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On a Variational Inequality to some Nonlinear Evolution Equations

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§1. Introduction

In this paper we are concerned with a variational inequality in the frame work of escalated energy space to some classes of nonlinear evolution equations. Especially, as typical examples, we consider the problems for semilinear parabolic equations and porus media type equations. In the below, we propose initial-value problems and construct functions called modified (m)-solutions which satisfy the problem in a certain sense, eventually we prove these functions satisfy the variational problem defined in this paper. This study is the continuation of the works [1-4].

§2. Evolution equations in a Hilbert space.

Let us consider evolution equations in a real separable Hilbert space H to cover general classes of nonlinear partial differential equations to which we can apply the analysis in this paper. For this purpose, we treat

$$(*)_0 \begin{cases} \dfrac{\partial u(t)}{\partial t} + a(t)N(u(t)) + b(t)u(t) = f(t), \quad 0 < t < T \\ u(0) = u_0 \in H \end{cases}$$

where $u(t) : (0,T] \to H$ is unknown, $a(t)$, $b(t)$, $N(u)$, $f(t)$ are functions satisfying conditions stated below. Let A be a densely defined positive operator in H which has cpmapct inverse A^{-1}. Then we put $\{\phi_j\}$ as a complete system in H which are eigen functions of A^{-1}. Let V be a set of all finite linear combinations of $\{\phi_j\}$ and put $V_k\,(k=1,2,\cdots)$ as a closure of V by the topology of inner product $(A^k\cdot,\cdot\cdot)$ $(A^k\cdot,\cdot\cdot)_H \equiv (\cdot,\cdot\cdot)_k$ and norm $(\cdot,\cdot)_k \equiv |\cdot|_k^2$. We now make some definitions.

<u>Definition 1.</u> A function $u(t) : [0,T] \to H$ is said to be an (m)-solution of $(*)_0$

$\Longleftrightarrow$ (1) $u(t) \in C^1(0,T;\ V_m) \cap L^\infty(0,T;\ V_{m+1})$

(2) $u(t)$ satisfies

$$\begin{cases} \dfrac{d}{dt}(u(t),\phi) + a(t)(N(u(t)),\phi) + b(t)(u(t),\phi) = (f(t),\phi) \\ \qquad 0 < t < T\ , \quad \phi \in V \\ u(0) = u_0 \end{cases}$$

<u>Definition 2.</u> A function $v(t) : [0,T] \to H$ is said to be a modified (m)-solution of $(*)_0$ controlled by $K(t)$ ($\in C[0,T]$, positive in $[0,T]$)

$\Longleftrightarrow$ (1) $v(t) \in C(0,T;\ V_m)$ (2) $v(0) = u_0\ (|u_0|_m^2 < K(0))$

(3) $|v(t)|_m^2 \leqq K(t)$, $t \in [0,T]$ (4) if $^\exists[\tau_1,\tau_2] \subset [0,T]$, s.t. $|v(t)|_m^2 < K(t)$, $\tau_1 \leqq t \leqq \tau_2$ then

$$\frac{d}{dt}(u(t),\phi) + a(t)(N(u(t)),\phi) + b(t)(u(t),\phi) = (f(t),\phi)$$

$\tau_1 \leqq t \leqq \tau_2$, $\phi \in V$.

<u>Definition 3.</u>

$$D_m(K(t)) \equiv \{v(t) \in C(0,T;\ V_m);\ |v(t)|_m^2 \leqq K(t),\ 0 \leqq t \leqq T\}$$

<u>Definition 4.</u>

$$M^+ \equiv \{\Phi(\xi)\ :\ [0,\infty) \to [0,\infty)\ ;\ \text{non-decreasing continuous}\}$$

We propose our problem :

<u>Problem.</u> Find $u(t) \in D_m(K(t))$ which satisfies

$$(*)\quad \begin{cases} (u'(t) + a(t)N(u(t)) + b(t)u(t) - f(t),\ w(t) - u(t))_m \geqq 0 \\ \quad 0 < {}^\forall t < T,\quad {}^\forall w(t) \in D_m(K(t)) \\ u(0) = u_0 \quad (|u_0|_m^2 < K(0)) \end{cases}$$

Here we put assumptions for $a(t)$, $b(t)$, $N(u)$, and $f(t)$.

