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On the uniqueness of viscosity solutions of second order PDE's with constraints

（制限条件をもつ２階偏微分方程式の粘性解の一意性について）

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In this note we shall present some uniqueness results of viscosity solutions of second order partial differential equations with constraint conditions.

First we shall give a definition of viscosity solutions for general nonlinear second order elliptic equations. Consider

(*) $$F(D^2u,\ Du,\ u,\ x) = 0 \qquad \text{in} \quad \Omega \subset \mathbf{R}^N,$$

where F is defined and continuous on $S^N \times \mathbf{R}^N \times \mathbf{R} \times \Omega$ (S^N denotes the set of $N\times N$ symmetric matrices) satisfiing the following ellipticity condition:

$$F(A,p,t,x) \geqq F(B,p,t,x) \quad \text{if} \quad A \leqq B;$$
$$A,\ B \in S^N,\ p \in \mathbf{R}^N,\ t \in \mathbf{R},\ x \in \Omega.$$

<u>Definition</u>. Let $u \in C(\Omega)$; u is said to be a viscosity solution of (*) if and only if we have for all $\varphi \in C^2(\Omega)$:

(i) for each local maximum point x_0 of $u - \varphi$, we have:

$$F(D^2\varphi(x_0),\ D\varphi(x_0),\ u(x_0),\ x_0) \leqq 0;$$

(ii) for each local minimum point x_0 of $u - \varphi$, we have:

$$F(D^2\varphi(x_0),\ D\varphi(x_0),\ u(x_0),\ x_0) \geqq 0. \blacksquare$$

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary and

$$L^p u = - \sum_{i,j=1}^{N} a^p_{ij} u_{x_i x_j} + \sum_{i=1}^{N} b^p_i u_{x_i} + c^p u, \quad p = 1, 2, \ldots, m$$

be uniformly elliptic operators with smooth coefficients.

We shall treat the following three kinds of problems:

(1) Minimax equation;

$$\min\{\max\{-\Delta u + u - f, u - \psi_1\}, u - \psi_2\} = 0 \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

(2) Hamilton-Jacobi-Bellman equation with a gradient constraint;

$$\max\{L^1 u - f^1, L^2 u - f^2, \ldots, L^m u - f^m, |Du| - g\} = 0 \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

(3) System of minimax equations;

$$\min\{\max\{L^p u^p - f^p, u^p - u^{p+1} - K\}, u^p - u^{p+1} + k\} = 0 \quad \text{in} \quad \Omega,$$

$$u^p = 0 \quad \text{on} \quad \partial\Omega, \quad p = 1, 2, \ldots, m, \quad u^{m+1} = u^1,$$

where K and k are given positive constants.

For these problems, we can prove the existence of solutions by the so-called penalty method. Our basic idea to prove the uniqueness of viscosity solutions is to compair any viscosity solution with the solution which we have established by the penalty method.

Let $\beta \in C^\infty(\mathbf{R})$ be a function such that

$$\beta(t) = 0 \quad \text{if} \quad t \leqq 0, \qquad \beta(t) = t - 1 \quad \text{if} \quad t \geqq 2,$$

$$\beta'(t) \geqq 0, \qquad \beta''(t) \geqq 0 \quad \text{on} \quad \mathbf{R},$$

and we set $\beta_\varepsilon(t) = \gamma_\varepsilon(t) = \beta(t/\varepsilon)$ for each $\varepsilon > 0$.

§1. A minimax equation

Consider a minimax equation

$$
(1.1)\quad \begin{cases} \min\{\max\{-\Delta u + u - f,\ u - \psi_1\},\ u - \psi_2\} = 0 \quad \text{a.e. in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}
$$

where f, ψ_1 and ψ_2 are smooth functions satisfying $\psi_2 \leqq \psi_1$ in Ω and $\psi_2|_{\partial\Omega} \leqq 0 \leqq \psi_1|_{\partial\Omega}$.

It is convenient to formulate the notion of viscosity solutions of (1.1) in the following manner. We say that $u \in C(\overline{\Omega})$ is a viscosity solution of (1.1) if both (A) and (B) hold:

(1.2)
(A) $\psi_2 \leqq u \leqq \psi_1$ in Ω, $u|_{\partial\Omega} = 0$.

