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On sufficient conditions for the Leopoldt conjecture

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Introductions.

Let p be a prime number. Let k be an algebaic number fields of finite degree over the rational number field $\mathbf{Q}$. Let S be a non-empty finite set of prime divisors of k, and let $U_S = \prod_{\mathfrak{p}\in S} U_{\mathfrak{p}}$, where $U_{\mathfrak{p}}$ is the unit group of the completion of k at $\mathfrak{p}$. Let E be the global unit group of k and $\iota : E \longrightarrow U_S$ be the canonoical injection. We denote by E_S the topological closure of $\iota(E)$ in U_S. Let P be the set of all prime divisors of k dividing p. Let r be the $\mathbf{Z}$-free rank of E and r_p be the $\mathbf{Z}_p$-free rank of E_P. Then it is equivalent to $r = r_p$ that the Leopoldt conjecture holds for (k , p). Put $\delta_p = \delta(k , p) = r - r_p$. Then $\delta_p \geq 0$ and it is the defect value for the Leopoldt conjecture. We study the conditions for $\delta_p = 0$ in this paper. Now we explain notation which uses in this paper and state main results of this paper. Let ζ_p be a primitive p-th root of unity and let $K = k(\zeta_p)$. We denote by $\overline{S}$ the set of all prime divisors of K which divide the prime divisors of k contained in S. Let C be the divisor class group of K and D_S be its subgroup generated by all divisor classes containing the prime divisors in $\bar{S}$. Put $C_S = C/D_S \cdot C^p$. Put $G = \mathrm{Gal}(K/k)$. Let $\omega : G \longrightarrow \mathbf{Z}_p^{\times}$ be a homomorphism to the multiplicative group of the ring of p-adic integers $\mathbf{Z}_p$ defined by $\xi_p^{\tau} = \xi_p^{\omega(\tau)}$ for any $\tau \in G$. Put $\varepsilon_{\omega} = (\Sigma_{\tau \in G}\ \omega(\tau)\tau^{-1}) / |G|$, which is an idempotent element of group ring $\mathbf{Z}_p[G]$ associated with ω. Since C_S is a $\mathbf{Z}_p[G]$-module, we set $C_{S,\omega} = \varepsilon_{\omega}(C_S)$. For an abelian group A, we denote by $t_p(A)$ the maximal p-group contained in the maximal torsion subgroup of A. We prove in Theorem 1 that $\delta_p = 0$ is equivalent to existence of a finite set S of prime divisors of k satisfying the following three conditions. (1) $S \supset P$. (2) $C_{S,\omega} \simeq \{1\}$. (3) $p\text{-rank}(t_p(E_S)) = p\text{-rank}(t_p(E))$. Put $e_P = \#\{\ \mathfrak{p} \mid \mathfrak{p} \in S \text{ and } \xi_p \in k_{\mathfrak{p}}\}$. We estimate δ_p in Theorem 2 and see that $\delta_p \leq p\text{-rank}(C_{P,\omega}) + e_P - p\text{-rank}(t_p(E))$ holds. We prove in Proposition 1 that the condition (3) holds if $e_P = p\text{-rank}(t_p(E))$. If the condition (2) also holds for P, the Leopoldet conjecture is true. This was known in Gras [2] , Gillard [1] and Sands [8]. Miki [6] showed that the following two conditions (4) and (5) are equivalent. (4) the Galois group of the maximal p-ramified p-abelian extension of k is torsion free and the Leopoldt conjecture holds for (k , p). (5) the condition (2) holds for P and $e_P = p\text{-rank}(t_p(E))$ holds. We assume in Theorem 4 that k is totally imaginary if $p = 2$ and prove that the Galois group over k of the maximal p-extension of k which is unramified outside S is a pro-p-free group if and only if the condition (2) holds for S and e_S = p-rank$(t_p(E))$.

1. the ncessary and sufficient conditiuon of $\delta(k : p) = 0$.

Let N be the set of the natural numbers. Let i, j be elements of N. Let A be an abelian group. Put $A_i = A/A^{p^i}$ for $i \in$ N. For $i \geq j$, we denote by $\varphi_{i,j}$ a homomorphism from A_i to A_j defined by $\varphi_{i,j}(a \cdot A^{p^i}) = a \cdot A^{p^j}$ for $a \in A$. Then $(A_i , \varphi_{i,j})$ is a projective system of abelian groups. We denote by $\overline{A}$ its projective limit, which is a $\mathbf{Z}_p$-module. Put $r(A) = \dim_{\mathbf{Q}_p}(\overline{A} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$, where we denote by $\mathbf{Q}_p$ the field of p-adic numbers. We observe that $r = r(E)$ and $r_p = r(E_P)$. We abbreviate $r(E_S)$ to r_S.

Theorem 1. $\delta(k : p) = 0$ *is equivalent to existence of the finite set* S *of prime divisors of* k *satisfying the following three conditions.* (1) $S \supset P$. (2) $C_{S,\omega} \simeq \{1\}$. (3) p-*rank*$(t_p(E_S))$ = p-*rank*$(t_p(E))$.

To prove this theorem, we need some lemmas.

Lemma 1. $r_P = r_S$ *if* $P \subset S$.

Proof. Let $\pi : U_S \longrightarrow U_P$ be the canonical projection. Let S' be the subset of S consisting of all elements which are not contained in P. The kernel of π is $U_{S'}$. Let V_1 resp. V_2 be any open subgroup of U_P resp. $U_{S'}$. We have $E \cdot V_1 \cdot V_2 = E_S \cdot V_1 \cdot V_2$ since $V_1 \cdot V_2$ is an open subgroup of U_S. Hence $\pi(E) \cdot V_1 = \pi(E_S) \cdot V_1$. Therefore the topological closure in U_P of $\pi(E)$ is equal to that of $\pi(E_S)$. Since E_S is compact, $\pi(E_S)$ is also compact. Thus we have $E_P = \pi(E_S)$. Put $V(\mathfrak{p}) = t_p(k_{\mathfrak{p}}^{\times})$ for $\mathfrak{p} \in S$. Put $p^m = \max\{ |V(\mathfrak{p})| \mid \mathfrak{p} \in S' \}$. Then we have

$$(U_{S'} \cap E_S) \cdot (E_S)^{p^n} / (E_S)^{p^n} \simeq t_p(U_{S'} \cap E_S).$$

Hence we have an exact sequence

$$1 \longrightarrow t_p(U_{S'} \cap E_S) \longrightarrow E_S/E_S^{p^n} \xrightarrow{\pi_n} E_P/E_P^{p^n} \longrightarrow 1,$$

where π_n is a homomorphism defined by $\pi_n(\varepsilon \cdot E_S^{p^n}) = \pi(\varepsilon) \cdot E_P^{p^n}$ for $\varepsilon \in E_S$. We take the projective limit of this exact sequence. Then we see $r_P = r_S$ by the definition. Q. E. D.

