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Theta functions on the hyperbolic three space

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To introduce our topic, let's consider the Dedekind eta-function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \quad q = \exp(2\pi i \tau)$$

defined for all τ in the upper half plane H. This function was introduced by Euler who proved the beatiful identity

$$\eta(\tau) = q^{1/24} \sum_{n\in\mathbb{Z}} (-1)^n q^{n(3n-1)/2} \quad .$$

The series on the right is a theta series, and the identity is a special case of a more general formula known as the Jacobi tripel product identity. The behavior of the theta series under the modular group $SL_2\mathbb{Z}$ can be analyzed with the help of the Poisson summation formula. In this way, Hermite proved the following theorem,

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sqrt{\frac{c\tau+d}{i}}\, \eta(\tau) \quad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$$

with a 24-th root of unity $\varepsilon(A)$, $\varepsilon(A)^{24} = 1$. Hermite also established a formula for $\varepsilon(A)$ as a Gaussian sum and gave a closed expression for $\varepsilon(A)$ in terms of the quadratic residue symbol. If we think of $\eta(\tau)$ as an infinite product, then the formula of Hermite is highly nontrivial.

On the other hand, from the definition as infinite product it is obvious that $\eta(\tau) \neq 0$ does not vanish for all τ in H, but it is far from obvious that the theta series has the same property. Because H is simply connected and $\eta(\tau) \neq 0$, we know that $\log \eta(\tau)$ is well defined if we fix the branch of the logarithm by requiring $\log \eta(i)$ to be a real number.

The next contribution was made by Dedekind (and that's apparently the reason why η is named after him):

$$\log \eta\left(\frac{a\tau+b}{c\tau+d}\right) - \log \eta(\tau) = \frac{1}{2}\log\left(\frac{c\tau+d}{i}\right) + \frac{\pi i}{12}\,\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{if} \quad c>0, \quad \text{where}$$

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{c} - \frac{3}{c}\sum_{k(c)}{}' \cot\pi\left(\frac{ak}{c}\right)\cot\pi\left(\frac{k}{c}\right) \quad \text{for} \quad c \neq 0.$$

where the sum on the right is called a Dedekind sum. As a corolary, we get

$$\Phi(A_1) + \Phi(A_2) + \Phi(A_3) = \operatorname{sign}(c_1 c_2 c_3)$$
$$\text{if} \quad A_1A_2A_3 = 1 \, , \quad A_j = \begin{pmatrix} * & * \\ c_j & * \end{pmatrix} \in SL_2\mathbb{Z} \; .$$

In other words, if we forget the correctional term sign of $(c_1c_2c_3)$ on the right hand side, then Φ is an homomorphism of $SL_2\mathbb{Z}$ because $\Phi(A^{-1}) = -\Phi(A)$. But there are no rational valued homomorphisms of $SL_2\mathbb{Z}$ (which do not vanish identically), therefore the correctional term is necessary.

The real reason why Dedekind sums appear in connection with $\log \eta(\tau)$ is that $\log \eta(\tau)$ is the integral of an Eisenstein differential form,

$$4\pi i \log \eta(\tau) = \int_\tau E(z)\,dz, \quad E(z) = \sum_{m}\sum_{\substack{n \\ (m,n)\neq 0}} (mz+n)^{-2},$$

because Dedekind sums are the periods of Eisenstein series. The last equality follows easily from the infinite product for $\eta(\tau)$.

If we compare the formulas of Dedekind and Hermite, we get the remarkable relation

$$\varepsilon(A) = \exp\left(\frac{\pi i}{12}\,\Phi(A)\right) \quad \text{for} \quad A \in SL_2\mathbb{Z},$$

establishing a relation between the quadratic residue symbol and the Dedekind sum. Although this formula can be proved directly (using the lemma of Gauss in the theory of quadratic residues), in our view this is only the tip of an iceberg, the iceberg being Euler's representation of $\eta(\tau)$ as a theta series.