(B) Let $\varphi \in C^2(\Omega)$:

(i) if $u - \varphi$ attains its local maximum at $x_0 \in \Omega$ and $\psi_2(x_0) < u(x_0)$, then $-\Delta\varphi + u \leqq f$ at x_0,

(ii) if $u - \varphi$ attains its local minimum at $x_0 \in \Omega$ and $u(x_0) < \psi_1(x_0)$, then $-\Delta\varphi + u \geqq f$ at x_0. ■

Since (1.1) is equivalent to the following variational inequalities with bilateral constraints:

$$
\begin{aligned}
&\psi_2 \leqq u \leqq \psi_1 \quad \text{in } \Omega,\ u|_{\partial\Omega} = 0, \\
&-\Delta u + u = f \quad \text{on } \{x \in \Omega \mid \psi_2 < u < \psi_1\}, \\
&-\Delta u + u \geqq f \quad \text{on } \{x \in \Omega \mid u = \psi_2\}, \\
&-\Delta u + u \leqq f \quad \text{on } \{x \in \Omega \mid u = \psi_1\},
\end{aligned}
$$

it is known (Bensoussan et Lions [1], Chapter 3, Section 5) that there exists a solution $u \in W^{2,r}(\Omega)$ with $r > N$ which is a limit of solutions u_ε of the penalized equation

$$
(1.3)\quad \begin{cases} -\Delta u_\varepsilon + u_\varepsilon + \beta_\varepsilon(u_\varepsilon - \psi_1) - \beta_\varepsilon(\psi_2 - u_\varepsilon) = f \quad \text{in } \Omega, \\ u_\varepsilon|_{\partial\Omega} = 0. \end{cases}
$$

<u>Theorem 1.1.</u> Let $v \in C(\overline{\Omega})$ be a viscosity solution of (1.1). Then we have $u = v$ in $\overline{\Omega}$.

Proof. We prove only $v \leqq u$ in Ω because the inequality in the opposite direction can be proved similarly.

If $v \leqq u$ in Ω does not hold, there exist a subsequence x_{ε_j} (we denote simply x_ε) and $x_0 \in \Omega$ such that

$$\text{(1.4)}\quad \begin{aligned} &\text{(i)} \quad x_\varepsilon \longrightarrow x_0 \quad \text{as} \quad \varepsilon \to 0, \\ &\text{(ii)} \quad (v - u_\varepsilon)(x_\varepsilon) = \max_{x \in \overline{\Omega}} (v - u_\varepsilon)(x), \\ &\text{(iii)} \quad (v - u)(x_0) = \max_{x \in \overline{\Omega}} (v - u)(x) > 0. \end{aligned}$$

Since $\psi_2 \leqq u < v$ at x_0, we have $\psi_2 < v$ and $u_\varepsilon < v \leqq \psi_1$ near x_0. Then (1.2 - i) implies

$$\text{(1.5)} \qquad -\Delta u_\varepsilon + v \leqq f \quad \text{at} \quad x_\varepsilon.$$

We also have $\beta_\varepsilon(u_\varepsilon - \psi_1) = 0$. Hence from (1.3) we get

$$\text{(1.6)} \qquad -\Delta u_\varepsilon + u_\varepsilon \geqq f \quad \text{at} \quad x_\varepsilon.$$

Combining (1.5), (1.6) and letting $\varepsilon \to 0$, we have

$$(v - u)(x_0) \leqq 0$$

which is a contradiction.■

§2. HJB equation with a gradient constraint

For given non-negative functions f^p, $p = 1,\dots,m$ and g, we consider the Dirichlet problem

$$\text{(2.1)} \quad \begin{aligned} &\max\{L^1u - f^1, \ \dots \ , L^m u - f^m, |Du| - g\} = 0 \quad \text{in} \quad \Omega, \\ &u|_{\partial\Omega} = 0. \end{aligned}$$

Here Du is the gradient of a function u.

Evans [3] was the first to treat the equation with a gradient constraint in the case $m = 1$ in (2.1). Relaxing the restrictions in [3], Ishii and Koike [10] have proved the existence of solutions in the space $W^{2,\infty}(\Omega)$ and the uniqueness in the class $W^{2,r}_{\ell oc}(\Omega) \cap C(\overline{\Omega})$ with $r > N$.

On the other hand, the Hamilton-Jacobi-Bellman (HJB) equation has been treated by many authors. Using a system of variational inequalities, Evans and Friedman [7], Lions [12] and Evans et Lions [8] have proved the existence of solutions in the space $W^{2,\infty}(\Omega)$ for uniformly elliptic HJB equations. Moreover, Evans [5], [6] has proved the existence of classical solutions for uniformly elliptic HJB equations (see also Gilbarg and Trudinger [9], Chapter 17). By defining an appropriate notion of weak or viscosity solution, Lions [13] has obtained the uniqueness in the space $C(\overline{\Omega})$, with the aid of stochastic representation of solutions. In [13], it is not assumed that the operators are uniformly elliptic, but rather that they contain zero-th order terms with strictly positive coefficients. Note that our equation (2.1) is a non-uniformly elliptic HJB equation without zero-th order term.