Put $A_S^{(2)} = (E \cap (E_S)^p)/E^p$.

Lemma 2. *Suppouse* $S \supset P$. *Then we have*

$$\delta_p = p\text{-}rank(t_p(E_S)) - p\text{-}rank(t_p(E)) + p\text{-}rank(A_S^{(2)}).$$

Proof. Let X be a set of representatives of all left coset in E with respect to E^p. Then X is a finite set.

Put $F = \bigcup_{\varepsilon \in X} \varepsilon \cdot E_S^p$. Then F is compact because it is a finite union of compact sets $\varepsilon \cdot E_S^p$. Hence we have $F = E_S$ because $E_S \supset F \supset E$. Let $f : E/E^p \longrightarrow E_S/E_S^p$ be a homomorphism defined by $f(\varepsilon \cdot E) = \varepsilon \cdot E_S^p$ for $\varepsilon \in X$. Then f is surjective and $\ker(f) = \{ \varepsilon \cdot E^p \mid \varepsilon \in E_S^p\} = (E \cap E_S^p)/E^p$. Therefore we have $E/(E \cap E_S^p) \simeq E_S/E_S^p$. Since E/E^p is an elementary p-abelian group, we have $E/E^p \simeq E/(E \cap E_S^p) \oplus A_S^{(2)}$. Hence $E/E^p \simeq E_S/E_S^p \oplus A_S^{(2)}$. Since $r = p\text{-rank}(E/E^p) - p\text{-rank}(t_p(E))$ and $r_S = p\text{-rank}(E_p/E_p^p) - p\text{-rank}(t_p(E_S))$, We have

$$\delta_p = r - r_S = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) + p\text{-rank}(A_S^{(2)}).$$

Q. E. D.

We define $U_k^S(p)$ by

$U_k^S(p) = \{ \alpha \in k^\times \mid$ There exists an ideal $\mathfrak{a}$ of k such that $\mathfrak{a}^p = (\alpha)$, and $\alpha \in (k_\mathfrak{p}^\times)^p$ for any $\mathfrak{p}.\}$,

where we denote by $k_\mathfrak{p}$ the completion of k at $\mathfrak{p}$. We denote by $\overline{S}$ the set $\{ \mathfrak{P} \mid \mathfrak{P}$ is a prime divisor of K which divides a prime divisor of k contained in $S.\}$. Let $U_K^{\overline{S}}(p)$ be a set of elements of $K^\times$ such that there exists an ideal $\mathfrak{A}$ of K satisfying $\mathfrak{A}^p = (\alpha)$, and $\alpha \in (K_\mathfrak{P}^\times)^p$ for any $\mathfrak{P} \in \overline{S}$, where we denote by $K_\mathfrak{P}$ the completion of K at $\mathfrak{P}$.

Lemma 3. $U_k^S(p) \simeq N_{K/k}(U_K^{\overline{S}}(p)) \cdot (K^\times)^p/(K^\times)^p$,

where we denote by $N_{K/k}$ *the norm map from* K *to* k.

Proof. Let $\alpha \in U_K^{\overline{S}}(p)$ and $\mathfrak{A}$ be an ideal of K such that $\mathfrak{A}^p = (\alpha)$. We denote by $J_{\mathfrak{p}}$ the semi-local product $\prod_{\mathfrak{P}|\mathfrak{p}} K_{\mathfrak{P}}^{\times}$. Since $N_{K/k}(\mathfrak{A})^p = N_{K/k}((\alpha))$ and $N_{K/k}(\alpha) \in N_{K/k}(J_{\mathfrak{p}}^p) \subset (k_{\mathfrak{p}}^{\times})^p$, we have $N_{K/k}(U_K^{\overline{S}}(p)) \subset U_k^S(p)$. Since $(U_k^S(p))^{[K:k]} \subset N_{K/k}(U_K^{\overline{S}}(p)) \subset U_k^S(p)$ and $(p, [K:k]) = 1$, we have $U_k^S(p)\cdot(K^{\times})^p = N_{K/k}(U_K^{\overline{S}}(p))\cdot(K^{\times})^p$. Let $j : k^{\times}/(k^{\times})^p \longrightarrow K^{\times}/(K^{\times})^p$ be a homomorphism defined by $j(\alpha\cdot(k^{\times})^p) = \alpha\cdot(K^{\times})^p$ for $\alpha \in k^{\times}$. Let $\overline{k}$ be the algebraic closure of k, and let μ_p be the group of all p-th roots of unity in K. Since we have the isomorphisms $H^1(\overline{k}/k, \mu_p) \simeq k^{\times}/(k^{\times})^p$ and $H^1(\overline{k}/K, \mu_p) \simeq K^{\times}/(K^{\times})^p$, we see that the injection of the cohomology groups Inj : $H^1(\overline{k}/k, \mu_p) \longrightarrow H^1(\overline{k}/k, \mu_p)$ induces j. Hence $\ker(j) \simeq H^1(K/k, \mu_p)$ by the exact sequence of the restriction and the injection of the first cohomology groups. Since $[K:k]$ is prime to p, we have $\ker(j) \simeq \{1\}$. Thus $U_k^S(p)/(k^{\times})^p$ is isomorphic to $U_k^S(p)\cdot(K^{\times})^p/(K^{\times})^p$ by j. Therefore we have $U_k^S(p)/(k^{\times})^p \simeq N_{K/k}(U_K^{\overline{S}}(p))\cdot(K^{\times})^p/(K^{\times})^p$.

