

On  $p$ -adic Galois representations  
attached to the elliptic curves over  $\mathbb{F}_p[[t]]$

Yasushi Fujiwara  
(University of Tokyo)

藤原 靖  
(東大. 理 )

§1. Introduction

Let  $R = \mathbb{F}_p[[t]]$  and  $K = \mathbb{F}_p((t))$ . Let  $E$  be an elliptic curve over  $R$  whose Hasse invariant is a uniformizer of  $R$ . Then the special fiber  $E_0 = E \otimes_{\mathbb{F}_p} \mathbb{F}_p$  is supersingular and the generic fiber  $E \otimes_{\mathbb{F}_p} K$  is ordinary.

Let  $\bar{K}$  denote the algebraic closure of  $K$ . As  $E \otimes_{\mathbb{F}_p} K$  is ordinary,  $p$ -power torsion of  $E(\bar{K})$  is isomorphic to the group  $\mathbb{Q}_p/\mathbb{Z}_p$ . Considering the Galois action on  $p$ -power torsion of  $E(\bar{K})$ , we get a Galois representation  $g : \text{Gal}(K^{\text{sep}}/K) \rightarrow \mathbb{Z}_p^\times$ . In this report, we shall investigate the continuous homomorphism  $\rho : K^\times \rightarrow \mathbb{Z}_p^\times$  which is associated with  $g$  via reciprocity map of local class field theory. For simplicity, we suppose that  $p \geq 5$ . By the method of Lubin-Tate [7], we first show the following

**Proposition 1.** Let the situation be as above. Then there exists a canonical local  $\mathbb{F}_p$ -algebra automorphism  $u$  of  $R$  which preserves the continuous homomorphism  $\rho$ , i.e.  $\rho(f(u(t))) = \rho(f(t))$  for any  $f \in K^\times$ .

Let  $\rho^0$  denote the restriction of  $\rho$  to  $R^\times$ . By the above proposition,  $\text{Ker } \rho^0$  contains the group  $\left\{ \frac{f(u(t))}{f(t)} \mid f \in K^\times \right\}$ . Our

main result is

Theorem. The kernel of the homomorphism  $\rho^0 : R^\times \rightarrow \mathbb{Z}_p^\times$  is the closure of the group  $\left\{ \frac{f(u(t))}{f(t)} \mid f \in K^\times \right\}$ .

We shall give a sketch of the proof of these results. For the full proof, cf. [2]. (In [2], we deal with formal groups over  $R$  which are "generic". It is easily checked that the formal group of  $E$  is generic).

We here explain the motivation of this report. Let  $k = \mathbb{F}_p(\lambda)$  with  $p \neq 2$ , where  $\lambda$  is an indeterminate. Let  $E_\lambda/k$  be an elliptic curve defined by the equation  $y^2 = x(x-1)(x-\lambda)$ . As  $E_\lambda/k$  is ordinary,  $p$ -power torsion of  $E_\lambda(\bar{k})$  is isomorphic to  $\mathbb{Q}_p/\mathbb{Z}_p$ . By considering the Galois action on  $p$ -power torsion of  $E_\lambda(\bar{k})$ , we obtain a Galois representation

$$r : \text{Gal}(k^{\text{sep}}/k) \rightarrow \mathbb{Z}_p^\times.$$

For any place  $w$  of  $k$ , denote by  $r_w$  the restriction of  $r$  to the decomposition group at  $w$ .

If the reduction of  $E_\lambda$  at  $w$  is not supersingular, we can completely determine the representation  $r_w$ . In the case that  $E_\lambda$  has ordinary reduction at  $w$ , the representation  $r_w$  is unramified and the value of the Frobenius element is the reciprocal of the unit root of the zeta function of the special fiber at  $w$  (Theorem 4.2.2 of [6]). In the case that  $E_\lambda$  has bad reduction at  $w$  (i.e.  $w = (\lambda)$ ,  $(\lambda-1)$ , or  $(\lambda^{-1})$ ), the situation is as follows (cf. Lemma 4.2.1 of [6]): At the place  $w = (\lambda)$ ,  $E_\lambda$  has split or non-split multiplicative reduction according as  $p \equiv 1 \pmod{4}$  or  $p \equiv 3 \pmod{4}$ . Thus  $r_w$  is trivial if  $p \equiv 1 \pmod{4}$  and is quadratic unramified if  $p \equiv 3 \pmod{4}$ . At the place  $w = (\lambda-1)$ ,  $E_\lambda$  has split multiplicative reduction, so the representation  $r_w$  is

trivial. At the place  $w = (\lambda^{-1})$ ,  $E_\lambda$  has additive reduction. By an easy calculation, we see that  $E_\lambda$  has split multiplicative reduction over  $\mathbb{F}_p(((-\lambda^{-1})^{\frac{1}{2}}))$  and that  $r_w$  is a quadratic character corresponding to the extension  $\mathbb{F}_p(((-\lambda^{-1})^{\frac{1}{2}}))/\mathbb{F}_p((\lambda^{-1}))$ .