(H.1.1) $a(t) \in C^1[0,T],\ a(t) \geqq 0,\ b(t) \in C^1[0,T].$

(H.1.2) $f(t) \in C^0(0,T;\ V_{m+1}) \cap C^1(0,T;\ V_m)$

(H.2.0) $N(w)\ :\ V \to \bigcap_{j=1}^{\infty} V_j,\ {}^\exists\alpha > 0\ ;\ N(\lambda w) = \lambda^\alpha N(w),\ \lambda \in R$

(H.2.1) ${}^\exists\Phi_1 \in \mathbb{M}^+\ ;\ (N(w),w)_m \geqq -\Phi_1(|w|_m),\ w \in V$

(H.2.2) ${}^\exists\Phi_2 \in \mathbb{M}^+\ ;\ (N(w),w)_{m+1} \geqq -\Phi_2(|w|_m)|w|_{m+1}^2,\ w \in V$

(H.2.3) ${}^\exists\Phi_3 \in \mathbb{M}^+\ ;\ (N(w),w)_{m+2} \geqq -\Phi_3(|w|_{m+1})|w|_{m+2}^2,\ w \in V$

(H.2.4) ${}^\exists\Phi_4 \in \mathbb{M}^+\ ;\ |N(w)|_m \leqq \Phi_4(|w|_{m+2}),\ w \in V$

(H.3.1) ${}^\exists\Phi_5 \in \mathbb{M}^+\ ;\ (\frac{d}{dt}N(u(t)),\ \frac{d}{dt}u(t))_m \geqq -\Phi_5(|u(t)|_{m+2})|\frac{du(t)}{dt}|_m^2$

(H.3.2) ${}^\exists\Phi_6 \in \mathbb{M}^+\ ;\ |(\frac{d}{dt}N(u(t)),\ \frac{d}{ds}u(s))_{m-1}| \leqq \Phi_6(|u(t)|_{m+2})|\frac{du}{dt}|_m|\frac{du}{ds}|_m$

$u(t) \in C^0(0,T;\ V_{m+2}) \cap C^1(0,T;\ V_m)$

(H.4) $N(w_k) \to N(w)$ in H if $w_k \to w$ in V_m $(k \to \infty)$

Then we have :

<u>Theorem 1.</u> Under the above assumptions and $u_0 \in V_{m+3}$, $|u_0|_m^2 < K$ (positive constant), there exists a function $u(t) \in C(0,T;\ V_{m+1}) \cap C^1(0,T;\ V_m) \cap L^\infty(0,T;\ V_{m+2}) \cap D_m(K)$ satisfying (*). Moreover if we assume

$$\text{(H.3.3)} \quad \exists \Phi_7,\ \Phi_8 \in \mathbb{M}^+ \ ;\ (N(u)-N(v),\ u-v)_m \geqq -\{\Phi_7(|u|_m) + \Phi_8(|v|_m)\}|u-v|_m^2 ,$$

then the uniqueness holds.

Let $\gamma(t) \in C^1[0,T]$ with $\gamma(0) = 1$ and $\gamma(t) > 0$ in $[0,T]$. For $a(t)$, $b(t)$, α and $f(t)$ appeared above, we put

$$\tilde{a}(t) \equiv \tilde{a}(t)(\gamma(t))^{\alpha-1},\ \tilde{b}(t) \equiv (\gamma'(t)/\gamma(t) + b(t)),\ \tilde{f}(t) \equiv \frac{f(t)}{\gamma(t)}$$

and put $K(t) \equiv \frac{K}{\gamma(t)}$, then we know that $\tilde{u}(t) = \frac{u(t)}{\gamma(t)}$ where $u(t)$ is the solution of (*) satisfies

$$(\tilde{*}) \begin{cases} (\tilde{u}'(t) + \tilde{a}(t)(N(\tilde{u}(t)) + \tilde{b}(t)\tilde{u}(t) - \tilde{f}(t),\ \tilde{w}-\tilde{u}(t))_m \geqq 0 \\ \quad 0 < {}^\forall t < T, \quad {}^\forall \tilde{w}(t) \in D_m(K(t)) \\ \tilde{u}(0) = u_0 . \end{cases}$$

Therefore, as a special case we have:

<u>Theorem 2.</u> Under the same assumptions as in Theorem 1, there exists a solution $u(t)$ for the problem :

$$(*) \begin{cases} (u'(t) + N(u(t)),\ w(t) - u(t))_m \geqq 0 \\ \quad 0 < {}^\forall t < T, \ {}^\forall w(t) \in D_m(K(t)) \\ u(0) = u_0 \ \ (|u_0|_m^2 < K(0)). \end{cases}$$