We make the following assumptions on L^p:

(2.2) $\quad a^p_{ij}\xi_i\xi_j \geqq \theta|\xi|^2$

for some $\theta > 0$, all $\xi \in R^N$ and $p = 1, \ldots, m$,

(2.3) $\quad a^p_{ij},\ b^p_i,\ c^p \in C^2(\overline{\Omega})$

for $p = 1, \ldots, m$ and $1 \leqq i,\ j \leqq N$,

(2.4) $\quad c^p \geqq c_0$

for some constant $c_0 > 0$ in Ω, $p = 1, \ldots, m$,

(2.5) $\quad a^p_{ij} = a^p_{ji}$

for $p = 1, \dots, m$, $1 \leqq i, j \leqq N$.

On given functions f^p, g on Ω, we impose the following assumptions:

(2.6) $$f^p,\ g \in C^2(\overline{\Omega})$$

for $p = 1, \dots, m$,

(2.7) $$f^p,\ g \geqq 0$$

in Ω for $p = 1, \dots, m$.

Under these assumptions we may state our main theorem.

Theorem 2.1. We have:

(i) Under the assumptions (2.2) - (2.7), there exists a solution $u \in W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$ of the equation

(2.8) $$\begin{cases} \max\{L^1 u - f^1, \dots, L^m u - f^m, |Du| - g\} = 0 \quad \text{a.e. in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

(ii) If, in addition, $g > 0$ in Ω, then the solution of (2.8) is unique in the class $C^1(\Omega) \cap C(\overline{\Omega})$, where the solution is understood as a viscosity solution satisfying the boundary condition.■

2.1. Approximate systems and a priori estimates

We consider the following approximate systems:

(2.9) $$\begin{cases} L^p u^p_\varepsilon + \beta_\varepsilon(|Du^p_\varepsilon|^2 - g^2) + \gamma_\varepsilon(u^p_\varepsilon - u^{p+1}_\varepsilon) = f^p \quad \text{in } \Omega, \\ u^p_\varepsilon|_{\partial\Omega} = 0, \quad p = 1, \dots, m, \quad \text{where } u^{m+1}_\varepsilon = u^1_\varepsilon. \end{cases}$$

In the following we shall state some a priori estimates for solutions u^p_ε, $p = 1, \dots, m$, of (2.9) which are independent of $\varepsilon > 0$. The proofs of these estimates are found in [15].

Lemma 2.2. ([15, Lemma 4.1]) We have

(2.10) $$0 \leqq u_\varepsilon^P \leqq C \quad \text{in} \quad \Omega,$$

(2.11) $$0 \leqq \frac{\partial u_\varepsilon^P}{\partial n} \leqq C \quad \text{on} \quad \partial\Omega. \blacksquare$$

Here and hereafter capital C denotes various constants depending on known constants and $\partial/\partial n$ denotes the inward normal derivative on $\partial\Omega$.

Lemma 2.3. ([15, Lemma 4.2]) We have

(2.12) $$\|u_\varepsilon^P\|_{W^{1,\infty}(\Omega)} \leqq C. \blacksquare$$

Lemma 2.4. ([15, Lemma 4.3]) We have

(2.13) $$\|u_\varepsilon^P\|_{W^{2,\infty}_{\ell oc}(\Omega)} \leqq C. \blacksquare$$

2.2. Proof of Theorem 2.1

Lemma 2.5. There exists a solution u of (2.8) belonging to $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$.

Proof. From a priori estimates in the preceding section, we can choose a sequence ε_j (which we simply denote ε) such that

$$u_\varepsilon^P \longrightarrow u^P \quad \text{in} \quad C(\overline{\Omega}), \quad Du_\varepsilon^P \longrightarrow Du^P \text{ compact uniformly in } \Omega,$$
$$D^2 u_\varepsilon^P \longrightarrow D^2 u^P \quad \text{weakly in} \quad L^r_{\ell oc}(\Omega) \quad \text{with} \quad r < \infty.$$

Since $\gamma_\varepsilon(u_\varepsilon^P - u_\varepsilon^{P+1})$ are locally bounded, it follows that u^P defined in (2.14) satisfy $u^1 = \dots = u^m \equiv u \in W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$. We shall prove that u solves (2.8).