As for the case that  $E_\lambda$  has supersingular reduction at  $w$ , although some important results are obtained ( e.g. Gross [3], Igusa [5] ), the situation does not seem so clear. Our point of view is to consider the continuous homomorphism  $\rho_w : k_w^\times \rightarrow \mathbb{Z}_p^\times$  ( $k_w$  denotes the completion of  $k$  at  $w$ ) which is associated with  $r_w$  via reciprocity map of local class field theory. If  $E_\lambda$  has supersingular reduction at the place  $w$  of degree 1, our results apply to the homomorphism  $\rho_w$  ( cf. Igusa [4] ). The author hopes that this report might shed some light on the global representation  $r : \text{Gal}(k^{\text{sep}}/k) \rightarrow \mathbb{Z}_p^\times$ .

## §2. Some invariance of $\rho$

In this section, we do not impose any condition on a prime  $p$  and shall prove Proposition 1 in a little generalized form. For the connection with the formulation given in the previous section, we refer to the beginning of §3.

We shall use the terminologies and results of formal groups (cf. Fröhlich [1]). For any elliptic curve  $A$ , we denote by  $\hat{A}$  its formal group.

To obtain local  $\mathbb{F}_p$ -algebra automorphisms of  $R$  which preserve  $\rho$ , we introduce the functor  $M$  whose affine algebra is non-canonically isomorphic to  $R$ . Let  $\mathcal{C}$  denote the category of complete noetherian local  $\mathbb{F}_p$ -algebras with residue field  $\mathbb{F}_p$ . Let  $S$  be any object of  $\mathcal{C}$ . The pair  $(G, \iota)$  consisting of a formal group  $G$  over  $S$  and an isomorphism  $\iota : G \otimes_{\mathbb{F}_p} \hat{E}_0$  will be

called a rigidified lifting of  $\hat{E}_0$  to  $S$ . Define the equivalence relation  $\sim$  of rigidified liftings of  $\hat{E}_0$  to  $S$  by setting  $(G, \iota) \sim (G', \iota')$  if and only if there exists an  $S$ -isomorphism  $\varepsilon : G \cong G'$  such that  $\iota = \iota' \circ \varepsilon_0$ , where  $\varepsilon_0$  denotes the special fiber of  $\varepsilon$ . We define the formal moduli functor  $M$  of  $\hat{E}_0$  as a covariant functor that associates to any object  $S$  of  $\mathcal{C}$  the set of  $\sim$ -equivalence classes of rigidified liftings of  $\hat{E}_0$  to  $S$ .

Lubin and Tate [7] has shown that the functor  $M$  is pro-representable and that the affine algebra  $P$  of  $M$  is non-canonically isomorphic to  $R = \mathbb{F}_p[[t]]$ . In other words, they constructed a formal group  $\mathcal{F}/P$  such that  $\mathcal{F} \otimes_{\mathbb{F}_p} = \hat{E}_0$  with the following property: For any rigidified lifting  $(G, \iota)$  of  $\hat{E}_0$  to  $S$ , there corresponds a unique local  $\mathbb{F}_p$ -algebra homomorphism  $v : P \rightarrow S$  which makes  $(G, \iota)$  and  $(\mathcal{F}_{P, v} \otimes S, \text{identity})$  are  $\sim$ -isomorphic ( We denote by  $\mathcal{F}_{P, v} \otimes S$  the formal group obtained from  $\mathcal{F}$  by making a scalar extension  $v$  ). By the assumption that the Hasse invariant of  $E$  is a uniformizer of  $R$ , the element  $(\hat{E}, \text{identity})$  corresponds to an isomorphism  $\lambda : P \cong R$  ( cf. Proposition 1 of [2] ).

