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On Matukuma's Equation and Related Topics.

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1. Introduction

This is a joint work with Wei-Ming Ni (Univ. of Minnesota).

In 1930, Matukuma, an astrophysicist, proposed the following mathematical model to describe the dynamics of a globular cluster of stars,

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0, \quad x \in \mathbf{R}^3, \tag{M}$$

where $p > 1$, u represents the gravitational potential (therefore $u > 0$), $\rho = -\frac{1}{4\pi}\Delta u = \frac{1}{4\pi} \cdot \frac{1}{1+|x|^2} \cdot u^p$ represents the density and $\int_{\mathbf{R}^3} \rho\, dx$ represents the total mass (for details, see [9]). His aim was to improve the model proposed earlier by Eddington in [3],

$$\Delta u + \frac{1}{1+|x|^2} e^{2u} = 0, \quad x \in \mathbf{R}^3, \tag{E}$$

where u, $\rho = -\frac{1}{4\pi}\Delta u = \frac{1}{4\pi} \cdot \frac{1}{1+|x|^2} e^{2u}$, and $\int_{\mathbf{R}^3} \rho\, dx$ represents the gravitational potential, the density, and the total mass, respectively. (Since Matukuma's original paper was written in Japanese, we include a brief derivation of (M) and (E) in Appendix for the reader's convenience.)

Since the globular cluster has the radial symmetry, positive radial entire solution (i.e. solutions with $u(x) = u(|x|) > 0$ for all $x \in \mathbb{R}^3$) are of particular interest, and the equations (M) and (E) reduce to ordinary differential equations

$$\begin{cases} u_{rr} + \frac{2}{r} u_r + \frac{1}{1+r^2} u^p = 0 \quad (r > 0), \\ u(0) = \alpha, \; u_r(0) = 0, \end{cases} \tag{M_α}$$

and

$$\begin{cases} u_{rr} + \frac{2}{r} u_r + \frac{1}{1+r^2} e^{2u} = 0 \quad (r > 0), \\ u(0) = \alpha, \; u_r(0) = 0, \end{cases} \tag{E_α}$$

respectively, where $\alpha > 0$. For each $\alpha > 0$, we denote the the global unique solution of (M_α) (or (E_α)) by $u = u(r;\alpha)$. Studying the structure of solutions of (M_α), Matukuma conjectured

(i) <u>if</u> $p < 3$, <u>then</u> $u(r;\alpha)$ <u>has a finite zero for every</u> $\alpha > 0$,

(ii) <u>if</u> $p = 3$, <u>then</u> $u(r;\alpha)$ <u>is a positive entire solution with finite total mass for every</u> $\alpha > 0$,

(iii) <u>if</u> $p > 3$, <u>then</u> $u(r;\alpha)$ <u>is a positive entire solution</u>

with infinite total mass for every $\alpha > 0$.

In 1938, Matukuma found an interesting exact solution of (M) (see [10])

$$u(r;3^{1/2}) = 3^{1/2}(1+r^2)^{-1/2} \qquad (p=3), \qquad (S)$$

which confirms part of his conjecture.

It turns out that the equation (M_α) is more delicate than Matukuma had expected. In answer to his conjecture, we prove that

(i) if $1 < p < 5$, then $u(r;\alpha)$ has a finite zero for every sufficiently large $\alpha > 0$,

(ii) if $1 < p < 5$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every sufficiently small $\alpha > 0$,

(iii) if $p \geq 5$, then $u(r;\alpha)$ is a positive entire solution with infinite total mass for every $\alpha > 0$.

The conclusions above follow from our main results stated in Section 2 below. (Set $K(r) = 1/(1+r^2)$, $n = 3$, and $\sigma = 0$ in Theorem 2, $\ell = -2$ and $c = 1$ in Theorem 5, and $\sigma = 0$ in Theorem 6.) It is rather interesting to note that not only the exponent p but also the initial value α has vital influence on the behavior of the solution $u(r;\alpha)$.

In studying the equation (M), we consider a more general equation which in particular contains the following equation

$$\Delta u + K(r)\, u^p = 0, \tag{P}$$

where $K > 0$ on $(0,\infty)$ and $p > 1$. Under various decaying hypotheses on K, we establish existence results for positive entire solutions (see Section 2 below).