Example. We can apply the result to the case $H = L^2(\Omega)$ (Ω is a bounded smooth domain in $\mathbb{R}^n$), $A = -\Delta$ with $D(A) = \mathring{H}^1(\Omega) \cap H^2(\Omega)$ for the following problem :

$$\begin{cases} u'(t) = \Delta(S(x)u^{2p+1}), & 0 < t < T \\ u(0) = u_0 \end{cases}$$

where p is a positive integer and $S(x)$ is a nonnegative smooth function which and the derivatives vanish on the boundary depending on the power p and dimension so that the function $N(u) = -\Delta(S(x)u^{2p+1})$ satisfies the assumptions (H.2) ~(H.4) and (H.3.3). Though we leave the proofs to the readers, we can refer the ideas in [3].

Note. For the semilinear case

$$\begin{cases} u' = \Delta u + a(x)u^p \\ u(0) = u_0 \end{cases}$$

where $a(x)$ is a smooth function with analogous properties as the above $S(x)$, the discussion is more simple.

§3. Outline of the proof of Theorem 1.

We employ Galerkin approximate procedure with penalty scheme. Let us consider

$$(*)_{\varepsilon j} \begin{cases} (u'_{\varepsilon j} + a(t)N(u_{\varepsilon j}) + b(t)u_{\varepsilon j} + \dfrac{\varepsilon}{K-|u_{\varepsilon j}|_m^2} u_{\varepsilon j}, \phi) = (f(t), \phi) \\ \quad 0 < t < T, \quad \phi \in [\phi_1, \phi_2, \cdots, \phi_j] \\ u_{\varepsilon j}(0) = u_{0j} \equiv \sum_{k=1}^{j} c_k \phi_k \to u_0 \quad \text{strongly in } V_{m+3}. \end{cases}$$

Then from the assumption $|u_0|_m^2 < K$, we have a local smooth solution $u_{\varepsilon j}(t)$, and in order we can prove

$$|u_{\varepsilon j}|_m^2 < K \quad (t \in [0,T]), \qquad |u_{\varepsilon j}|_{m+2} \leqq C(u_0,T) < \infty,$$

$$|u'_{\varepsilon j}|_m \leqq C(u_0,T) < \infty$$

and moreover

$$\frac{\varepsilon}{K-|u_{\varepsilon j}|_m^2} \leqq C(u_0,T) < \infty,$$

$$|u'_{\varepsilon j}(t_1) - u'_{\varepsilon j}(t_2)|_m^2 \leqq C(u_0,T)|t_1-t_2|$$

for $0 \leqq t_1, t_2 \leqq T$. From these estimates we have a function

$$u(t) \in C(0,T;\ V_{m+1}) \cap L^\infty(0,T;\ V_{m+2}) \cap D_m(K)$$

$$u'(t) \in C(0,T;\ V_m)$$

and $\chi(t) \in L^\infty(0,T)$, $\chi(t) \geqq 0$ satisfying

$$(u'(t) + a(t)N(u) + (b(t) + \chi(t)u,\ \phi) = (f(t),\phi)$$

$$0 < t < T\ , \quad \phi \in V$$

$$u(0) = u_0 .$$

Here we note that $\chi(t) = 0$ if

$$|u(t)|_m^2 < K \quad (\text{Cf. } [4]).$$

Thus, we can assert that

$$(u'(t) + a(t)N(u) + b(t)u - f,\ w-u)_m = \chi(t)(u,\ u-w)_m$$

for $w \in D_m(K)$.

Consequently,

$$(u'(t) + a(t)N(u) + b(t)u - f,\ w-u)_m \geqq \chi(t)(|u|_m^2 - |u|_m|w|_m)$$
$$\geqq 0$$

This shows $u(t)$ satisfies the variational problem $(*)$.

Note. We can show that the function $u(t)$ is a modified (m)-solution of $(*)_0$ controlled by $K(t) \equiv K$.

Note. For the uniqueness question, we can answer by putting $U(t)$ as an another solution of $(*)$ with the same properties, and setting

$$w_1(s) = \begin{cases} U(s), & 0 \leqq s \leqq t \\ u(s), & t < s \leqq T, \end{cases} \qquad w_2(s) = \begin{cases} u(s), & 0 \leqq s \leqq t \\ U(s), & t < s \leqq T \end{cases}$$

and substituting them in the equalities.

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