Let  $\varphi$  be an automorphism of  $\hat{E}_0$  over  $\mathbb{F}_p$ . Then  $\varphi$  naturally acts on the functor  $M$  as follows :

$$(G, \iota) \rightarrow (G, \varphi \circ \iota).$$

Denote by  $\bar{\varphi}$  the automorphism of  $M$  defined above. Then we easily have

$$\text{i) } \overline{\varphi \circ \psi} = \bar{\varphi} \circ \bar{\psi} \quad \varphi, \psi \in \text{Aut}_{\mathbb{F}_p} \hat{E}_0$$

$$\text{ii) } \bar{\varphi} = \text{identity} \iff \varphi \in \mathbb{Z}_p^\times.$$

Let  $\bar{\varphi}^*$  be the local  $\mathbb{F}_p$ -algebra automorphism of  $P$  which induces the automorphism  $\bar{\varphi}$  on  $M$ . As the ring  $\text{End}_{\mathbb{F}_p} \hat{E}_0$  is

generated over  $\mathbb{Z}_p$  by the Frobenius endomorphism  $F$ , the group

$\text{Aut}_{\mathbb{F}_p} \hat{E}_0$  is commutative. Thus we have

$$\overline{\varphi \circ \psi}^* = \overline{\psi}^* \circ \overline{\varphi}^* = \overline{\varphi}^* \circ \overline{\psi}^*.$$

For any  $\varphi \in \text{Aut}_{\mathbb{F}_p} \hat{E}_0$ , we denote by  $\mathcal{F}_\varphi/P$  the formal group obtained from  $\mathcal{F}/P$  by making a scalar extension  $\overline{\varphi}^*$ .

Lemma 1 (Lubin-Tate [7]). There exists a  $P$ -isomorphism

$$\varphi^\dagger : \mathcal{F} \rightarrow \mathcal{F}_\varphi \text{ such that } \varphi^\dagger \otimes \mathbb{F}_p = \varphi.$$

Proof. By the definition of  $\overline{\varphi}^*$ , the pairs  $(\mathcal{F}, \varphi)$  and  $(\mathcal{F}_\varphi, \text{identity})$  are  $\sim$ -equivalent. Thus there exists an isomorphism  $\varphi^\dagger : \mathcal{F} \rightarrow \mathcal{F}_\varphi$  such that  $\varphi = (\text{identity of } \hat{E}_0) \circ (\varphi^\dagger \otimes \mathbb{F}_p)$ . Q.E.D.

For  $\varphi \in \text{Aut}_{\mathbb{F}_p} \hat{E}_0$ , define a local  $\mathbb{F}_p$ -algebra automorphism  $u_\varphi$  of  $R$  by setting  $u_\varphi = \lambda \cdot \overline{\varphi}^* \cdot \lambda^{-1}$ . Denote by  $E_\varphi/R$  the elliptic curve obtained from  $E/R$  by making a scalar extension  $u_\varphi$ . By Lemma 1 we get an  $R$ -isomorphism  $\tilde{\varphi} : \hat{E} \rightarrow \hat{E}_\varphi$  such that  $\tilde{\varphi} \otimes \mathbb{F}_p = \varphi$ . Thus the following proposition is almost obvious.

Proposition 1. For any  $\varphi \in \text{Aut}_{\mathbb{F}_p} \hat{E}_0$  and  $f \in K^\times$ , we have

$$\rho(f(u_\varphi(t))) = \rho(f(t)).$$

Proof. Denote by  $\rho_\varphi : K^\times \rightarrow \mathbb{Z}_p^\times$  the continuous homomorphism attached to  $E_\varphi$ . Then we have  $\rho_\varphi(f(u_\varphi(t))) = \rho(f(t))$ . On the other hand, since  $\hat{E}$  and  $\hat{E}_\varphi$  are isomorphic over  $R$ , we easily conclude that  $\rho = \rho_\varphi$ . Thus  $\rho(f(u_\varphi(t))) = \rho(f(t))$ . Q.E.D.

## §3. Main result

In this section, we assume that  $p \neq 2, 3$ . Then the Frobenius endomorphism  $F$  of  $\hat{E}_0$  satisfies the relation  $F^2 + p = 0$  in  $\text{End}_{\mathbb{F}_p} \hat{E}_0$  and the group  $\text{Aut}_{\mathbb{F}_p} \hat{E}_0 / \mathbb{Z}_p^\times$  is a free  $\mathbb{Z}_p$ -module generated by  $1 - F$ . For  $\varphi = 1 - F$ , we shall abbreviate  $u_\varphi$  by  $u$ .