Q. E. D.

Put $L = K(\sqrt{\alpha} \mid \alpha \in U_k^S(p))$. Then L is the maximal elementary p-abelian extension of K such that any prime divisor contained in $\overline{S}$ is completely decomposed and $\mathrm{Gal}(L/K)$ is isomorphic to C_S by class field theory. Thus we identify C_S with $\mathrm{Gal}(L/K)$.

Lemma 4. $Hom(C_{S,\omega}, \mu_p) \simeq N_{K/k}(U_K^{\overline{S}}(p))\cdot(K^{\times})^p/(K^{\times})^p$, *where we denote by* μ_p *the group of all p-th roots of unity contained in* K.

Proof. Denote by $\overline{\alpha}$ the left coset $\alpha\cdot(K^{\times})^p$ for $\alpha \in U_K^{\overline{S}}(p)$. Let $x \in C_S$. Set $< \overline{\alpha} , x > = \sqrt{\alpha}^{x-1}$. Then it defines a non-degenerate pairing on $(U_K^{\overline{S}}(p)/(K^{\times})^p) \times C_S$. Let $\sigma \in G$. Then $< \sigma(\overline{\alpha}) , \sigma(x) > = < \overline{\alpha} , x >^{\omega(\sigma)}$. Hence we have $< \overline{\alpha} , \varepsilon_{\omega}(x) >^m = < N_G(\overline{\alpha}) , x >$, where we denote by N_G the norm map of a G-module, and $m = [K:k]$. Let $H = \{\alpha \in U_K^{\overline{S}}(p) \mid N_G(\alpha) \in (K^{\times})^p \}$. Then we see that $H/(K^{\times})^p$ is the annihilator of $C_{S,\omega}$. Therefore we have

$$\mathrm{Hom}(C_{S,\omega}, \mu_p) \simeq N_{K/k}(U_K^{\overline{S}}(p))\cdot(K^{\times})^p/(K^{\times})^p,$$

since $\mathrm{Hom}(C_{S,\omega}) \simeq U_K^{\overline{S}}(p)/H\cdot(K^{\times})^p \simeq N_G(U_K^{\overline{S}}(p))\cdot(K^{\times})^p/(K^{\times})^p$.

Q. E. D

Corollary. $U_K^S(p)/(k^{\times})^p \simeq C_{S,\omega}$.

Proof of Theorem 1. By Corollary to Lemma 4, we see that $U_K^S(p) = (k^{\times})^p$ is equivalent to $C_{S,\omega} = \{1\}$. Assume that S satisfies the condition (1), (2) and (3) of Theorem 1. Since $U_k^S(p) = (k^{\times})^p$, we have $A_S^{(2)} = \{1\}$ by the proof of Collorary to Lemma 7. Hence $\delta_p = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E))$. Therefore $\delta_p = 0$ by the condition (3). Conversely we assume that $\delta_p = 0$. Let S be a finite set of prime divisors of k

containing P. Since $t_p(E_S) \supset t_p(E)$, we have p-rank$(t_p(E_S))$ - p-rank$(t_p(E)) \geq 0$. Hence we have p-rank$(t_p(E_S))$ - p-rank$(t_p(E)) = 0$ by Lemma 2. The condition (3) holds for any S containing P. We take a sufficiently large set S so that $C_{S,\omega} \simeq \{1\}$. Then the condition (2) holds. Q. E. D.

2. Some sufficient conditions for the conditions of Theorem 1.

Put $e_P = \#\{ \mathfrak{p} \mid \mathfrak{p} \in P \text{ and } \zeta_p \in k_{\mathfrak{p}} \}$.

Proposition 1. *p-rank$(t_p(E_P))$ = p-rank$(t_p(E))$ if e_P = p-rank$(t_p(E))$ or if $\zeta_p \notin k_{\mathfrak{p}}$ for any $\mathfrak{p} \in P$.*

Proof. Since $e_P \geq$ p-rank$(t_p(E_P)) \geq$ p-rank$(t_p(E))$, we have p-rank$(t_p(E_P))$ = p-rank$(t_p(E))$ if e_P = p-rank$(t_p(E))$. We see that e_P = p-rank$(t_p(E))$ holds if $\zeta_p \in k$ and $|P| = 1$ or if $\zeta_p \notin k_{\mathfrak{p}}$ for any $\mathfrak{p} \in P$.

Q. E. D.

Proposition 2. *Suppose that $\zeta_p \notin k_{\mathfrak{p}}$ for any $\mathfrak{p} \in P$ and that the p-class field tower of K is finite. Then the Leopoldt conjecture holds for (k, p).*

Proof. Let $K = K_0 \subsetneq K_1 \subsetneq K_2 \cdots \subsetneq K_n = K_{n+1}$ be the p-class field tower of K. Then K_n/k is a Galois extension. Since K_n/K is a p-extnsion and K/k is a cyclic extension whose extension degree is prime to p, there exists $\tau \in \mathrm{Gal}(K/k)$ whose order is just $[K:k]$. Let M be the fixed field of τ in K_n. Since $[K_n:M] = [K:k]$ and $K_n \supset M(\zeta_p)$, we have $[K_n:M(\zeta_p)] \mid [K:k]$. On the other hand, since $M(\zeta_p) \supset K$, we have $[K_n:M(\zeta_p)] \mid [K_n:K]$. Since $[K:k]$ is prime to $[K_n:K]$, we have $[K_n:M(\zeta_p)] = 1$. Hence $K_n = M(\zeta_p)$. Let P_M be the set of all prime divisors of M dividing p. Then we see that the condition (2) of Theorem 1 holds for (M, P_M). Let $\mathfrak{P} \in P_M$ and $M_{\mathfrak{P}}$ be the completion of M at $\mathfrak{P}$. Let $\mathfrak{p}$ be a prime divisor of k divided by $\mathfrak{P}$. Since $\zeta_p \notin k_{\mathfrak{p}}$ and $M_{\mathfrak{P}}/k_{\mathfrak{p}}$ is a p-extension, we have $\zeta_p \notin M_{\mathfrak{P}}$. By Proposition 1, we have that the condition (3) holds for (M, P). Therefore the Leopoldt conjecture holds for M. Hence it also holds for k. Q. E. D.