The equation (P) has come up both in geometry and in physics, and has been a subject of extensive studies for some time. When $K \equiv 1$, equation (P) is known as the Lane-Emden equation in astrophysics. (It is also sometimes referred to as the Emden-Fowler equation due to the fundamental contribution of Fowler [4] on this equation.) In this context, u corresponds to the density of a single star, and positive radial solutions of (P) in balls with zero Dirichlet boundary data are of particular interest (see [1]). In 1973, Hénon [5] proposed a model to study the 'rotating steller systems' and derived (P) with $K(r) = r^{\ell}$. This case (and its generalizations) has been studied by various authors (see [13] and the references therein). At the critical exponent $p = (n+2)/(n-2)$, equation (P) arises in the problem of finding conformal Riemannian metrics with prescribed scalar curvatures K. In this context, u corresponds to the conformal factor and the exact solution (S) has also been known for quite some time which represents the usual metric on the standard sphere.

The first general and systematic study of (P) seems due to Ni [12]. It is proved there that if K grows faster than or equal to $r^{(n-2)(p-1)-2}$ at ∞, then (P) possesses no positive entire solutions. (In fact, a slightly more general integral condition is obtained in [12].) Existence results in the case K decays faster than r^{-2} at ∞ have also been established in [12] and slightly improved by [6]-[8], [11] later. (In fact, all these results deal with $K = K(x)$, not just radial cases.) Other existence and non-existence results are also discussed in [12]. Recently, Ding and Ni [2] have proved a striking result for the case $p = (n+2)/(n-2)$ and K is a perturbation of the constant 1. However almost all the methods developed in those papers do not seem to apply to the case where K decays slower than or equal to r^{-2} at ∞ (which includes (M)) and $p < (n+2)/(n-2)$ (except Proposition 4.1 in [12]). Our results (see Section 2 below) cover most of the cases left open by previous works.

For the Eddington's equation (E), it is seen that it has no positive entire solutions. It is interesting to note that the Eddington's equation (E) also comes up in Riemannian geometry. When $n = 2$, it is related to problem of finding conformal metrics with prescribed Gaussian curvatures (in this case, the solution u is not required to be positive in the geometric problem).

2. Main results

We consider the equation

$$u_{rr} + \frac{n-1}{r} u_r + K(r)(u^+)^p = 0 \quad (r > 0), \quad u(0) = \alpha > 0 \quad (P_\alpha)$$

where $u^+ = \max\{u, 0\}$. We shall assume throughout this paper that $n \geq 3$ although many of our results and methods do apply to the case $n < 3$.

We now collect the hypotheses which will be assumed under various circumstance (but not simultaneously). We introduce

$$(K.1) \quad \begin{cases} K \in C((0,\infty)), \ K \geq 0 \ \text{ on } \ (0,\infty), \ \text{ and} \\ K(r) = O(r^\sigma) \ \text{ at } \ r = 0, \end{cases}$$

$$(K.2) \quad K(r) = c\, r^\sigma + o(r^\sigma) \ \text{ at } \ r = 0,$$

$$(K.3) \quad \liminf_{r\to\infty} \{r^{-\ell} K(r)\} > 0,$$

$$(K.4) \quad K(r) = c\, r^\ell + o(r^\ell) \ \text{ at } \ r = \infty,$$

$$(K.5) \quad \lim_{r\to\infty} \{r(r^{-\ell} K(r))_r\} = 0,$$

$$(K.6) \quad (r^\sigma K(r))_r \leq 0 \ \text{ on } \ (0,\infty),$$

where σ, ℓ and c are constants satisfying

$$\sigma > -2, \ \ell \geq -2, \ c > 0.$$

Theorem 1. *Suppose that* (K.1) *holds. Then there exists a unique solution* $u(r;\alpha) \in C([0,\infty)) \cap C^2((0,\infty))$ *of* (P_α), *and* $u = u(r;\alpha)$ *satisfies the following identity,*

$$\frac{n-2}{2} R^{n-1} u(R) u_r(R) + \frac{1}{2} R^n u_r(R)^2 + \frac{R^n}{p+1} K(r)\, u(r)^{p+1}$$
$$= \int_0^R \left\{ \left(\frac{n}{p+1} - \frac{n-2}{2} \right) (u^+)^{p+1} + \frac{1}{p+1} r K_r(r) (u^+)^{p+1} \right\} r^{n-1}\, dr \qquad (2.1)$$

where R *is an arbitrary positive number.*

Remark 2.1. It is well-known that such Pohozaev-type identity holds for the case $K \in C([0,\infty))$. The above theorem claims that it also holds under the weaker assumption (K.1).