First we note that $L^P u_\varepsilon^P - f^P \leqq 0$ a.e. in Ω. Hence we have $L^p u - f^p \leqq 0$ a.e. in Ω, $p = 1, \ldots, m$. Since $\beta_\varepsilon(|Du_\varepsilon^p|^2 - g^2)$ are also locally bounded, we get

(2.15) $\max\{L^1 u - f^1, \ldots, L^m u - f^m, |Du| - g\} \leqq 0$ a.e. in Ω.

To prove the inequality in the opposite direction, it is sufficient to show that u is a viscosity supersolution of (2.8). Let $\varphi \in C^2(\Omega)$ and assume that $u - \varphi$ takes its local strict minimum at $x_0 \in \Omega$. We shall show

(2.16) $$\max_{p=1,\ldots,m}\{-a_{ij}^p \varphi_{ij} + b_i^p \varphi_i + c^p u - f^p, |D\varphi| - g\} \geqq 0 \quad \text{at} \quad x_0.$$

Here and hereafter we use the summation convention and denote the derivatives with respect to x_i by subscript i.

Since $|D\varphi(x_0)| \geqq g(x_0)$ implies (2.16), we may assume $|D\varphi(x_0)| < g(x_0)$. Since u_ε^p converges to u uniformly, there exists a sequence $\{x_\varepsilon^p\} \subset \Omega$ such that

(i) $\lim_{\varepsilon \to 0} x_\varepsilon^p = x_0$ for any $p = 1, \ldots, m$,

(ii) $u_\varepsilon^p - \varphi$ attains its local minimum at x_ε^p,

(iii) $|D\varphi(x_\varepsilon^p)| \leqq g(x_\varepsilon^p)$.

For each ε, let $p(\varepsilon)$ be such that

(2.17) $$(u_\varepsilon^{p(\varepsilon)} - \varphi)(x_\varepsilon^{p(\varepsilon)}) = \min_{p=1,\ldots,m} (u_\varepsilon^p - \varphi)(x_\varepsilon^p).$$

Since p varies in a finite set, there exists $\bar{p}$ which appears infinitely many times in (2.17). Consider such $\bar{p}$ and ε such that $p(\varepsilon) = \bar{p}$. Then we have $\beta_\varepsilon(|Du_\varepsilon^{\bar{p}}|^2 - g^2) = 0$ and $\gamma_\varepsilon(u_\varepsilon^{\bar{p}} - u_\varepsilon^{\bar{p}+1}) = 0$ at $x_\varepsilon^{\bar{p}}$. Since $u_\varepsilon^{\bar{p}}$ is also a viscosity supersolution of (2.9), we get

$$-a_{ij}^{\bar{p}} \varphi_{ij} + b_i^{\bar{p}} \varphi_i + c^{\bar{p}} u_\varepsilon^{\bar{p}} \geqq f^{\bar{p}} \quad \text{at} \quad x_\varepsilon^{\bar{p}}.$$

Passing to the limit as $\varepsilon \to 0$, along which we take $\bar{p} = p(\varepsilon)$, we have (2.16).■

Lemma 2.6. Assume $g > 0$ in Ω. Then the viscosity solution of (2.8) is unique in the class $C^1(\Omega) \cap C(\overline{\Omega})$.

Proof. By Lemma 2.5 we have a solution u belonging to $W^{2,\infty}_{\ell oc}(\Omega) \cap W^{1,\infty}(\Omega)$ and approximate solutions u^p_ε which converge to u along a subsequence. In the following we fix such a u and convergent approximate solutions $u^p_{\varepsilon_j}$ (simply we denote u^p_ε).

Let v be any viscosity solution of (2.8) which belongs to $C^1(\Omega) \cap C(\overline{\Omega})$.

First we claim that $v \leqq u$ in Ω. If not, there exist $x_0 \in \Omega$ and p_0 such that

$$(2.18) \qquad (v - u^{p_0}_\varepsilon)(x_0) = \max_{\substack{x \in \overline{\Omega} \\ p=1,\dots,m}} (v - u^p_\varepsilon)(x) > 0.$$

Since v is a viscosity subsolution, we have

$$(2.19) \qquad \begin{aligned} &-a^{p_0}_{ij} u^{p_0}_{\varepsilon,ij} + b^{p_0}_i u^{p_0}_{\varepsilon,i} + c^{p_0} v \leqq f^{p_0} \quad \text{at } x_0, \\ &|Du^{p_0}_\varepsilon(x_0)| \leqq g(x_0). \end{aligned}$$

