

Dedekind sums and special values of L-functions

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1. The purpose of this note is to supply the author's paper [2] with a remark and an example. Let  $K$  be an imaginary quadratic field with class number one (This assumption on the class number is used only in the argument after the formula (9).) and  $F$  be a quadratic extension of  $K$ . Consider a Grössencharacter  $\psi$  of  $F$  defined modulo an ideal  $\mathfrak{m}$  of  $F$  such that

$$(1) \quad \psi((\mu)) = \overline{N_{F/K}(\mu)}$$

for every integer  $\mu$  of  $F$  with  $\mu \equiv 1 \pmod{\mathfrak{m}}$  and let

$$L(s, \psi) = \sum_{(\mathfrak{a}, \mathfrak{m})=1} \psi(\mathfrak{a}) N(\mathfrak{a})^{-s} \quad (\operatorname{Re}(s) > 3/2)$$

where  $\mathfrak{a}$  runs over all integral ideals of  $F$  which is prime to  $\mathfrak{m}$ . In the following we shall give an expression of  $L(1, \psi)$  in terms of Dedekind sums of Sczech [4] and give an example of calculation of the value  $L(1, \psi)$ . If we have an elliptic curve  $E$  defined over  $F$  which has complex multiplication by an order of  $K$ , the complex conjugate  $\overline{\psi}_E$  of the Grössencharacter  $\psi_E$  of  $F$  associated with  $E$  is of the type mentioned above. Since the Hasse-Weil zeta function of  $E$  over  $F$  is equal to

$L(s, \psi_E) \cdot L(s, \overline{\psi_E})$ , our study on the value  $L(1, \psi)$  will have some interest in view of the well-known conjecture of Birch and Swinnerton-Dyer.

2. First let us review some of the results of [4] and [2]. Let  $O = O_K$  be the ring of integers of  $K$  and  $D$  the discriminant of  $K$ . For a non-negative integer  $k$  and two complex numbers  $p$  and  $q$ , put

$$G(s, k, p, q) = \sum_{\substack{m \in O \\ m+q \neq 0}} e\left(\frac{m\overline{p}}{\sqrt{D}}\right) (\overline{m+q})^k |m+q|^{-2s-k}$$

for  $\text{Re}(s) > 1$ , where  $e(z) = \exp(2\pi i(z + \overline{z}))$  and  $\text{Im}(\sqrt{D}) > 0$ .

This can be continued analytically to the whole  $s$ -plane.

Functions  $G_k(p) := G(k/2, k, 0, p)$  and  $G(p) := \frac{2\pi i}{\sqrt{D}} G(0, 2, 0, p)$  are

periodic with respect to the lattice  $O$  and can be expressed by means of the Weierstrass elliptic functions (cf. [4]). For two integers  $a$  and  $c$  in  $O$  with  $c \neq 0$  and two complex numbers  $p$  and  $q$ , define

$$D(a, c; p, q) = -\frac{1}{c} \sum_{r \in L/cL} G_1\left(\frac{a(r+p)}{c} + q\right) G_1\left(\frac{r+p}{c}\right).$$

Furthermore for every matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $SL(2, O)$ , let

$$\begin{aligned} \Phi(A)(p, q) = & -\left(\frac{a}{c}\right) G(p) - \left(\frac{d}{c}\right) G(p^*) - \frac{a}{c} G_0(p) G_2(q) \\ & - \frac{d}{c} G_0(p^*) G_2(q^*) - D(a, c; p, q) \end{aligned}$$

if  $c \neq 0$ , and

$$\Phi(A)(p, q) = - \overline{\left(\frac{b}{d}\right)} G(p) - \frac{b}{d} G_0(p)G_2(q)$$

if  $c = 0$ . Here  $(p^*, q^*) := (p, q)A = (ap+cq, bp+dq)$ . In particular, we have

$$(2) \quad \Phi \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} (p, q) = -\bar{d} G(q) - d G_0(q)G_2(p-dq) - G_1(p)G_1(q)$$

and

$$(3) \quad \Phi \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} (p, q) = 0$$

for every unit  $\xi$  contained in  $K$ . One of the main results of [4] is the formula

$$(4) \quad \Phi(AB)(p, q) = \Phi(A)(p, q) + \Phi(B)((p, q)A)$$

which holds for arbitrary matrices  $A$  and  $B$  in  $SL(2, O)$ .

Take  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O)$  with  $(a+d)^2 \neq 0, 1, 4$  and

$p, q \in K$  which satisfy

$$(5) \quad (p, q)A \equiv (p, q) \pmod{O^2}.$$

Then we can consider the L-series  $L(A, s; p, q)$  as follows.

