KNOTTED ATTRACTING PERIODIC ORBITS

IN SILNIKOV BIFURCATIONS

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1. INTRODUCTION

Let X be a C^1-vector field on $\mathbb{R}^3$ which satisfies the following conditions;

(i) X(0)=0,

(ii) DX(0) has eigenvalues $\lambda, -\mu \pm i\omega$ with $0 < \mu < \lambda, \omega \neq 0$,

(iii) $W^s(0)$ and $W^u(0)$ intersect at a homoclinic orbit Γ.

We call such a vector field a Silnikov system. A Silnikov system has infinitely many periodic orbits arbitrally close to the homoclinic orbit Γ[3]. Moreover the knot types of these periodic orbits and their linking numbers around Γ determines μ/λ. Therefore μ/λ is a modulus, that is , if X and X' are two Silnikov systems with $\mu/\lambda \neq \mu'/\lambda'$, then they are not topologically equivalent.[4][5]. If a Silnikov system X has eigenvalues $\lambda, -\mu \pm i\omega$ with $0 < \mu < \lambda < 2\mu$ and $\omega \neq 0$, then one can perturb X to X' so that X' has an attracting periodic orbit arbitrally close to Γ. Attracting periodic orbits in $\mathbb{R}^3$ are the simplest attractors which can have non trivial embedding types. So

it would be important to ask whether μ/λ gives any restrictrions on the possible embedding types of attracting periodic orbits of perturbed systems. In this paper, we prove that:

THEOREM   *Let X and X' be Silnikov systems with homoclinic orbits Γ and Γ' and eiginvalues $\lambda, -\mu \pm i\omega$ and $\lambda', -\mu' \pm i\omega'$ respectively. Suppose that $0 < \mu < \lambda < 2\mu$ and $\mu/\lambda < \mu'/\lambda'$.*

Let X_s be a 1-parameter family of vector fields with $X_0 = X$. Let X_s satisfies the transversality condition defined later. Then there exists a $\delta > 0$ such that for any $\varepsilon > 0$ there exists a positive $s < \varepsilon$ such that

   (i) *X_s has an attracting periodic orbit γ in the ε-neighbourhood of Γ, and*

   (ii) *any δ-perturbation X'' of X' has no periodic orbit in the δ-neighbourhood of Γ' whose embedding type into the δ-neighbourhood is the same of that of γ.*

To prove the theorem we use the attracting periodic orbits which trips three times arong Γ.

2. ANALYTIC STUDY OF ATTRACTING PERIODIC ORBITS

Let S be the unit cylinder;

$$S = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1, \ |x_3| \leq 1\}.$$

We use (r, θ, z)-coordinate in the unit cylinder S;

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z.$$

Let SS be the side surface of S;

$$SS = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = 1, \ |x_3| \leq 1\}.$$

Let X_s be a 1-parameter family of C^1-vector fields on $\mathbb{R}^3$ with X_0 Silnikov. To avoide the technical difficalties, we suppose the following;

(*) there exists a 1-parameter family of diffeomorphisms

which linearlizes X_s in the unit cylinder.

Take a sufficiently small r_1 so that any X_s which starts at a point p in

$$\Sigma^2 = \{(r,\theta,1) \mid r \leq r_1\}$$

crosses SS at a point $f_s(p)$, and the mapping

$$f_s: p \in \Sigma^2 \longrightarrow f_s(p) \in SS$$

is a diffeomorphism onto its image. For each s let hight(s) denote the z-coordinate of the point at which $W^u(0)$ crosses SS. We suppose the transversality condition;

$$\frac{d}{dt}\text{hight}(s)\big|_{s=0} \neq 0.$$

Let L_s be the map SS $\longrightarrow \{(r,\theta,1) \mid r \leq 1\}$ induced by the linear vector field $X_s | S$. Let AP(s,ε) be the set of attracting periodic orbit γ of X_s which satisfies

(i) γ is in the ε-neighbourhood of Γ,

(ii) γ crosses SS at three points, say $P_1(\gamma)$, $P_2(\gamma)$ and $P_2(\gamma)$

with z-coordinates $m_1(\gamma), m_2(\gamma)$ and $m_3(\gamma)$,

(iii) $f_s \circ L_s(p_i) = P_{i+1}$ (mod 3) and $m_1(\gamma) < m_2(\gamma) < m_3(\gamma)$.