Theorem 2. *Suppose that* (K.1) *and* (K.2) *hold, and* $p < (n+2+2\sigma)/(n-2)$. *Then there exists* $\alpha_1 > 0$ *such that for every* $\alpha \geq \alpha_1$, $u(r;\alpha)$ *has a finite zero* on $[0,\infty)$.

Theorem 3. *Suppose that* (K.1) *holds with* $\sigma = 0$, $K(r) = O(r^{-2})$ *at* $r = \infty$ *and* $p > 1$. *Then there exists* α_0 *such that for every* $\alpha \in (0,\alpha_0]$, $u(r;\alpha)$ *is positive on* $[0,\infty)$ *and*

$$\lim_{r\to\infty} u(r;\alpha) = 0$$

Theorem 4. *Suppose that* (K.1), (K.3) *and* (K.5) *hold, and* $p > (n+2+2\ell)/(n-2)$. *Then there exists* α_0 *such that for every* $\alpha \in (0,\alpha_0]$, $u(r;\alpha)$ *is positive on* $[0,\infty)$ *and* $\lim_{r\to\infty} u(r;\alpha) = 0$.

Theorem 5. *Suppose that* (K.1), (K.4) *and* (K.5) *hold, and* $p > (n+2+2\ell)/(n-2)$. *Then there exists* $\alpha_0 > 0$ *such that for every* $\alpha \in (0,\alpha_0]$, $u(r;\alpha)$ *is positive on* $[0,\infty)$, $\lim_{r\to\infty} u(r;\alpha) = 0$, *and* $\int_0^\infty K(r)u(r)^p r^{n-1} dr = \infty$.

Theorem 6. *Suppose that* (K.1), (K.6) *hold, and* $p \geq (n+2+2\sigma)/(n-2)$ *Then for every* $\alpha > 0$, $u(r;\alpha)$ *is positive on* $[0,\infty)$. *Moreover if* $(r^{-\sigma}K(r))_r \gneq 0$ *on* $(0,\infty)$ *or* $p > (n+2+2\sigma)/(n-2)$, *then* $\int_0^\infty K(r)u(r)^p r^{n-1} dr = \infty$.

Remark 2.2. All the exponents appeared in Theorems 1-6 are sharp. This may be seen by constructing suitable examples.

Remark 2.3. These results are announced in [14]. The details are shown [15].

3. Concluding remarks

We have given partial answers to the Matukuma's conjecture in Section 1. Concerning the structure of the solutions of (M_α) with $1 < p < 5$, our analysis is still incomplete and we suspect that the following conjecture hold : for every $1 < p < 5$, there exists a unique $\alpha_p > 0$ such that

(i) <u>if</u> $\alpha > \alpha_p$, <u>then</u> $u(r;\alpha)$ <u>has a finite zero</u>,

(ii) <u>if</u> $\alpha = \alpha_p$, <u>then</u> $u(r;\alpha)$ <u>is a positive entire solution with finite total mass</u>,

(iii) <u>if</u> $0 < \alpha < \alpha_p$, <u>then</u> $u(r;\alpha)$ <u>is a positive entire solution with infinite total mass</u>.

In other words, we conjecture that for $1 < p < 5$ there is a unique positive entire solution of (M_α) with finite total mass.

We have done several numerical computations for the initial value problem (M_α) by using Runge-Kutta method. The independent variable transformation, $r = s/(1-s)$, is used for reducing the interval $[0,\infty)$ to $[0,1)$. Fig. 1 (resp. Fig. 2) shows the spatial pattern of $U(s;\alpha) = u(r;\alpha)$, $r = s/(1-s)$ (resp. $V(s;\alpha) = u(r;\alpha)/\alpha$, $r = s/(1-s)$) , and the zero, where $u(r;\alpha)$ is a solution of (M_α) with $n = 3$, $p = 3$, and the interval $[0,\infty)$ is reduced to $[0,1)$.