There are two distinct complex numbers  $\alpha$  and  $\alpha'$  such that

$$A \begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha' \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}$$

and  $\varepsilon = c\alpha + d$  is a unit of the quadratic extension field  $F := K(\alpha)$  of  $K$  which satisfies  $N_{F/K}(\varepsilon) = 1$ . We assume  $|\varepsilon| > 1$  by changing  $\alpha$  and  $\alpha'$  if necessary. Let, for  $\text{Re}(s) > 3/2$ ,

$$(6) \quad L(A, s; p, q) = \sum_{\mu \in (M_{p, q} - \{0\}) / \langle \varepsilon \rangle} \overline{N_{F/K}(\mu)} N(\mu)^{-s}$$

where

$$M_{p, q} := (O+q)\alpha + O+q.$$

Note that  $\varepsilon M_{p,q} = M_{p,q}$  by the assumption (5). The L-series (6) has an analytic continuation to the whole  $s$ -plane and the following formula is proved in [2]:

$$(7) \quad (\alpha - \alpha') L(A, s; p, q) = \Phi(A).$$

3. Returning to the situation of section 1, take an complete set of representatives  $(\alpha_1, \dots, \alpha_h)$  of the absolute ideal class group of  $F$  such that  $(\alpha_j, \mathfrak{m}) = 1$  and a unit  $\varepsilon$  with  $|\varepsilon| > 1$ ,  $N_{F/K}(\varepsilon) = 1$  and  $\varepsilon \equiv 1 \pmod{\mathfrak{m}}$ . Then,

$$\begin{aligned} L(s, \psi) &= \sum_{j=1}^h \sum_{(\mu) \subset \alpha_j} \alpha_j^{-1} \psi(\mu \alpha_j) N(\mu \alpha_j)^{-s} \\ &= [O_F^\times : \langle \varepsilon \rangle]^{-1} \sum_{j=1}^h \psi(\alpha_j) N(\alpha_j)^{-s} \sum_{\mu \in \alpha_j^{-1} / \langle \varepsilon \rangle} \psi((\mu)) N(\mu)^{-s} \end{aligned}$$

where  $O_F^\times$  denotes the group of all units of  $F$ . Define, for every  $\mu \in F$  which is prime to  $\mathfrak{m}$ , the root of unity  $\chi(\mu)$  by

$$(8) \quad \psi((\mu)) = \chi(\mu) \overline{N_{F/K}(\mu)}.$$

If  $\mu$  is congruent multiplicatively to 1 modulo  $\mathfrak{m}$ , then  $\chi(\mu) = 1$ . It follows that

$$\begin{aligned} &\sum_{\mu \in \alpha_j^{-1} / \langle \varepsilon \rangle} \psi((\mu)) N(\mu)^{-s} \\ (9) \quad &= \sum_{\substack{v \in \alpha_j^{-1} / \mathfrak{m} \alpha_j^{-1} \\ (v, \mathfrak{m}) = 1}} \chi(v) \sum_{\mu \in (v + \mathfrak{m} \alpha_j^{-1}) / \langle \varepsilon \rangle} \overline{N_{F/K}(\mu)} N(\mu)^{-s}. \end{aligned}$$

Because the class number of  $K$  is one, there exist  $\beta_j$  and  $\gamma_j$  in  $F$  such that

$$\mathfrak{m} \alpha_j^{-1} = O_K \beta_j + O_K \gamma_j \quad (j = 1, \dots, h).$$

Put  $\alpha_j = \beta_j/\gamma_j$  and define the matrix  $A_j$  in  $SL(2, O_K)$  by

$$\varepsilon \begin{pmatrix} \alpha_j \\ 1 \end{pmatrix} = A_j \begin{pmatrix} \alpha_j \\ 1 \end{pmatrix}.$$

For each  $\nu$  in  $\alpha_j^{-1}$ , denote by  $p_j(\nu)$  and  $q_j(\nu)$  the elements of  $K$  such that

$$\nu = p_j(\nu)\beta_j + q_j(\nu)\gamma_j.$$

Then the summation over  $\mu$  in (9) is equal to

$$\overline{N_{F/K}(\gamma_j)} N(\gamma_j)^{-s} L(A_j, s; p_j(\nu), q_j(\nu))$$

and we have

$$(10) \quad L(1, \psi)$$

$$= [O_F^\times : \langle \varepsilon \rangle]^{-1} \sum_{j=1}^h \psi(\alpha_j) N(\alpha_j)^{-1} N_{F/K}(\gamma_j)^{-1} (\alpha_j - \alpha_j')^{-1}$$

$$\times \sum_{\substack{\nu \in \alpha_j^{-1}/m\alpha_j \\ (\nu, m)=1}} \chi(\nu) \Phi(A_j)(p_j(\nu), q_j(\nu)).$$