The second inequality in (2.19) implies $\beta_\varepsilon(|Du^{p_0}_\varepsilon|^2 - g^2) = 0$ at x_0 and (2.18) implies $\gamma_\varepsilon(u^{p_0}_\varepsilon - u^{p_0+1}_\varepsilon) = 0$ at x_0. Then, from (2.9), we have

$$-a^{p_0}_{ij} u^{p_0}_{\varepsilon,ij} + b^{p_0}_i u^{p_0}_{\varepsilon,i} + c^{p_0} u^{p_0}_\varepsilon = f^{p_0} \quad \text{at } x_0.$$

Subtracting this from the first inequality in (2.19) we get

$$c^{p_0}(x_0)(v - u^{p_0}_\varepsilon)(x_0) \leqq 0,$$

which is a contradiction.

Next we show that $\rho u \leqq v$ in Ω for $0 < \rho < 1$. If not, there exist $\rho \in (0,1)$ and $x_0 \in \Omega$ such that

$$(2.20) \qquad (v - \rho u)(x_0) = \min_{x \in \overline{\Omega}} (v - \rho u) < 0.$$

Since $v \in C^1(\Omega)$, we have $|Dv(x_0)| = \rho|Du(x_0)| < g(x_0)$. Then there exists a ball U with center x_0 satisfying

$$(2.21) \qquad |Dv| < g \quad \text{in} \quad U.$$

This implies that v is a viscosity supersolution of

$$(2.22) \qquad \max_{p=1,\ldots,m} \{L^p v - f^p\} = 0 \quad \text{in} \quad U.$$

Consequently v is a viscosity solution of (2.22) in U. Considering (2.22) with boundary condition $\phi = v|_{\partial U}$, it is known (Evans [5], [6], Gilbarg and Trudinger [9] Chapter 17) that (2.22) has a smooth solution. On the other hand, it is also known (Lions [13]) that the viscosity solution of (2.22) is unique. Therefore we can conclude that v is the smooth solution of (2.22) in U.

By a selection lemma, there exists a measurable function $p: U \longrightarrow \{1, \ldots, m\}$ such that

$$L^{p(x)} v - f^{p(x)} = 0 \quad \text{a.e. in} \quad U.$$

Since u is a subsolution of (2.22) we have

$$(2.23) \qquad L^{p(x)} (v - \rho u)(x) - (1 - \rho) f^{p(x)} \geqq 0 \quad \text{a.e. in} \quad U.$$

On the other hand, by Bony's maximum principle, we get

$$(2.24) \qquad \lim_{x \to x_0} \text{ess inf} \, (-a_{ij}^{p(x)} (v - \rho u)_{ij} + b_i^{p(x)} (v - \rho u)_i) \leqq 0.$$

Combining (2.23) and (2.24), we have

$$c^{p(x_0)}(x_0)(v - \rho u)(x_0) - (1 - \rho) f^{p(x_0)}(x_0) \geqq 0$$

which contradicts (2.20).

Since ρ is arbitraly in $(0,1)$, we have $v \equiv u$ in Ω.

This completes the proof.■

By Lemmas 2.5 and 2.6 we have completed the proof of Theorem 2.1.■

§3. System of minimax equations

In this section we consider the following system of minimax equations:

$$(3.1)\quad \begin{aligned} &\min\{\max\{L^p u^p - f^p,\ u^p - u^{p+1} - K\},\ u^p - u^{p+1} + k\} = 0 \quad \text{in} \quad \Omega, \\ &u^p|_{\partial\Omega} = 0, \qquad p = 1, \ldots, m, \quad u^{m+1} = u^1. \end{aligned}$$

This system is equivalent to the following system of variational inequalitits with bilateral obstacles:

$$(3.2)\quad \begin{aligned} &u^{p+1} - k \leqq u^p \leqq u^{p+1} + K \quad \text{in} \quad \Omega, \quad u^p|_{\partial\Omega} = 0, \\ &L^p u^p = f^p \quad \text{if} \quad u^{p+1} - k < u^p < u^{p+1} + K \\ &L^p u^p \leqq f^p \quad \text{if} \quad u^p = u^{p+1} + K, \\ &L^p u^p \geqq f^p \quad \text{if} \quad u^p = u^{p+1} - k, \end{aligned}$$

for $p = 1, \ldots, m$ where $u^{m+1} = u^1$.

In [14], the author have treated this system in the space $W^{2,r}(\Omega)$ $(1 \leqq r < \infty)$ and proved the uniqueness of the solution in this space by using a stochastic representation of the solution. The new argument here is to prove the uniqueness of the solution in the class of viscosity solutions.