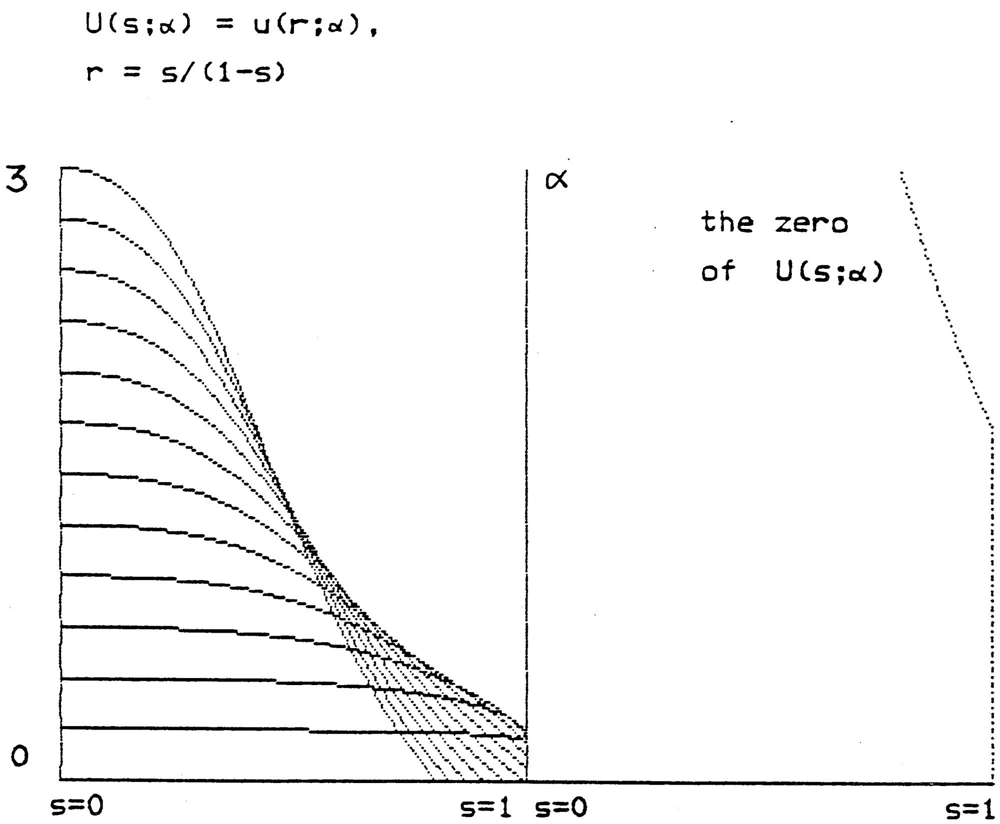

Fig. 1

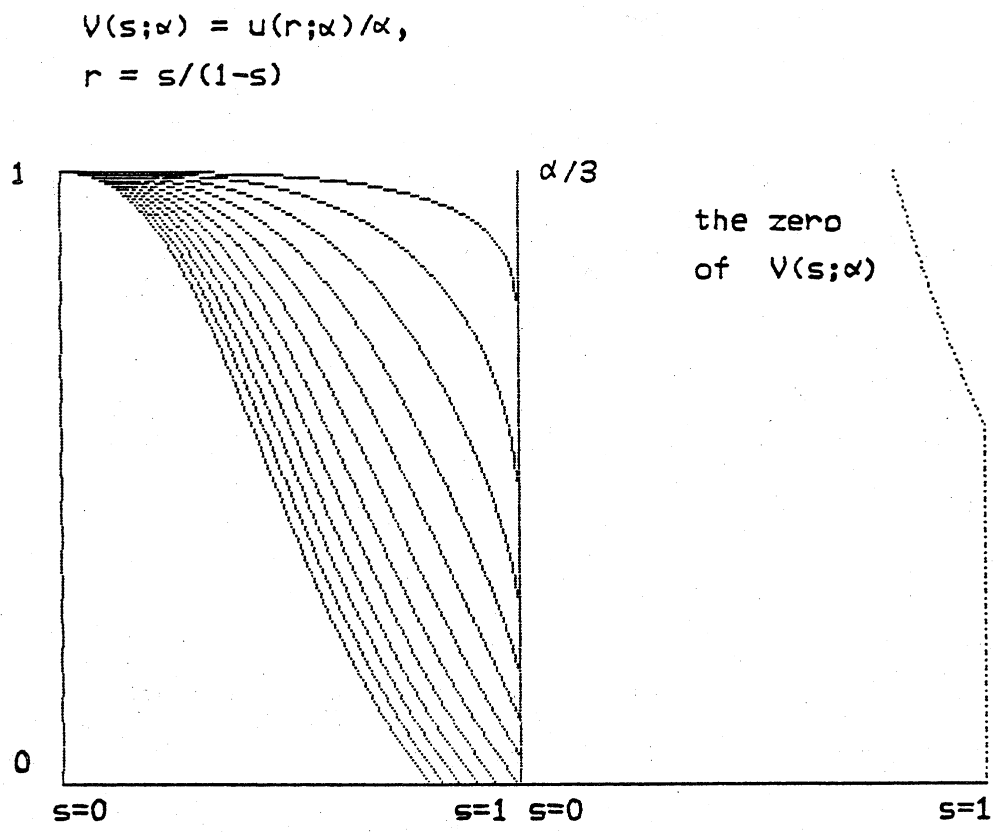

Fig. 2

Appendix.