Propositin 3. *Let k_0 be an algebraic number field such that $[k : \mathbf{Q}]$ is finite. Suppose that k_0 is a cyclic extension of k of degree p. Let S_0 be the finite set of prime divisors of k_0 such that the condition (2) of Theorem 1 holds for (k_0, S_0). Let S be the set of all prime divisors of k which divide prime divisors contained in S_0. Put $K_0 = k_0(\zeta_p)$. Let R be the set of primes divisors $\mathfrak{p}$ of k_0 satisfying the following two conditions. (1) $\mathfrak{p}$ is contained in S_0 or an extension of $\mathfrak{p}$ to K is ramified at K/K_0. (2) $\mathfrak{p}$ is completely decomposed at K_0.*

Then the condition (2) of Theorem 1 for (k_0, S_0) implies that for (k, S) if $R = \phi$.

Put $X_\omega = \varepsilon_\omega(X)$ for a $\mathbf{Z}_p[G]$-module X. To prove this proposition, we need the following two lemmas.

Lemma 5. *Let* $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ *be an exact sequence of* $\mathbf{Z}_p[G]$*-modules. Then we have the exact sequence*

$0 \longrightarrow N_\omega \longrightarrow M_\omega \longrightarrow P_\omega \longrightarrow 0.$

Proof. We regard N as a submodule of M. We have an exact sequence $0 \longrightarrow N \cap M_\omega \longrightarrow M_\omega \longrightarrow P_\omega \longrightarrow 0$. Let y be an element of M such that $\varepsilon_\omega(y) \in N$. Then $\varepsilon_\omega(y) = \varepsilon_\omega \cdot \varepsilon_\omega(y) \in N_\omega$ since $\varepsilon_\omega \cdot \varepsilon_\omega = \varepsilon_\omega$. Hence $N \cap M_\omega \subset N_\omega$. Since $N \subset N \cap M_\omega$, we have $N_\omega = N \cap M_\omega$. Q. E. D.

Let L be the class field of K whose Galois group over K is isomorphic to C_S by class field theory. We denote by L^* the maximal abelian extension of k contained in L. Let J_K be the idèle group of $K = k(\xi_p)$ and U_K be its unit group. Put $W_K = U_K \cdot \prod_{\mathfrak{P} \in \overline{S}} K_{\mathfrak{P}}^\times$, where we denote by $\overline{S}$ the set of all prime divisors of K dividing prime divisors contained in S_0. Then we have $C_S \simeq J_K / W_K \cdot (J_K)^p \cdot K^\times$. Put $K_0 = k_0(\xi_p)$.

Lemma 6. *Let* σ *be a generator of* $Gal(K/K_0)$. *Then we have*

$$C_S/(C_S)^{\sigma-1} \simeq N_{K/K_0}(J_K) \cdot K_0^\times / N_{K/K_0}(W_K \cdot (J_K^p)) \cdot K_0^\times$$

Proof. We identify $\mathrm{Gal}(L/K)$ to C_S. Let M be the fixed field of $C_S^{\sigma-1}$ in L. $\mathrm{Gal}(M/K_0)$ is an abelian group since K/K_0 is a cyclic extension. Hence $L^* \supset M$. Put $H = \mathrm{Gal}(L/L^*)$. Then $H \supset (C_S)^{\sigma-1}$ because σ acts trivially on $\mathrm{Gal}(L^*/K)$. Hence $L^* \subset M$. Thus we have $L^* = M$ and $\mathrm{Gal}(L^*/K) \simeq C_S/(C_S)^{\sigma-1}$. On the other hand, we have $\mathrm{Gal}(L^*/K) \simeq N_{K/K_0}(J_K)\cdot K_0^\times/N_{K/K_0}(W_K\cdot(J_K^p))\cdot K_0^\times$ by translation theorem of class field theory, because $\mathrm{Gal}(L/K) \simeq J_K/W_K\cdot(J_K)^p\cdot K^\times$. Q. E. D.

Proof of Proposition 3. We denote by $\mathfrak{p}$ resp. $\mathfrak{P}$ a prime divisor of K_0 resp. K. We denote by $K_{0\,\mathfrak{p}}$ resp. $K_\mathfrak{P}$ the completion of K_0 resp. K at $\mathfrak{p}$ resp. $\mathfrak{P}$. Let $U_\mathfrak{p}$ resp. $U_\mathfrak{P}$ be its unit group. Let $\mathfrak{p}_0$ be a prime divisor of k_0. We define $V_{\mathfrak{p}_0}$ and $W_{\mathfrak{p}_0}$ by

$$V_{\mathfrak{p}_0} = \prod_{\mathfrak{p}|\mathfrak{p}_0} K_{0\,\mathfrak{p}}^\times, \qquad W_{\mathfrak{p}_0} = \prod_{\mathfrak{P}|\mathfrak{p}_0} K_\mathfrak{P}^\times \qquad \text{if } \mathfrak{p}_0 \in S_0,$$

$$V_{\mathfrak{p}_0} = \prod_{\mathfrak{p}|\mathfrak{p}_0} U_{\mathfrak{p}_0}, \qquad W_{\mathfrak{p}_0} = \prod_{\mathfrak{P}|\mathfrak{p}_0} U_\mathfrak{P} \qquad \text{if } \mathfrak{p}_0 \notin S_0.$$

For each $\mathfrak{p}$, we choose a prime divisor $\mathfrak{P}$ of K dividing $\mathfrak{p}$ and denote by $Z_\mathfrak{p}$ the decomposition group of $\mathfrak{P}$ in $\mathrm{Gal}(K/K_0)$. If $\mathfrak{P}$ is ramified at K/K_0, then $Z_\mathfrak{p}$ is also the inertia group of $\mathfrak{P}$. Hence we have by class field theory

(2.1) $V_{\mathfrak{p}_0}/N_{K/K_0}(W_{\mathfrak{p}_0}) \simeq \prod_{\mathfrak{p}|\mathfrak{p}_0} Z_{\mathfrak{p}}$.