2. Now let $\Gamma = SL_2\mathcal{O}_K$ be the unimodular group of an imaginary quadratic field K. This group acts on the hyperbolic three space H^3 which can be conveniently defined as the quaternionic upper halfspace of quaternions $\tau = x + jy$ with $x \in \mathbb{C}$, $y > 0$, $j^2 = -1$, $jx = \bar{x}j$. Using this definition, the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ on H^3 can be written in the familiar way

$$\tau \longrightarrow (a\tau + b)\,(c\tau + d)^{-1}.$$

To define theta functions on H^3, we need an integral ideal L in $\mathcal{O}_K$ of norm $N = N(L) \equiv 1(2)$, and determinant $\Delta = \Delta(L) = N\sqrt{D}$ where D is the discriminant of K. For simplicity we assume that $D \equiv 1(8)$, but with small modifications everything is valid for arbitrary discriminants.

Then

$$\vartheta(\tau) = \sqrt{y} \sum_{\mu \in L+\frac{N}{2}} \exp\left(-\frac{2\pi|\mu|^2 y}{|\Delta|} + \pi i \operatorname{tr}\left(\frac{\mu^2 x + \mu N}{\Delta}\right)\right)$$

is an analogue of the Dedekind eta-function for imaginary quadratic fields. The reason for this claim is the number of properties $\vartheta(\tau)$ shares with $\eta(\tau)$. First of all, we have

<u>Theorem</u>: $\vartheta(\tau)$ does not vanish for all τ in H^3.

Because the hyperbolic three-space H^3 is simply connected, this implies that $\log \vartheta(\tau)$ is a well defined function on H^3 if we require $\log \vartheta(j)$ to be a real number. But the analogy goes further than that. As is well known, $\eta(\tau)$ is a cuspform, that means $\eta(\tau)$ vanishes exponentially if we approach $i\infty$ or a rational boundary point of the upper half plane H. The same is true for $\vartheta(\tau)$, it vanishes exponentially if τ appraoches $j\infty$ or a number in $K \subseteq \mathbb{C}+j\cdot 0$.

How to prove the non-vanishing of $\vartheta(\tau)$? To this end we embedd H^3 in the Siegel upper half space $\mathfrak{h}_2$ of genus 2.

Then $\vartheta(\tau)$ is the restriction of the Riemann theta function $\theta(T)$ (defined on $\mathfrak{h}_2$) to $H^3 \subseteq \mathfrak{h}_2$. Now the zero locus of $\theta(T)$ can be determined exactly (and in an elementary way). It consists of certain Humbert surfaces in the Siegel space. These surfaces are defined by some quadratic equations. All we have to do is to verify that H^3 does not meet these surfaces (in fact, it meets these surfaces exactly in the cusps).

Concerning the behavior of $\vartheta(\tau)$ under Γ, we have the following

<u>Theorem</u>: $\vartheta(A\tau) = \chi(A)\vartheta(\tau)$ with $\chi(A)^8 = 1$
for all $A \in x\Gamma x^{-1}$, where $x \in \begin{pmatrix} \bar{L} & \bar{L} \\ L & L \end{pmatrix}$, $\det(x) = N$.

In other words, up to an eighth root of unity χ, the theta function $\vartheta(\tau)$ is invariant under the action of Γ. This can be proved in the same way as Hermite got his result. With some care, the proof also gives an explicit formula for the eighth root of unity $\chi(A)$ in terms of the quadratic residue symbol $\left(\frac{2a}{cL/\bar{L}}\right)$ in K. If $cN \equiv 1(2)$, then

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \left(\frac{2a}{cL/\bar{L}}\right) \exp\left(\frac{\pi i}{4} \operatorname{tr}\left(\frac{(a+d)cN^2}{\Delta}\right)\right)$$

$$\text{where } \gamma = \begin{cases} 1, & cN \equiv \pm 1(4) \\ \pm i, & cN \equiv \pm\sqrt{D}(4). \end{cases}$$

The case $cN \not\equiv 1(2)$ can easily be reduced to the case $cN \equiv 1(2)$, so that this formula gives a complete determination of χ. The formula is useful not only for calculational, but also for theoretical purposes as we will see in a moment.