On the other hand, Lenhart and Belbas [11] considered a system of equations with unilateral constraints

$$(3.3)\quad \begin{aligned} &\max\{L^p u^p - f^p,\ u^p - M^p u\} = 0 \quad \text{in} \quad \Omega, \\ &u^p|_{\partial\Omega} = 0, \ p = 1, \ldots, m \end{aligned}$$

and proved the existence and uniqueness of the solution in $W^{2,\infty}(\Omega)$. Here, $u = \{u^1, \ldots, u^m\}$ and $M^p u$ is defined as follows:

$$M^p u = \min\{ u^q + K(p,q) \mid 1 \leqq q \leqq m,\ q \neq p\},$$

where $K(p,q)$ are given nonnegative constants.

As was explained in [11], [14], these systems are closely related to some stochastic control problems (see [11], [14] or [7] for the detail). As a generalization of these systems we can treat the following system:

$$(3.4)\quad \begin{cases} m^p u \leqq u^p \leqq M^p u \quad \text{in} \quad \Omega, \quad u^p|_{\partial\Omega} = 0, \\ L^p u^p = f^p \quad \text{if} \quad m^p u < u^p < M^p u, \\ L^p u^p \leqq f^p \quad \text{if} \quad u^p = M^p u, \\ L^p u^p \geqq f^p \quad \text{if} \quad u^p = m^p u, \end{cases}$$

for $p = 1, \ldots, m$. Here, $m^p u$ is defined by

$$m^p u = \max\{ u^q - k(p,q) \mid 1 \leqq q \leqq m,\ q \neq p\}$$

where $k(p,q)$ are given nonnegative constants. However, to make the idea clear, we shall treat the simpler preblem (3.1). The system (3.4) will be treated in [16].

We make the following assumptions on L^p:

$$(3.5)\quad a^p_{ij}\xi_i\xi_j \geqq \theta|\xi|^2$$

for some $\theta > 0$, all $\xi \in \mathbf{R}^N$ and $p = 1, \ldots, m$,

$$(3.6)\quad a^p_{ij},\ b^p_i,\ c^p \in C^1(\overline{\Omega})$$

for $p = 1, \ldots, m$, and $1 \leqq i,j \leqq N$.

There exists a sufficiently large constant c_0 such that

$$(3.7)\quad c^p \geqq c_0$$

for all $p = 1, \ldots, m$. The size of c_0 is determined only by

θ and the sizes of a^p_{ij}, b^p_i and their derivatives (see [14, Lemma 4.3]).

On given functions f^p, $p = 1, \dots, m$, we assume

(3.8) $$f^p \in C^1(\bar{\Omega}).$$

Let K, k be given positive constants satisfying the following "no loop of zero cost" condition:

(3.9) $$\frac{k}{K} \neq \frac{m - q}{q} \quad \text{for } q = 1, \dots, m-1.$$

Our main result in this section is the following:

<u>Theorem 3.1.</u> Under the assumptions (3.5) - (3.9), we have:

(i) there exists a viscosity solution $u = \{u^1, u^2, \dots, u^m\}$ of the system of inequalities

(3.1) $$\begin{aligned} &\min\{\max\{L^p u^p - f^p,\ u^p - u^{p+1} - K\},\ u^p - u^{p+1} + k\} = 0 \quad \text{in } \Omega, \\ &u^p|_{\partial\Omega} = 0, \qquad p = 1, \dots, m, \quad u^{m+1} = u^1. \end{aligned}$$

each of which belongs to $W^{1,\infty}(\Omega)$ $(1 \leqq r < \infty)$.

(ii) the viscosity solution of (3.1) is unique in $C(\bar{\Omega})$.■

The notion of viscosity solution of (3.1) is defined as follows which is a modification of original definition in Crandall and Lions [2] and Lions [13].

<u>Definition.</u> A set of functions $u = \{u^1, \dots, u^m\}$, $u^p \in C(\bar{\Omega})$, $p = 1, \dots, m$ is said to be a viscosity solution of (3.1) if and only if both (3.9) and (3.10) hold:

(3.9) $$u^{p+1} - k \leqq u^p(x) \leqq u^{p+1} + K \quad \text{in } \Omega \quad \text{and} \quad u^p|_{\partial\Omega} = 0,$$

for $p = 1, \dots, m$.