4. An example. Let  $K = \mathbb{Q}(i)$  and  $F = K(\rho) = \mathbb{Q}(\xi_{12})$  where  $i = \xi_4$ ,  $\rho = \xi_3$  with  $\xi_n = \exp(2\pi i/n)$ . The class number of  $F$  is one and the group  $O_F^\times$  is generated by  $\xi_{12}$  and  $\varepsilon_0 = 1 + \xi_{12}$  (cf. Kuroda [3]). Consider the elliptic curve

$$E : Y^2 = X^4 - \varepsilon_0^2$$

defined over  $F$  which has complex multiplication by  $O_K$ .

Denote by  $\psi = \psi_E$  the Grössencharacter of  $F$  associated with  $E$ .

Then, for every integral ideal  $\alpha$  of  $F$  prime to 2, we have

$$\psi(\alpha) = \left( \frac{\varepsilon_0}{\alpha} \right) \theta(N_{F/K}(\alpha))$$

from Davenport and Hasse [1] and Weil [5]. Here  $\left( \frac{\varepsilon_0}{\alpha} \right)$  means the quadratic residue symbol in  $F$  and, for every integral ideal  $b$  of  $K$  prime to  $2$ ,  $\theta(b)$  denotes the generator of  $b$  which is congruent to  $1$  modulo  $(1+i)^3$ . The Grössencharacter  $\bar{\psi}$  satisfies the condition (1) with  $\pi = 4O_F$ . Taking  $\varepsilon = -(2+\sqrt{3})^2 = \rho\varepsilon_0^4$ ,  $\alpha_1 = O_F$  and  $\beta_1 = 4\alpha$  ( $\alpha := \xi_6$ ),  $\gamma_1 = 4$  in the notation of section 3, we see, from (10),

$$L(1, \bar{\psi}) = \frac{-i}{2^8 \cdot 3\sqrt{3}} \sum_{\nu \in (O_F/(4))^{\times}} \overline{\chi(\nu)} \Phi(A) \left( \frac{p(\nu)}{4}, \frac{q(\nu)}{4} \right),$$

where

$$A := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1+i \end{pmatrix}^4,$$

the character  $\chi$  of  $(O_F/(4))^{\times}$  is determined by the relation

$$\psi((\mu)) = \chi(\mu) N_{F/K}(\mu) \quad (\mu \in O_F, (\mu, 2)=1),$$

and integers  $p(\nu)$  and  $q(\nu)$  of  $K$  are defined by

$$\nu = p(\nu)\alpha + q(\nu)$$

for every  $\nu \in O_F$ . Decomposing  $A$ , for example, as

$$A = - \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} C^2$$

with

$$C = i \begin{pmatrix} 1 & i \\ -i & 1+i \end{pmatrix}^2 = \begin{pmatrix} 2i & -2-i \\ 2+i & -2+i \end{pmatrix}$$

$$= - \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

the values  $\Phi(A) \left( \frac{p(\nu)}{4}, \frac{q(\nu)}{4} \right)$  can be evaluated by (2), (3), (4) and the 4-division values of  $G_1$ ,  $G_2$  and  $G$  which are listed in the table 1 below. The calculation shows

$$L(1, \bar{\psi}_E) = \frac{1}{2\sqrt{6}} \Omega^2,$$

where  $\Omega$  is the positive number such that  $O_K \Omega$  is the period lattice of the Weierstrass  $p$ -function satisfying  $p'^2 = 4p^3 - 4p$ .

Table 1

$\alpha$	$G_1(\alpha)\Omega^{-1}$	$G_2(\alpha)\Omega^{-2}$	$G(\alpha)\Omega^{-2}$
0	0	0	0
$\frac{1+i}{2}$	0	0	0
$\frac{1}{2}$	0	1	$\frac{1}{2}$
$\frac{1+i}{4}$	$\frac{-1+i}{2}$	-i	$-\frac{i}{4}$
$\frac{1}{4}$	$\frac{-1-\sqrt{2}}{2}$	$1+\sqrt{2}$	$\frac{1+2\sqrt{2}}{8}$
$\frac{1+2i}{4}$	$\frac{1-\sqrt{2}}{2}$	$1-\sqrt{2}$	$\frac{1-2\sqrt{2}}{8}$

## References

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