for any prime divisor $\mathfrak{p}_0$ of k_0. We consider the $\mathbf{Z}_p[G]$-module structure of this group. We see $Z_{\mathfrak{p}} \simeq \{1\}$ if $\mathfrak{p}$ is decomposed at K/K_0 or if $\mathfrak{p}_0$ is not contained in S_0 and $\mathfrak{p}$ is not ramified at K/K_0. Now we consider the group of (2.1) for $\mathfrak{p}_0$ whose extension $\mathfrak{P}$ to K is not decomposed if $\mathfrak{p}_0 \in S_0$ or which is ramified at K/k if $\mathfrak{p}_0 \notin S_0$. Let $\mathfrak{p}$ be a fixed prime divisor of K_0 dividing $\mathfrak{p}_0$. We denote by $G_{\mathfrak{p}_0}$ be the decomposition group of $\mathfrak{p}$ in $Gal(K_0/k_0)$. Let $G = \cup_{i=1}^{t} \sigma_i \cdot G_{\mathfrak{p}_0}$ be the decomposition of G to left cosets. We assume $\sigma_1 \in G_{\mathfrak{p}_0}$. Let τ_1 be a generator of $Z_{\mathfrak{p}}$. Then $\tau_i = \sigma_i \cdot \tau_1 \cdot \sigma_i^{-1}$ is a generator of $Z_{\mathfrak{p}^{\sigma_i}}$. We use the additive notation for $Z_{\mathfrak{p}}$ in the followings. Then

$$\prod_{\mathfrak{p}|\mathfrak{p}_0} Z_{\mathfrak{p}} \simeq \{ (n_1 \cdot \tau_1, n_2 \cdot \tau_2, \cdots, n_t \cdot \tau_t) \mid n_i \in \mathbf{Z}/p\mathbf{Z} \text{ for } i = 1, \cdots, t\}.$$

Let $\varphi : \prod_{\mathfrak{p}|\mathfrak{p}_0} Z_{\mathfrak{p}} \longrightarrow (\mathbf{Z}/p\mathbf{Z})[G/G_{\mathfrak{p}}]$ be a $\mathbf{Z}_p[G]$-isomorphism defined by $\varphi((n_1 \cdot \tau_1, \cdots, n_t \cdot \tau_t)) = \sum_{i=1}^{t} n_i \cdot \sigma_i \cdot G_{\mathfrak{p}}$. Then we have

(2.2) $(\prod_{\mathfrak{p}|\mathfrak{p}_0} Z_{\mathfrak{p}})_\omega \simeq ((\mathbf{Z}/p\mathbf{Z})[G/G_{\mathfrak{p}}])_\omega$.

This module is $\{0\}$ if $G_{\mathfrak{p}} \not\simeq \{1\}$, and is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ if $G_{\mathfrak{p}} \simeq \{1\}$. By (2.1) and (2.2), we have $(V_{\mathfrak{p}_0}/N_{K/K_0}(W_{\mathfrak{p}_0}))_\omega \simeq \{1\}$ if and only if $\mathfrak{p}_0$ is not completely decomposed at

K_0/k_0. Hence we have $(V_{\mathfrak{p}_0}/N_{K/K_0}(W_{\mathfrak{p}_0}))_\omega \simeq \{1\}$ for any $\mathfrak{p}_0$ because $R = \phi$. Put $V_{K_0} = \prod_{\mathfrak{p}_0} V_{\mathfrak{p}_0}$ and $W_K = \prod_{\mathfrak{p}_0} W_{\mathfrak{p}_0}$, where $\mathfrak{p}_0$ runs through the set of all prime divisors of k_0. Then we have

$$(2.3)\quad (V_{K_0}/N_{K/K_0}(W_K))_\omega \simeq \{1\}.$$

We define $\mathbf{Z}_p[G]$-modules N, M, P, Z, Y by

$$N = V_{K_0}\cdot N_{K/K_0}(J_K^p)\cdot K_0^\times / N_{K/K_0}(W_K\cdot(J_K^p))\cdot K_0^\times,$$

$$M = V_{K_0}\cdot(J_{K_0}^p)\cdot K_0^\times / N_{K/K_0}(W_K\cdot(J_K^p))\cdot K_0^\times,$$

$$P = V_{K_0}\cdot(J_{K_0}^p)\cdot K_0^\times / V_{K_0}\cdot N_{K/K_0}(J_K^p)\cdot K_0^\times,$$

$$Y = N_{K/K_0}(J_K)\cdot K_0^\times / N_{K/K_0}(W_K\cdot(J_K^p))\cdot K_0^\times,$$

$$Z = J_{K_0}/N_{K/K_0}(W_K\cdot(J_K^p))\cdot K_0^\times.$$

Then we have exact sequences of $\mathbf{Z}_p[G]$-modules

$$(2.4)\quad 1 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 1,$$

$$(2.5)\quad 1 \longrightarrow M \longrightarrow Z \longrightarrow Z/M \longrightarrow 1,$$

$$(2.6)\quad 1 \longrightarrow Y \longrightarrow Z \longrightarrow J_{K_0}/N_{K/K_0}(J_K)\cdot K_0^\times \longrightarrow 1.$$