(3.10) Let $\varphi \in C^2(\Omega)$:

(i) if $u^p - \varphi$ attains its local maximum at x_0 and $u^{p+1}(x_0) - k < u^p(x_0)$, then

$$- a^p_{ij}\varphi_{ij} + b^p_i\varphi_i + c^p u^p \leqq f^p \quad \text{at} \quad x_0,$$

(ii) if $u^p - \varphi$ attains its local minimum at x_0 and $u^p(x_0) < u^{p+1}(x_0) + K$, then

$$- a^p_{ij}\varphi_{ij} + b^p_i\varphi_i + c^p u^p \geqq f^p \quad \text{at} \quad x_0. \blacksquare$$

3.1. Approximate systems and a priori estimates

We consider the following approximate systems:

$$\text{(3.11)} \quad \begin{cases} A^p u^p_\varepsilon + \beta_\varepsilon(u^p_\varepsilon - u^{p+1}_\varepsilon - K) - \beta_\varepsilon(u^{p+1}_\varepsilon - k - u^p_\varepsilon) = f \quad \text{in} \quad \Omega, \\ u^p_\varepsilon|_{\partial\Omega} = 0, \quad p = 1, \ldots, m, \quad u^{m+1} = u . \end{cases}$$

In the following, we shall state some a priori estimates on u^p_ε. The proofs of these estimates are found in [14].

<u>Lemma 3.1.</u> ([14, Lemmas 3.1 and 4.2]) We have

$$\|u^p_\varepsilon\|_{C(\Omega)} + \|u^p_\varepsilon\|_{W^{1,\infty}(\partial\Omega)} \leqq \text{Const.} \blacksquare$$

<u>Lemma 3.2.</u> ([14, Lemma 4.3]) We have

$$\|u^p_\varepsilon\|_{W^{1,\infty}(\Omega)} \leqq \text{Const.} \blacksquare$$

3.2. Proof of Theorem 3.1

Using these estimates, we can find a viscosity solution of (3.1). First, by the same argument as in [14, Theorem 5.2], we have:

<u>Lemma 3.3.</u> There exists $u^p \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$, $p = 1, \ldots, m$, such that $u^{p+1} - k \leqq u^p(x) \leqq u^{p+1} + K$ in Ω. $\blacksquare$

<u>Lemma 3.4.</u> The set of functions $u = \{u^1, \ldots, u^m\}$ which was obtained in Lemma 3.3 is a viscosity solution of (3.1).

Proof. From Lemma 3.3 it is obvious that u satisfies (3.9). We shall prove (3.10). Let $\varphi \in C^2(\Omega)$ and assume that $u^P - \varphi$ takes its local maximum at $x_0 \in \Omega$ and $u^{P+1}(x_0) - k < u^P(x_0)$. Without loss of generality we may assume that x_0 is a strict maximum point of $u^P - \varphi$ in Ω.

Since u^P_ε converges to u^P uniformly, there exists a sequence $\{x^P_\varepsilon\} \subset \Omega$ such that

$$
\text{(3.12)}\quad
\begin{aligned}
&\text{(i)} \quad \lim_{\varepsilon \to 0} x^P_\varepsilon = x_0,\\
&\text{(ii)} \quad u^P_\varepsilon - \varphi \text{ attains its local maximum at } x^P_\varepsilon,\\
&\text{(iii)} \quad u^P_\varepsilon(x^P_\varepsilon) \geqq u^{P+1}_\varepsilon(x_\varepsilon) - k.
\end{aligned}
$$

Note that (3.12-iii) implies that

$$\text{(3.13)}\qquad \beta_\varepsilon(u^{P+1}_\varepsilon - k - u^{P+1}_\varepsilon) = 0 \qquad \text{at } x^P_\varepsilon$$

and

$$\text{(3.14)}\qquad \beta_\varepsilon(u^P_\varepsilon - u^{P+1}_\varepsilon - K) = 0 \qquad \text{in } \Omega.$$

Substituting (3.13) and (3.14) into (3.11), we get

$$\text{(3.15)}\quad - a^P_{ij} u^P_{\varepsilon,ij} + b^P_i\, u^P_{\varepsilon,i} + c^P u^P_\varepsilon \leqq f^P \qquad \text{at } x^P_\varepsilon.$$

On the other hand since (3.12-ii) implies

$$(u^P_\varepsilon - \varphi)_{ij} \leqq 0, \quad (u^P_\varepsilon - \varphi)_i = 0 \qquad \text{at } x^P_\varepsilon,$$

we have

$$\text{(3.16)}\quad - a^P_{ij} \varphi^P_{\varepsilon,ij} + b^P_i\, \varphi^P_{\varepsilon,i} + c^P u^P_\varepsilon \leqq f^P \qquad \text{at } x^P_\varepsilon.$$

Passing to the limit as $\varepsilon \to 0$, we can conclude (3.10-i). The opposite inequality (3.10-ii) is obtained by the same argument.■

To prove the uniqueness of viscosity solutions, we choose a viscosity solution $u = (u^1, \ldots, u^m)$ which was obtained in Lemma 3.4 and fix it.