Let  $U^1$  denote the group of principal units  $1 + tR$  of  $R$ . To prove our theorem, we shall introduce a  $\mathbb{Z}_p[[T]]$ -module structure on  $U^1$ . Denote  $u \cdots u$  ( $p^n$ -times) by  $u^{[n]}$ . For any nonnegative integer  $v$ , define  $R_v = R/t^{v+1}R$  and  $E_v = E \otimes_{\mathbb{Z}_p} R_v$ . Via the canonical projection, we regard  $\text{End}_{R_v} \hat{E}_v$  as a subring of  $O = \text{End}_{\mathbb{F}_p} \hat{E}_0$ . The next lemma enables us to reduce the study of an automorphism  $u$  of  $R$  to that of endomorphism of  $\hat{E}_n$ .

Lemma 2. Let  $n$  and  $v$  be nonnegative integers. Then the following conditions are equivalent.

$$\text{i) } u^{[n]}(t) \equiv t \pmod{t^{v+1}}$$

$$\text{ii) } \text{End}_{R_v} \hat{E}_v \supset \mathbb{Z}_p + p^n O.$$

Cf. Lemma 1 of [2].

Let  $i_n = \text{ord}_R(u^{[n]}(t) - t)$ , where  $\text{ord}_R$  denotes the normalized additive valuation of  $R$ . By using Lemma 2, we can determine  $i_0$  and  $i_1$ .

Lemma 3. i)  $i_0 = 2$  and  $i_1 = 2p + 2$ .

$$\text{ii) } i_n \geq (2p + 2)p^{n-1} \text{ for any } n \geq 1.$$

Cf. Proposition 3, Lemma 2 and Proposition 4 of [2].

We regard  $U^1$  as a  $Z_p[[T]]$ -module by setting  $(Tf)(t) = \frac{f(u(t))}{f(t)}$  for  $f \in U^1$ . Since  $((T+1)^{p^n} - 1)f(t) = \frac{f(u^{[n]}(t))}{f(t)}$ , we have

$$((T+1)^{p^n} - 1)U^1 \subset 1 + t^{i_n}R.$$

By ii) of Lemma 3 and the fact that  $U^1$  is complete,  $U^1$  becomes a module over  $\varprojlim Z_p[[T]]/((T+1)^{p^n} - 1) = Z_p[[T]]$ . Let  $A = Z_p[[T]]$ . By i) of Lemma 3, we can determine the  $A$ -module structure of  $U^1$ .

Proposition 2.  $U^1$  is a free  $A$ -module of rank 2 generated by any elements  $\alpha$  and  $\beta$  of  $U^1$  such that  $\text{ord}_R(\alpha - 1) = 1$  and  $\text{ord}_R(\beta - 1) = 1 + p$ .

Cf. Proposition 5 of [2].

From this, we can now prove our theorem. Denote by  $V$  the closure of the group  $\left\{ \frac{f(u(t))}{f(t)} \mid f \in K^\times \right\}$ . By Igusa [5], we have  $\rho^0(R^\times) = Z_p^\times$ , so  $\text{Ker } \rho^0 \subset U^1$  and  $\rho^0(U^1) = 1 + pZ_p$ . Since  $1 + pZ_p$  is a free  $Z_p$ -module of rank 1, it suffices to show that the  $Z_p$ -module  $U^1/V$  is also free of rank 1.

As  $f \in K^\times$  is written in the form  $f = at^{n_x}h$  ( $a \in F_p^\times$ ,  $n \in Z$ ,  $h \in U^1$ ), we have

$$\left\{ \frac{f(u(t))}{f(t)} \mid f \in K^\times \right\} = \left\{ \left( \frac{u(t)}{t} \right)^z \times \frac{h(u(t))}{h(t)} \mid z \in Z, h \in U^1 \right\}.$$

Thus we get

$$V = \left\{ \left( \frac{u(t)}{t} \right)^z \times \frac{h(u(t))}{h(t)} \mid z \in Z_p, h \in U^1 \right\}.$$

Take any element  $\beta$  of  $U^1$  such that  $\text{ord}_R(\beta - 1) = 1 + p$ . As  $\text{ord}_R\left(\frac{u(t)}{t} - 1\right) = 1$  by i) of Lemma 3,  $U^1$  is a free  $A$ -module generated by  $\alpha = \frac{u(t)}{t}$  and  $\beta$ . Then  $V = Z_p\alpha + TU^1 = A\alpha + A(T\beta)$ ,

hence we have  $U^1/V = A\beta/A(T\beta) \cong A/TA = \mathbb{Z}_p$ . Q.E.D.

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