Since N is a homomorphic image of $V_{K/K_0}/N_{K/K_0}(J_K)\cdot K_0^\times$ as $\mathbf{Z}_p[G]$-module, we see $N_\omega \simeq \{1\}$ by Lemma 5 and (2.3). Let f : $J_{K_0}/N_{K/K_0}(J_K)\cdot K_0^\times \longrightarrow P$ be a $\mathbf{Z}_p[G]$-homomorphism defined by $f(\alpha\cdot N_{K/K_0}(J_K)\cdot K_0^\times) = \alpha^p\cdot V_{K_0}\cdot N_{K/K_0}(J_K^p)\cdot K_0^\times$. Then f is a surjection. Since $J_{K_0}/N_{K/K_0}(J_K)\cdot K_0$ is a trivial G-module, we have $(J_{K_0}/N_{K/K_0}(J_K)\cdot K_0^{\times})_{\omega} \simeq \{1\}$. Hence $\text{image}(f)_{\omega} = P_{\omega} \simeq \{1\}$ by Lemma 5. Then we have $M_{\omega} \simeq \{1\}$ by Lemma 5 and (2.4). Hence we have $Z_{\omega} \simeq (Z/M)_{\omega}$ by Lemma 5 and (2.5). Since it is equivalent to $(J_{K_0}/V_{K_0}\cdot(J_{K_0}^{p})\cdot K_0^{\times})_{\omega} \simeq \{1\}$ that the conondition (2) of Theorem 1 holds for (k_0 , S_0), we have $(Z/M)_{\omega} \simeq \{1\}$ by the assumption of Proposition. Hence we see $Z_{\omega} \simeq \{1\}$. Therefore $Y_{\omega} \simeq \{1\}$ by (2.6). Since $Y \simeq C_S/(C_S)^{\sigma-1}$ by Lemma 6, we have $(C_S/C_S^{\sigma-1})_{\omega} \simeq \{1\}$. Since $\mathrm{Gal}(K/k_0)$ is an abelian group, we have $\sigma\cdot\tau = \tau\cdot\sigma$ for any $\tau \in G$. Hence $\sigma\cdot\varepsilon_{\omega} = \varepsilon_{\omega}\cdot\sigma$. Thus $(C_S^{\sigma-1})_{\omega} = (C_{S,\omega})^{\sigma-1}$. Therefore we have $C_{S,\omega}/(C_{S,\omega})^{\sigma-1} \simeq \{1\}$ because $(C_S/C_S^{\sigma-1})_{\omega} \simeq C_{S,\omega}/(C_{S,\omega})^{\sigma-1}$. This implies $C_{S,\omega} \simeq \{1\}$. Q. E. D.

3. Some theorems concerned with $\delta_p(k : p)$.

We define groups V_S, W_S and $A_S^{(1)}$ by

$V_S = \{ u \in U_S \mid u^p \in E_S \}$,

$W_S = \{ u \in U_S \mid u^p = 1 \}$,

$A_S^{(1)} = E \cap (U_S^p)/E \cap (E_S^p)$.

Lemma 8. $1 \longrightarrow W_S/W_S \cap t_p(E_S) \longrightarrow V_S/W_S \longrightarrow A_S^{(1)} \longrightarrow 1.$

Proof. Let $u \in V_S$. Then there exists $\delta \in E_S$ such that

$u^p = \delta$. Since $E\cdot(E_S^p) = E_S$, we have $\varepsilon \in E$ and $\delta_1 \in E_S$ such that $\delta = \varepsilon\cdot\delta_1^p$. Then we see $\varepsilon \in E \cap (U_S^p)$. Let f : $V_S \longrightarrow A_S^{(1)}$ be a homomorphsm defined by $f(u) = \varepsilon\cdot(E \cap (E_S^p))$. Then $\ker(f) = W_S\cdot E_S$. Let $\varepsilon \in E \cap (U_S^p)$. Then there exists $u \in U_S$ such that $u^p = \varepsilon$. We have $f(u) = \varepsilon\cdot(E \cap (E_S^p))$. Hence f is surjective. Since $W_S\cdot E_S/E_S \simeq W_S/W_S \cap t_p(E_S)$, we have an exact sequence

$$1 \longrightarrow W_S/W_S \cap t_p(E_S) \longrightarrow V_S/E_S \longrightarrow A_S^{(1)} \longrightarrow 0.$$

Q. E. D.

Let T be a finite set of prime divisors of k. We permit it in the case $T = \phi$. Let A be the maximal subgroup of the ideal class group of k whose exponent is divided by p. We define $U_k^T(p)$ for $T = \phi$ by $U_k^{\phi}(p) = \{\ \alpha \in k \mid$ There exists an ideal $\mathfrak{a}$ of k such that $\mathfrak{a}^p = (\alpha).\}$. We define a subgroup $A_T^{(0)}$ of A by

$$A_T^{(0)} = \{\ c \in A \mid c \text{ contains an ideal } \mathfrak{a} \text{ of } k \text{ such that } \mathfrak{a}^p = (\alpha) \text{ for some } \alpha \in U_k^T(p).\}.$$

Lemma 9. $U_k^T(p)/(E \cap (U_k^T(p))\cdot(k^{\times})^p \simeq A_T^{(0)}$.

Proof. Let $\alpha \in U_k^T(p)$. Then there exists an ideal $\mathfrak{a}$ of k such that $\mathfrak{a}^p = (\alpha)$. Let c be the divisor class containing $\mathfrak{a}$. This divisor class is contained in $A_S^{(0)}$. We define a homomorphism f : $U_k^T(p) \longrightarrow A_S^{(0)}$ by $f(\alpha) = c$. Then f is surjective by the definition of $A_S^{(0)}$. Since $\ker(f) = \{ \alpha \in U_k^T(p) \mid$ There exists $\beta \in k$ such that $(\alpha) = (\beta^p).\}$, we have $\ker(f) = (E \cdot (k^\times)^p) \cap U_k^T(p) = (E \cap U_k^T(p)) \cdot (k^\times)^p$. Therefore we see $\text{image}(f) \simeq U_k^T(p)/(E \cap U_k^T(p)) \cdot (k^\times)^p$.

Q. E. D.

Corollary. *Suppose S is a non-empty finite set of prime divisors of k. Then we have*

$$U_k^S(p)/(k^\times)^p \simeq A_S^{(0)} \oplus A_S^{(1)} \oplus A_S^{(2)}.$$

Proof. Since $E \cap U_k^S(p) = E \cap U_S^p$, we have a chain of abelian groups $U_k^S(p)/(k^\times)^p \supset (E \cap U_k^S(p)) \cdot (k^\times)^p/(k^\times)^p \supset (E \cap E_S^p) \cdot (k^\times)/(k^\times)^p$. The sequence of the quotient groups of this chain is isomorphic to $A_S^{(0)}$, $A_S^{(1)}$, $A_S^{(2)}$. Since the exponent of $U_k^S(p)/(k^\times)^p$ divides p, it is isomorphic to $A_S^{(0)} \oplus A_S^{(1)} \oplus A_S^{(2)}$.

Q. E. D.

Put $e_S = \{ \mathfrak{p} \mid \mathfrak{p} \in S$ and $\zeta_p \in k_{\mathfrak{p}}\}$. We see that e_S is equal to $p\text{-rank}(W_S)$.