Lemma 3.5. Let $v = (v^1, \ldots, v^m)$ be any viscosity solution of (3.1). Then we have $u = v$.

Proof. We shall prove $v^p \leqq u^p$ for all $p = 1, \ldots, m$. If not, we can find a convergent sequence $\{x_\varepsilon\}$ from the family of maximum points of $v^p - u_\varepsilon^p$ in $\bar{\Omega}$, which satisfies the following conditions:

There exist p_0, $1 \leqq p_0 \leqq m$, and $x_0 \in \Omega$ such that

$$\text{(i)} \quad \lim_{\varepsilon \to 0} x_\varepsilon = x_0,$$

$$\text{(3.17)} \quad \text{(ii)} \quad (v^{p_0} - u^{p_0})(x_0) = \max_{x,p} (v^p - u^p)(x) > 0,$$

$$\text{(iii)} \quad (v^{p_0} - u_\varepsilon^{p_0})(x_\varepsilon) = \max_{x \in \Omega} (v^{p_0} - u_\varepsilon^{p_0})(x).$$

First we shall claim that

$$\text{(3.18)} \quad u^{p_0}(x_0) = u^{p_0+1}(x_0) + K \quad \text{or} \quad v^{p_0}(x_0) = v^{p_0+1}(x_0) - k.$$

Assume the contrary. From the convergence and continuity of $u_\varepsilon^{p_0}$, we have

$$\text{(3.19)} \qquad u_\varepsilon^{p_0}(x_\varepsilon) \leqq u_\varepsilon^{p_0}(x_\varepsilon) + K.$$

Substituting this into (3.11), we get

$$\text{(3.20)} \qquad - a_{ij}^{p_0} u_{\varepsilon,ij}^{p_0} + b^{p_0} u_{\varepsilon,i}^{p_0} + c^{p_0} u_\varepsilon^{p_0} \geqq f^{p_0} \quad \text{at} \quad x_\varepsilon.$$

On the other hand, since v is a viscosity solution, (3.17-iii) implies

$$\text{(3.21)} \qquad - a_{ij}^{p_0} u_{\varepsilon,ij}^{p_0} + b_i^{p_0} u_{\varepsilon,i}^{p_0} + c^{p_0} v^{p_0} \leqq f^{p_0} \quad \text{at} \quad x_\varepsilon.$$

Combining (3.20) and (3.21) and letting $\varepsilon \to 0$, we have

$c^{p_0}(u^{p_0} - v^{p_0})(x_0) \geqq 0$, which contradicts (3.17-ii).

Hence we have (3.18).

On the other hand, from (3.17-ii), we get

$$v^{p_0}(x_0) - u^{p_0}(x_0) \geqq v^{p_0+1}(x_0) - u^{p_0+1}(x_0).$$

Since this implies

$$0 \geqq v^{p_0}(x_0) - v^{p_0+1}(x_0) - K$$
$$\geqq u^{p_0}(x_0) - u^{p_0+1}(x_0) - K,$$

we have $v^{p_0} = v^{p_0+1} + K$ at x_0 if $u^{p_0} = u^{p_0+1} + K$.

Therefore we can conclude that

$$v^{p_0}(x_0) = v^{p_0+1}(x_0) + K$$

(3.22) or

$$v^{p_0}(x_0) = v^{p_0+1}(x_0) - k.$$

We have also

(3.23) $(v^{p_0} - u^{p_0})(x_0) = (v^{p_0+1} - u^{p_0+1})(x_0) = \max_{x,p} (v^p - u)(x).$

We can repeat this argument starting from the paramarer p_0+1 and can get (3.22) and (3.23) for p_0+2.

Continuing this procedure, we can obtain (3.22) and (3.23) for all $p = 1, \ldots, m$. Summing up (3.22) for $p = 1, \ldots, m$, we have

$$\sum_{i=1}^{m} \kappa^p = 0$$

where $\kappa^p = K$ or $-k$.

Since this contradicts the assumption (3.9), we have $v^p \leqq u^p$ for all $p = 1, \ldots, m$.

The opposite inequality $u^p \leqq v^p$, $p = 1, \ldots, m$ is obtained by the same argument.■

These two lemmas complete the proof of Theorem 3.1.■

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