[u(t,x) - \sum_{i=1}^{n}\Big(\varphi_i(x)t^{\rho_i(x)} + \sum_{k=1}^{N}\sum_{h=0}^{mk}\varphi_{i,k,h}(x)t^{\rho_i(x)+k}(\log t)^h\Big)\Big]\Big|_K$$

$$\longrightarrow +0 \quad \text{in } \mathcal{E}^{\{s\}}(K) \ [\text{resp. } \mathcal{E}(K)]$$

as $t \longrightarrow +0$ for any $j \in \mathbf{Z}_+^n$,

where $\mathcal{E}^{\{s\}}(K)$ is the locally convex space of all functions $f(x) \in C^\infty(K)$ satisfying (2.1) for some $C>0$ and $h>0$ (see [4]).

§3. Remark.

(1) $\sigma=1$ is equivalent to the following:

$$p(j,\alpha) \geq \chi_1\alpha_1 + \ldots + \chi_n\alpha_n \quad \text{for any } (j,\alpha).$$

Hence, when $\sigma=1$, we can apply our C^∞ results to $P(t,x,t\partial_t,\partial_x)$.

(2) When P is of the form

$$P = (t\partial_t)^2-t^{2\varkappa}\partial_x^2+t^p a(t,x)\partial_x+b(t,x)(t\partial_t)+c(t,x),$$

σ is given by $\sigma=\max[1,(2\varkappa-p)/\varkappa]$. Therefore, if $p<\varkappa$, (A-2) is given by $1<s<(2\varkappa-p)/(\varkappa-p)$. This coincides with the example in [2,3,9].

(3) When P is of the form

$$P = (t\partial_t)^2-t^{2\varkappa_1}\partial_{x_1}^2 - \dots -t^{2\varkappa_n}\partial_{x_n}^2$$

$$+t^{p_1}a_1(t,x)\partial_{x_1}+ \dots +t^{p_n}a_n(t,x)\partial_{x_n}$$
$$+b(t,x)(t\partial_t)+c(t,x),$$

σ is given by $\sigma=\max[1,(2\varkappa_1-p_1)/\varkappa_1,\dots,(2\varkappa_n-p_n)/\varkappa_n]$.

(4) When $\varkappa_1=\dots=\varkappa_n(=k)$, σ is given by

$$\sigma = \max\left[1, \max_{\substack{j+|\alpha|<m \\ j<m}} \left(\frac{m-j-p(j,\alpha)/k}{m-j-|\alpha|}\right)\right].$$

(5) When $\varkappa_1=\dots=\varkappa_n$, Uruy [9] defines an index σ_u (≥ 1) and obtain Theorem 1 under the condition $1<s<(\sigma_u-1)/\sigma_u$. The relation between σ and σ_u is as follows.

(i) $1\leq\sigma\leq\sigma_u$ holds in general.

(ii) There are examples for which $1<\sigma<\sigma_u$ holds.

For example, in the case $P=(t\partial_t)((t\partial_t)^2-t^8\partial_x^2)+t\partial_x+t^3\partial_x^2$, we have $\sigma=9/4$ and $\sigma_u=5/2$. Hence, we can say that our condition (A-2) is better than that in [9].

References

[1] Bove, A. Lewis, J.E. and Parenti, C.: Cauchy Problem for Fuchsian hyperbolic operators, Hokkaido Math. J., 14 (1985), 175-248.

[2] Igari, K.: An admissible data class of the Cauchy problem for non-strictly hyperbolic operators. J. Math. Kyoto Univ., 21 (1981), 351-373.

[3] Ivrii, V. Ja.: Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity for Gevrey classes. Sib. Mat. Zh., 17 (1976), 1256-1270.

[4] Komatsu, H.: Ultradistributions, I. Structure theorems and a characterization. J. Fac. Sci. Univ. Tokyo, Sect. IA Math., 20 (1973), 25-105.

[5] Tahara, H.: Fuchsian type equations and Fuchsian hyperbolic equations. Japan. J. Math., 5 (1979), 245-347.

[6] Tahara, H.: Singular hyperbolic systems, I,II,III,IV,V. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 26 (1979), 213-238, 391-412; 27 (1980), 465-507; Japan. J. Math., 8 (1982), 297-308; J. Math. Soc. Japan, 36 (1984), 449-473.

[7] Tahara, H.: Cauchy problems for Fuchsian hyperboic equations in spaces of functions of Gevrey classes. Proc. Japan Acad., 61A (1985), 63-65.

[8] Tahara, H.: Asymptotic expansions of solutions of Fuchsian hyperbolic equations in spaces of functions of Gevrey classes. Proc. Japan Acad., 61A (1985), 255-258.

[9] Uryu, H.: Conditions for well-posedness in Gevrey classes of the Cauchy problems for Fuchsian hyperbolic operators. Publ. RIMS. Kyoto Univ., 21 (1985), 355-383.