From the theorem we conclude that $\chi = \chi_{Lx}$ is an homomorphism $\Gamma \longrightarrow \mathbb{Z}/8\mathbb{Z}$ because

$$\begin{aligned} \vartheta(AB\tau) &= \chi(AB)\,\vartheta(\tau) \\ &= \chi(A)\,\vartheta(B\tau) = \chi(A)\,\chi(B)\,\vartheta(\tau), \end{aligned}$$

and therefore (because $\vartheta(\tau)$ does not vanish)

$$\chi(AB) = \chi(A)\,\chi(B).$$

This property suggests the introduction of the following function $\varphi : \Gamma \longrightarrow \mathbb{Z}$,

$$\varphi(A) := \frac{4}{\pi i}\left[\log\vartheta(xAx^{-1}\tau) - \log\vartheta(\tau)\right].$$

Then the same reasoning as before tells us that φ is an homomorphism,

$$\varphi(AB) = \varphi(A) + \varphi(B) \quad \text{for } A,B \in \Gamma.$$

It is easy to see that φ is nontrivial; for example if $L = \mathcal{O}_K$ and $x = \mathrm{id}$, then

$$\varphi\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = \mathrm{tr}\left(\frac{b}{\sqrt{D}}\right) \quad \text{for } b \in \mathcal{O}_K.$$

How many different homomorphisms do we get this way? To answer this question remember that $\varphi = \varphi(L,x)$ depends on the choice of L and x. Denote by $\langle \varphi(L,x) \rangle$ the vector space over $\mathbb{Q}$ generated by all possible L and x.

<u>Theorem</u>: dim $\langle \varphi(L,x) \rangle$ = h = class number of K,

dim $\langle \varphi(L,x), L \text{ fixed} \rangle$ = # of genera in K.

The proof of this theorem rests on the fact that we can calculate the value of φ on parabolic matrices by expanding $\vartheta(\tau)$ near a cusp. Knowing the value of φ on parabolic matrices, we form the hxh-determinant det $(\varphi(U))$ where φ and U run through a system of suitably chosen homomorphisms resp. parabolic matrices. The crucial point is that it is possible to prove the non-vanishing of this determinant in an elementary way. This is remarkable because in a similar but earlier situation [3], we had to deduce this property from the non-vanishing of certain L-series at s=1.

3. To analyze φ further, we have to talk about the first cohomology group of Γ,

$$H^1(\Gamma, \mathbb{Q}) = \mathrm{Hom}\,(\Gamma^{ab}, \mathbb{Q})$$

which is nothing else than the $\mathbb{Q}$-vector space of all homomorphisms of Γ. This is a finite dimensional vector space which comes with a canomical decomposition

$$H^1(\Gamma) = H^1_c(\Gamma) \oplus H^1_e(\Gamma).$$

The subspace H^1_c (called the cuspidal part of H^1) can be defined as the subspace of all homomorphisms which vanish on parabolic matrices, whereas the so called Eisenstein-part H^1_e is generated by homomorphisms explicitly given by Dedekind sums [3], compare H. Ito's talk at this conference (without recourse to explicit formulas H^1_e can be defined as the largest subspace in the complement of H^1_c which is invariant under the action of Hecke). This subspace is well-understood; around 14 years ago, Serre proved that

<u>Theorem</u>: $\dim H^1_e(\Gamma) = h$

by using geometric properties of the so-called Borel-Serre compactification of the quotient H^3/Γ. If we decompose the theta homomorphisms φ as $\varphi = \varphi_c + \varphi_e$, where $\varphi_c \in H^1_c$ is the cuspidal, resp. $\varphi_e \in H^1_e$ the Eisenstein part of φ, then our results give another proof of Serre's theorem (by displaying h linear independent homomorphisms φ_e) because it is easy to see that $\dim H^1_e \leq h$.

Concerning H^1_c not much is known: N. Kramer [1] and J. Rohlfs [2] proved (by refining ideas of Serre and Harder) that

<u>Theorem</u>: $\frac{|D|}{24} - h < \dim H^1_c < \frac{1}{4}|D|^{3/2}$.