Note that AP(s,ε) may be empty.   Let

$$\eta(s,\varepsilon) = \inf \frac{\log m_3(\gamma)}{\log m_2(\gamma)}$$

where infimum is taken over all γ's in AP(s',ε) with $0 < s' \leq s$.

PROPOSITION 1.   *For any* $\varepsilon_0 > 0$ *there exists a* $\delta > 0$ *such that*

$$\eta(s,\delta) > \mu/\lambda - \varepsilon_0$$

for all $|s| < \delta$.

PROPOSITION 2. $\eta(s_0,\varepsilon_0) \leq \mu/\lambda$ *for sufficiently small* $s_0 > 0$ *and* ε_0.

2.   PROOF OF THE PROPOSITIONS.

Proof of proposition 1.   Let $c(s)=\mu/\lambda$.   If $\delta_0>0$ is sufficiently

small, then there exists a constant K such that

$$\text{hight}(s) - K(m_1(\gamma))^{c(s)} \leq m_2(\gamma),$$

$$m_3(\gamma) \leq \text{hight}(s)+K(m_2(\gamma))^{c(s)}$$

for all γ in $AP(s,\delta_0)$ with $0<s\leq\delta_0$.   Since $m_1(\gamma)<m_2(\gamma)$, we get

$$m_3(\gamma)\leq(1+2K)(m_2(\gamma))^{c(s)}.$$

Taking the logarizm, we get

$$\frac{\log m_3(\gamma)}{\log m_2(\gamma)} \geq c(s) + \frac{1+2K}{\log m_2(\gamma)}.$$

Taking a smaller δ, we get the required inequality.

Proof of proposition 2.   The proposition 2 would be prooved by a

standard but very messy arguments.

3. TOPOLOGICAL INVARIANT

The braid of the periodic orbit

in $AP(s,\varepsilon)$ is indicated in the fig 1,

where β is an unknown but fixed braid.

The numbers N_1 and N_2 goes to infinity

as ε goes to zero.   The ratio of N_1

and N_2 is estimated by the proposit-

ions 1 and 2, so the remaining task is

to relate the numbers N_1 and N_2 to some

topological invariants.   Hence the

following proposition completes the

proof of the theorem.

Let $\Delta_\gamma(t)$ be the Alexander polynomial

of the knot γ^2. $^)$ Let $n(\gamma)$ be the number

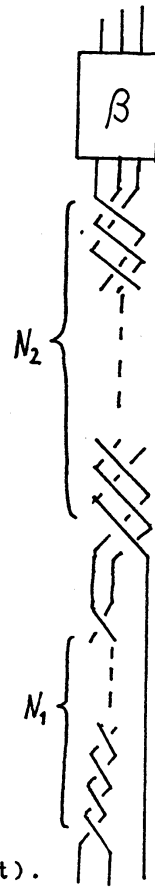of the non zero terms of the polynomial $\Delta_\gamma(t)$.

Fig 1

PROPOSITION 3. *There exists a constant* C *such that*

$$| \, m(t)-N_2 \, | < C$$

$$| \, \text{degree of } \Delta_\gamma(t) - (6N_1+N_2) \, | < C,$$

for all N_1 *and* N_2.

Proof. Let $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ be the Peron Mugnus representaion of the braid β[1]. Then the Alexander polynomial $\Delta_\gamma(t)$ is

$$. \quad . \quad (-1)^m t^{-(6n+m)}(a(t)d(t)-b(t)c(t)) + t^{-3n}b(t)\sum_{K=1}^{m}(-t)^{-k}$$

$$- (-1)^m t^{-(3n+m)}a(t) - t^{-3n}d(t) + 1,$$

where $n=N_1$ and $m=N_2$. Now the proposition is clear, since neither $b(t)$ nor $a(t)d(t)-b(t)c(t)$ is a zero polynomial.(Note that neither $a(1)d(1)-b(1)c(1)$ nor $b(-1)$ is zero).

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