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THETAS AND THEIR DIFFERENTIAL EQUATIONS

by Iacopo Barsotti (Padova)

I'll present a method for introducing theta functions by a local definition, hence unencumbered by periods; it leads, in turn, to differential equations (nonlinear of course) which have all and only theta functions as solutions; the method works in any characteristic, but I'll stick to characteristic zero (the diff. equations part for charact. p is still under construction, since differential algebra is ruefully inadequate).

Let k be an algebraically closed field of characteristic zero, and let A be a commutative group-variety of dimension n over k, connected and with degeneration locus A_d; the variety is, of course, projective, and A_d, of pure dimension n-1, is the set of the points of A which do not belong to the group. If $C = k(A)$, the composition law $\mu: A \times A \longrightarrow A$ has a "dual" $\mathbb{P}: C \longrightarrow Q(C \otimes C)$ = quotient field of $C \otimes_k C$, which is a coproduct and turns C into a cocommutative hyperfield. Let R be the completion of $k(O/A)$ = local ring on A of the zero point O, and let ω_i, for $i = 1, \ldots, n$, be a k-basis for the invariant differentials on A (they are of the first kind if A is abelian); when C is embedded in $Q(R)$, the ω's become exact, and have integrals u_i which can be selected in R^+ = maximal prime of R; then $u_1, \ldots, u_n$ form a regular set of parameters of R, and $R = k\{u\}$ = ring of power series in the u's, with coefficients in k and nonnegative integral exponents. Call $d_1, \ldots, d_n$ a k-basis for the invariant derivations on A dual to $u_1, \ldots, u_n$: $d_i u_j = \delta_{ij}$ (more simply, $d_i = \partial/\partial u_i$), and let $m \leqslant n$ be a positive integer such that there exist elements $c_{ij} \in k$ with the property that the constant field, in C, for the derivations $d_i' = \sum_1^n{}_j c_{ij} d_j$ $(i = 1, \ldots, m)$ is k [the condition for the existence of the c_{ij} is that A have no direct factor which is the product of more than m additive lines]; and finally consider the homomorphism of $k\{u\}$ into $k\{u_1', \ldots, u_m'\}$ which sends u_j into $\sum_1^m{}_i c_{ij} u_i'$. Because of the choice of the d_i', this homomorphism can be restricted-extended to an isomorphism of C into $Q(k\{u'\})$, and I shall consider C so embedded; the original d_i, u_i are abandoned, and the apex is dropped from d_i', u_i': $C \subset Q(k\{u_1, \ldots, u_m\})$, $d_i = \partial/\partial u_i$. The ring $k\{u\}$ has a coproduct $\mathbb{P}$ given by $\mathbb{P}u_i = u_i \hat{\otimes} 1 + 1 \hat{\otimes} u_i$ (but I'll rather use $u \longrightarrow u+v$), and this $\mathbb{P}$ induces in C its own $\mathbb{P}$ (because the u's are invariant integrals).

Now let X be a divisor on A, without components on the degeneration locus, and whose support does not contain 0 (unnecessary but convenient condition). On $A \times A$ consider the divisor $Y = X_{12}+X_{1,-2}-2X_1-X_2-X_{-2}$, where the symbolism is as follows: $A\times A \xrightarrow{\mu} A$ gives $X_{12} \xleftarrow{\mathrm{div}\,\mu} X$; $A \times A \xrightarrow{\iota \times (-\iota)} A \times A \xrightarrow{\mu} A$ gives $X_{1,-2} \longleftarrow X$ (ι is the identity mapping); $X_1 = X \times A$, $X_2 = A \times X$, $X_{-2} = A \times (\mathrm{div}-\iota)X$. Then $Y \sim 0$ (linearly equivalent to zero), as a particular case of the theorem of the square: set $Q = -P$ in $\sigma_{P+Q}X - \sigma_P X - \sigma_Q X + X \sim 0$. Hence there is a $g(u, v) \in Q(C \otimes C) \cap k\{u, v\} \subset Q(k\{u\} \otimes k\{v\}) \cap k\{u, v\}$ such that $\mathrm{div}_{A \times A} g = Y$, and we can require $g(0, 0) = 1$. By looking at the divisors, on $A \times A$, of both sides we can verify that

$$(1) \quad g(u+w, v)g(u-w, v)g^2(u, v) = g(u, v+w)g(u, v-w)g(v, w)g(-v, w).$$

I claim that a $g(u, v) \in k\{u, v\}$ which satisfies (1) must "split"; namely, there must exist a $\vartheta(u) \in k\{u\}$ (and we require $\vartheta(0) = 1$) such that

$$g(u, v) = \vartheta(u+v)\vartheta(u-v)/\vartheta^2(u)\vartheta(v)\vartheta(-v).$$

The proof is achieved by applying $d_u^q d_v^r d_w^s$ to the logarithm of (1), with q, r, s multiindices, and by setting $u = v = w = 0$; this gives relations among the coefficients in the power series $g(u, v)$, and the conclusion follows easily. Well, $\vartheta(u)$, from now on denoted by $\vartheta_X(u)$, is the <u>theta type</u> of X, or the <u>theta</u> of X when $m = n$; moreover, ϑ is holomorphic (which means that X is effective) if and only if $g(u, v) \in k\{u\} \otimes k\{v\}$. Naturally, when $k = \mathbb{C}$, $m = n$, A is abelian, and X is effective, $\vartheta_X(u)$ is the classical theta function whose set of zeroes has, on A, the image X.