Theorem 2. *We have for $S \supset P$,*

$$\delta_p = e_S + p\text{-}rank(C_{S,\omega}) - p\text{-}rank(t_p(E)) - p\text{-}rank(V_S/E_S) - p\text{-}rank(A_S^{(0)}).$$

Proof. By Lemma 2. we have

(3.1) $\delta_p = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) + p\text{-rank}(A_S^{(2)})$.

By Lemma 8, we have

(3.2) $p\text{-rank}(t_p(E_S)) = e_S - p\text{-rank}(W_S/E_S) + p\text{-rank}(A_S^{(1)})$

since $p\text{-rank}(t_p(E_S)) = p\text{-rank}(W_p \cap t_p(E_S))$.
We substitute $p\text{-rank}(t_p(E_S))$ in (3.1) by the right hand side of (3.2). Then

(3.3) $\delta_p = e_S - p\text{-rank}(V_S/E_S) + p\text{-rank}(t_p(E)) +$

$$(p\text{-rank}(A_S^{(1)}) + p\text{-rank}(A_S^{(2)})).$$

By Corollary to Lemma 9 and Corollary to Lemma 4,

(3.4) $p\text{-rank}(A_S^{(1)}) + p\text{-rank}(A_S^{(2)}) = p\text{-rank}(C_{S,\omega}) - p\text{-rank}(A_S^{(0)})$

We substitute $(p\text{-rank}(A_S^{(1)}) + p\text{-rank}(A_S^{(2)}))$ in (3.3) by the right hand side of (3.4). Then

$$\delta_p = e_S - p\text{-rank}(V_S/E_S) - p\text{-rank}(t_p(E)) + p\text{-rank}(C_{S,\omega})$$
$$- p\text{-rank}(A_S^{(0)}).$$

Q. E. D.

Corollary. *We have* $\delta_p \leq p\text{-}rank(C_{S,\omega}) + e_S - p\text{-}rank(t_p(E))$ *for* $P \subset S$.

Let k_S be the maximal p-extension of k unramified outside S. Put $G_S = \mathrm{Gal}(k_S/k)$ and $G_S^* = G_S/[G_S, G_S]$, where

$[G_S, G_S]$ is the comutator subgroup of G_S. Put $\hat{G}_S = \mathrm{Hom}(G_S^*, \mathbf{Q}/\mathbf{Z})$. Let f_p be an endomorphism of $\mathbf{Q}/\mathbf{Z}$ defined by $f_p(x) = p \cdot x$ for $x \in \mathbf{Q}/\mathbf{Z}$. We have an exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{f_p} \mathbf{Q}/\mathbf{Z} \longrightarrow 0.$$

Then we have the following cohomology long exact sequence

$$H^1(G_S, \mathbf{Q}/\mathbf{Z}) \xrightarrow{f_{p\,*}^{(1)}} H^1(G_S, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(G_S, \mathbf{Z}/p\mathbf{Z}) \longrightarrow$$

$$H^2(G_S, \mathbf{Q}/\mathbf{Z}) \xrightarrow{f_{p\,*}^{(2)}} H^2(G_S, \mathbf{Q}/\mathbf{Z}),$$

where we denote by $f_{p\,*}^{(1)}$ and $f_{p\,*}^{(2)}$ the induced homomorphism of the cohomology groups by f_p. Since G_S acts trivially on $\mathbf{Q}/\mathbf{Z}$, we have $H^1(G_S, \mathbf{Q}/\mathbf{Z}) \simeq \hat{G}_S$. Then $\mathrm{coker}(f_{p\,*}^{(1)}) = \hat{G}_S / p \cdot \hat{G}_S$ and $\ker(f_{p\,*}^{(2)}) = \{ c \in H^2(G_S, \mathbf{Q}/\mathbf{Z}) \mid p \cdot c = 0 \}$. Put $H^2(G_S, \mathbf{Q}/\mathbf{Z})_p = \ker(f_{p\,*}^{(2)})$ and $\mathrm{G}_{S,p}^* = \{ x \in G_S^* \mid x^p = 1 \}$. Then $G_{S,p}^*$ is equal to $\{ x \in G_S^* \mid x^p = 1 \}$. Put $\hat{G}_{S,p} = \mathrm{Hom}(G_{S,p}^*, \mathbf{Q}/\mathbf{Z})$. Then $\mathrm{coker}(f_{p\,*}^{(1)}) \simeq \hat{G}_S / p \cdot \hat{G}_S$ is equal to $\hat{G}_{S,p}$. Hence we have a short exact sequence

$$(3.5) \quad 0 \longrightarrow \hat{G}_{S,p} \longrightarrow H^2(G_S, \mathbf{Z}/p\mathbf{Z}) \longrightarrow H^2(G_S, \mathbf{Q}/\mathbf{Z})_p \longrightarrow 0 .$$

Put $g^S = p\text{-rank}(H^1(G_S, \mathbf{Z}/p\mathbf{Z}))$, $r^S = p\text{-rank}(H^2(G_S, \mathbf{Q}/\mathbf{Z}))$ and $t^S = p\text{-rank}(t_p(G_S^*))$.

Theorem 3. *Suppose that k is totally imaginary if $p = 2$. Suppose $P \subset S$. Let r_2 be the number of the complex places of k. Then we have*

(1) $p\text{-}rank(H^2(G_S,\mathbf{Z}/p\mathbf{Z})) = g^S - (r_2 + 1) = \delta_p + t^S,$

(2) $p\text{-}rank(H^2(G_S,\mathbf{Q}/\mathbf{Z})_p) = \delta_p.$

Proof. Let S_1 be the union of S and the set of all infinite prime divisors of k. We see $k_S = k_{S_1}$ in case of $p = 2$ because k is totally imaginary. If there exists an intermidiate field k' of k_{S_1}/k of finite degree over k such that the infinite prime divisors are ramified, then we see $[\tilde{k}':k] \equiv 0 \bmod 2$, where we denote by $\tilde{k}'$ the Galois closure of k'. Since $\tilde{k}' \subset k_{S_1}$, such k' dose not exist in case of $p \neq 2$. Thus $k_S = k_{S_1}$. Therefore we have, by Corollary 1 to the main theorem of Neumann [7],

$$\Sigma_{i=0}^{2} (-1)^i p\text{-rank}(H^i(G_S,\mathbf{Z}/p\mathbf{Z})) = -r_2.$$