Since h grows like $|D|^{1/2}$, this means that H^1_e is a rather tiny subspace of H^1 for large $|D|$, but surprisingly, no cuspidal homomorphism of Γ is known so far! Given this situation, the following three conjectures about φ_c could be of some interest.

For simplicity, we assume for the rest of the paper that $L = \mathcal{O}_K$ and $x = id$ is the identity matrix.

Recall that $\varphi = \varphi_c + \varphi_e$ is integral valued, but this is not necessarily true for φ_c and φ_e which are in general rational valued homomorphisms. The following table shows the first nontrivial denominators of φ_c resp. φ_e,

D	-55	-79	-87	-95	-103	-111	-119
	11	9	11	19	59	11	7

To find these numbers we have to determine a set of generators for Γ, and calculate φ_e on these generators. Determining generators for Γ is a time consuming job, but for φ_e there is a closed formula in terms of Dedekind sums (generalizing Dedekinds formula mentioned in the introduction) which is easy to implement on a computer.

Conjecture 1: The denominator of φ_c is odd.

This denominator is a divisor of a special value of a Hecke L-function. To explain that, let ψ be a Größencharacter of K satisfying

$$\psi((\alpha)) = \bar{\alpha}^2 .$$

Then $\psi \circ N_{H/K}$ is a Größencharacter of H, the Hilbert class-field of K. Let $L(\psi \circ N_{H/K}, s)$ be the corresponding L-function (defined over H), and set

$$\Omega = \left| \frac{4\pi}{(9D)^{1/4}} \frac{\eta(2\tau)}{\eta(\tau)} \right| , \tau = \frac{1+\sqrt{D}}{2} .$$

Using non-trivial results from complex multiplication, it can be shown that the number

$$X := 8\ (2\Omega)^{-2h}\ L\ (\psi \circ N_{H/K}, 2)$$

is an algebraic integer. On the other hand, from the explicit formula for φ_e it follows that the denominator of φ_c divides X . This gives

Theorem: If X is rel. prime to 2, then the conjecture 1 is true.

Assuming the conjecture 1 is true, we can state our second conjecture.

Conjecture 2: $\varphi_c \equiv 0\ (8)$.

This means that the eighth root of unity χ defined by the action of Γ on $\vartheta(\tau)$ can be written as in the classical case

$$\chi = \exp\left(\frac{2\pi i}{8}\varphi_e\right) .$$

Clearly, the second conjecture implies the first one. Both conjectures are supported by the following result.

Theorem: The behavior of φ, φ_e under the Hecke algebra is the same module 8.

For example, if the Hecke operator T_p corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, where p is a prime number in K, then

$$T_p(\varphi_e) = (p + \bar{p})\varphi_e,$$
$$T_p(\varphi) \equiv (p + \bar{p})\varphi\ (8).$$

The last congruence follows from the explicit formula for χ stated in section 2. If we would have a multiplicity one result modulo 8, then the conjectures 1 and 2 would follow from this theorem. But I am not sure whether such a result is known.

Finally, to state the third conjecture, let N be the normalizer of Γ in SL_2K. By a result of Hurwitz, the quotient N/Γ is isomorphic to the genera group of K. An element $x \in N$ acts on $H^1(\Gamma)$ by $\varphi \longrightarrow \varphi_x$,

$$\varphi_x(A) \doteq \varphi(xAx^{-1}) \quad \text{for} \quad \varphi \in H^1(\Gamma),\ A \in \Gamma.$$

Notice that this action is trivial for $x \in \Gamma$, so we have an action of the genera group on $H^1(\Gamma)$.

<u>Conjecture 3</u>: φ_c is an eigenvector of x for all $x \in N$.

In other words, it is $(\varphi_c)_x = \pm\varphi_c$. This property, if true, would indicate that φ_c is not just 8 times a "random" element in $H^1_c(\Gamma)$ as a pessimistic interpretation of the second conjecture could suggest, but an interesting cohomology class in $H^1_c(\Gamma)$ worth further studys.

For every fixed D, the conjectures can be verified by a finite calculation. This has been done for all $D \equiv 1(8)$, $|D| < 100$.

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