The next task is to define a theta type a priori, without A and X; it is enough to turn the tables: $\vartheta(u_1, \ldots, u_m) \in k\{u\}$ is a <u>theta type</u> if it satisfies the <u>prosthaferesis formula</u>

$$(2) \qquad \vartheta(u+v)\vartheta(u-v) \in Q(k\{u\} \otimes k\{v\});$$

it is holomorphic if it satisfies the holomorphic prosthaferesis, which is the same (2) without Q. The sequence of consequences of this definition is as follows:

A. Because of (2), the ratios of the coefficients (belonging to $k\{u\}$) of the left-hand-side of (2), seen as a power series in the v's, to one of them, say $\vartheta^2(u)$, generate, over k, a finitely generated field C.

B. The field C is generated, over k, by (finitely many) iterated derivatives, from second derivatives on, of $\log\vartheta(u)$, after assuming $\vartheta(0) = 1$.

C. The field C is a cocommutative hyperfield with respect to the $\mathbb{P}$ of $k\{u\}$: $\mathbb{P}C \in Q(C \otimes C)$.

D. Hence $C = k(A)$ for a commutative group-variety A.

E. $g(u, v) = \vartheta(u+v)\vartheta(u-v)/\vartheta^2(u)\vartheta(v)\vartheta(-v) \in Q(C \otimes C)$, and there esists a divisor X on A such that $X_{12}+X_{1,-2}-2X_1-X_2-X_{-2} = \mathrm{div}_{A \times A}\, g$; we say that $X = \mathrm{div}\,\vartheta$.

F. $\vartheta = \vartheta_X$.

A pleasant surprise is that any A, other than those with a direct factor which is the product of at least 2 additive lines, can be reached in this manner when m = 1. We shall accordingly use u for just one argument, and look more closely at the holomorphic prosthaferesis formula:

(3) $\quad \vartheta(u+v)\vartheta(u-v) \in k\{u\} \otimes k\{v\}$.

The absence of Q (which was present in (2)) permits to say more than just the finite generation of the field C; it permits to say that the ratios of coefficients mentioned in point A above form a finite -dimensional vector space P over k; this is actually a necessary and sufficient condition for the validity of (3). Let's then have a closer look at these ratios: set $\vartheta_r = (r!)^{-1}d^r\log\vartheta$, let $y_2, y_4, \ldots$ be indeterminates, and set

$$P_{2r}(y) = \Sigma_j\, 2^{|j|-1} y_2^{j_1} y_4^{j_2} \ldots y_{2r}^{j_r}, \text{ where } |j| \text{ means } \Sigma_i\, j_i$$

and j ranges over all the multiindices $(j_1, \ldots, j_r)$ such that $j_i \geqslant 0$ and $j_1+2j_2+\ldots+rj_r = r$. Then P is generated, over k, by finitely many among the $P_{2r}(\vartheta) = P_{2r}(\vartheta_2, \vartheta_4, \ldots, \vartheta_{2r})$.

It is now clear what the differential equations defining theta types are: just express each P_{2r}, when r is outside a certain finite set N, as a linear combination with coefficients in k of the P_{2s} for $s \in N$; this provides infinitely many differential equations, ordinary, nonlinear, in ϑ (actually in ϑ_2); but really any one of them can be so considered, the others becoming initial conditions; a posteriori, only finitely many of them are needed, so that the remaining ones are either redundant or incompatible. If one wants equations which will give all possible theta types whose spaces P are generated by the P_{2r} with $r \in N$, we can simply write the equations which express the vanishing of the wronskians of all the sets $\{P_{2i},\ i \in N \cup \{j\}\}$ when j ranges over the nonnegative integers not in N.

The theory provides all the usual trimmings of commutative group-varieties, such as their Picard variety, the maximal algebraic systems of divisors on them, and so on. I may mention two a posteriori connections with other theories:

1). By taking m = 3 instead of = 1 (in which case the P's will have 3 indices), a necessary and sufficient condition in order that A (abelian) be a jacobian with "theta divisor" X is that X be irreducible, and that $k(A) = k(\vartheta_2, \vartheta_3, \ldots)$ for a suitable ϑ for which the following linear relation among the $P_{???}(\vartheta)$ obtains:

$$12P_{400} + 3P_{020} - 2P_{101} + cP_{000} = 0 \text{ for some } c \in k.$$

This, of course, is not proved within the framework of the theory (but it would be desiderable to so prove it): it is a translation into my symbolism of the K dV equation [8].

2). In [9] one can find, after a suitable interpretation, the definition of a k-bilinear mapping $\tilde{\Delta}$ of $k\{u\} \otimes k\{u\}$ into $k\{u\}$ related to a given polynomial Δ, with coefficients in k, in the invariant derivations $d_i = \partial/\partial u_i$; the definition is as follows:

$\tilde{\Delta}(\varphi \otimes \psi) = (1 \otimes \varepsilon)(1 \otimes \Delta)[\varphi(u\otimes 1 + 1\otimes u)\,\psi(u\otimes 1 - 1\otimes u)]$, where

ε is the natural mapping of $k\{u\}$ onto k: $\varepsilon f(u) = f(0)$. This $\tilde{\Delta}$ is related to the P_{2r} by:

$$P_{2r}(\vartheta_2, \vartheta_4, \ldots \vartheta_{2r}) = (1/2)((2r)!)^{-1}\vartheta^{-2}\,\widetilde{d^{2r}}(\vartheta \otimes \vartheta).$$

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