Hence $p\text{-rank}(H^2(G_S,\mathbf{Z}/p\mathbf{Z})) = p\text{-rank}(H^1(G_S,\mathbf{Z}/p\mathbf{Z})) - (r_2 + 1)$. Since the $\mathbf{Z}_p$-free rank of G_S^* equals $r_2 + 1 + \delta_p$ by the theory of $\mathbf{Z}_p$-extensions, we see

$$p\text{-rank}(G_S^*/(G_S^*)^p) = (r_2 + 1 + \delta_p) + t^S.$$

Since $g^S = p\text{-rank}(H^1(G_S,\mathbf{Z}/p\mathbf{Z})) = p\text{-rank}(G_S^*/(G_S^*)^p)$, we have

$$p\text{-rank}(H^2(G_S,\mathbf{Z}/p\mathbf{Z})) = g^S - (r_2 + 1) = \delta_p + t^S.$$

Hence we have $p\text{-rank}(H^2(G_S,\mathbf{Q}/\mathbf{Z})_p) = \delta_p$ by (3.5), because $p\text{-rank}(\hat{G}_{S,p}) = p\text{-rank}(G_{S,p}^*) = p\text{-rank}(t_p(G_S^*))$.

Q. E. D.

Corollary. *Suppose* k *is totally imaginary if* $p = 2$. *Suppose* $P \subset S$. *Then* G_S *is a pro-p-free group if and only if* $\delta_p = 0$ *and* $t^S = 0$.

Proof. By Satz 4.12 of Koch [5], we have that G_S is the pro-p-free group if and only if $H^2(G_S, \mathbf{Z}/p\mathbf{Z}) = \{0\}$. By (1) of Theorem 3, we have that $H^2(G_S, \mathbf{Z}/p\mathbf{Z}) = \{0\}$ is equivalent to $\delta_p = t^S = 0$. Q. E. D.

Lemma 10. *Suppose* $P \subset S$. *Let* r_2 *be the number of complex places of* k. *Then we have*

$$g^S = e_S - p\text{-rank}(t_p(E)) + p\text{-rank}(C_{S,\omega}) + r_2 + 1.$$

Proof. Let J be the idèle group of k, and U be its unit group. Put $U(S) = \prod_{\mathfrak{p}\in S} U_{\mathfrak{p}}$, which is contained in U. By §3 in Miki [6], we have an exact sequence

$$(3.6) \quad 1 \longrightarrow U_k^S(p)/(k^\times)^p \longrightarrow U_k^\phi(p)/(k^\times)^p \longrightarrow U/U(S)\cdot U^p$$
$$\longrightarrow J/U(S)\cdot J^p\cdot k^\times \longrightarrow J/U\cdot J^p\cdot k^\times \longrightarrow 1.$$

Since $G_S^*/(G_S^*)^p \simeq J/U(S)\cdot J^p\cdot k^\times$ by class field theory, we have $p\text{-rank}(J/U\cdot J^p\cdot k^\times) = g_S$. We compute g^S by (3.6). We have $U_k^\phi(p)/E\cdot(k^\times)^p \simeq A_\phi^{(0)}$ by Lemma 9. Let h_p be the p-rank of the p-Sylow subgroup of the ideal class group of k. Then h_p

$= p\text{-rank}(J/U\cdot J^p\cdot k^\times)$ by the definition of h_p. Since h_p is also equal to $p\text{-rank}(A_\phi^{(0)})$, we have $p\text{-rank}(J/U\cdot J^p\cdot k^\times) = p\text{-rank}(A_\phi^{(0)})$. Hence

$$(3.7)\quad p\text{-rank}(J/U\cdot J^p\cdot k^\times) - p\text{-rank}(U_k^\phi(p)/(k^\times)^p) =$$

$$p\text{-rank}(J/U\cdot J^p\cdot k^\times) - (p\text{-rank}(U_k^\phi(p)/E\cdot(k^\times)^p) +$$

$$p\text{-rank}(E\cdot(k^\times)^p/(k^\times)^p)\ )$$

$$= p\text{-rank}(E\cdot(k^\times)^p/(k^\times)^p) = -\ p\text{-rank}(E/E^p).$$

We have $p\text{-rank}(U/U(S)\cdot U^p) = e_S + [k:\mathbf{Q}]$ because $U/U(S)\cdot U^p$ is isomorphic to U_S/U_S^p. Therfore by (3.6), (3.7) and corollary to Lemma 4, we have

$$g^S = p\text{-rank}(U_k^S(p)/(k^\times)^p) + (e_S + [k:\mathbf{Q}]) - p\text{-rank}(E/E^p)$$

$$= p\text{-rank}(C_{S,\omega}) + e_S + r_2 + 1 - p\text{-rank}(t_p(E)).$$

Q. E. D.

Theorem 4. *Put* $r^S = p\text{-rank}(H^2(G_S,\ \mathbf{Z}/p\mathbf{Z})$. *Suppose that* k *is totally imginary if* $p = 2$. *Suppose that* $P \subset S$. *Then* $r^S = 0$ *if and only if* $p\text{-rank}(C_{S,\omega}) = 0$ *and* $e_S = p\text{-rank}(t_p(E))$.

Proof. By (1) of Theorem 3 and Lemma 10, we have $r^S = 0$ holds if and only if $e_S - p\text{-rank}(t_p(E)) + p\text{-rank}(C_{S,\omega}) = 0$. Since $e_S - p\text{-rank}(t_p(E)) \geq 0$, we have $r^S = 0$ if and only if $e_S - p\text{-rank}(t_p(E)) = 0$ and $p\text{-rank}(C_{S,\omega}) = 0$. Q. E. D.

Remark. We see $e_S \geq p\text{-rank}(t_p(E_S)) \geq p\text{-rank}(t_p(E))$. Hence the condition (3) of Theorem 1 holds if $e_S = p\text{-rank}(t_p(E))$.

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