In this appendix we give a brief explanation of a derivation of the Eddington's equation and the Matukuma's equation, following Matukuma's paper [9]. Let $\varphi(x)$ be the gravitational potential of a globular cluster, and $\rho(x)$ be the density, where $x \in \mathbb{R}^3$. Then we have

$$\varphi(x) = \int_{\mathbb{R}^3} \frac{\rho(\xi)}{|x - \xi|} d\xi \tag{A.1}$$

which implies

$$\Delta\varphi = -4\pi\rho. \tag{A.2}$$

From the observation of a globular cluster it seems that ρ is radial and

$$\rho = \rho(r) \downarrow 0 \quad \text{as} \quad r \uparrow \infty,$$

where $r = |x|$. Several physical theories have been proposed to understand the dynamics of it. Here we shall focus on two theories, one is due to Eddington and the other is Matukuma's theory. The essential point is to drive the relation between ρ and φ. After the relation is determined, we obtain the equation for φ. Denote the distribution function of stellar velocities by f, then

$$\rho(x) = \int_{\mathbb{R}^3} f(v,x)\, dv. \tag{A.3}$$

Eddington assumed that the globular cluster has the following properties,

(i) <u>it is stationary (independent of the time)</u>,

(ii) <u>it has radial symmetry</u>,

(iii) <u>the distribution of stellar velocities everywhere follows Schwarzschild's ellipsoidal law</u>.

After some mathematical and physical considerations, he obtained that

$$f(v,x) = a \exp\{h^2(2\phi-|v|^2) - k^2|x \times v|^2\}, \quad x, v \in \mathbb{R}^3 \qquad (A.4)$$

where a, h, and k are positive constants. From (A.3) and (A.4), we see that

$$\rho = \text{const.}\ (h^2+k^2r^2)^{-1} \exp(2h^2\phi), \qquad (A.6)$$

and by (A.2) and (A.6) we have

$$\Delta\phi + C\ (h^2+k^2r^2)^{-1} \exp(2h^2\phi) = 0, \qquad (A.7)$$

where C is a positive constant. Putting

$$r = hk^{-1}s, \quad \phi = h^{-2}(\psi - 2^{-1}\log(h^2k^{-2}c)) \qquad (A.8)$$

we have

$$\Delta\psi + (1+s^2)^{-1} e^{2\psi} = 0. \qquad (A.9)$$

Thus we get the Eddington's equation (E). It is seen that (E_α) has no entire positive solution with finite total mass.

Therefore Matukuma tried to improve the Eddington's theory. Matukuma assumed that the globular cluster had the properties (i),(ii) and

(iii)′ the velocity distribution function f has the followinf form

$$f = f(2\varphi - |v| - k\,|x \times v|\,), \quad x, v \in \mathbf{R} \qquad (A.10)$$

and

$$f(\xi) = \begin{cases} \xi^m(\alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \cdots) & \text{for } \xi \geq 0, \\ 0 & \text{for } \xi \leq 0, \end{cases} \qquad (A.11)$$

where $m > 0$, $\alpha_0, \alpha_1, \alpha_2, \cdots \geq 0$ are constants.

Then we have, from (A.3),(A.10) and (A.11), that

$$\rho = \frac{1}{1+k^2r^2}\,\varphi^{m+3/2}\,(\beta_0 + \beta_1\varphi + \beta_2\varphi^2 + \cdots), \qquad (A.12)$$

where β_i $(i=0,1,2,\cdots)$ are non-negative constants. Thus it follows from (A.2) and (A.3) that

$$\Delta\varphi + \frac{\varphi^{m+3/2}}{1+k^2r^2}\{\beta_0 + \beta_1\varphi + \beta_2\varphi^2 + \cdots\} = 0,$$

where $m > 0$. After some change of variables, we get the equation

$$\Delta\psi + \frac{\psi^p}{1+r^2}\{c_0 + c_1\psi + c_2\psi^2 + \cdots\} = 0,$$

where $p > 1$, $c_0, c_1, \cdots \geq 0$ are non-negative constants. As a special case, setting $c_0 = 1$, $c_1 = c_2 = \cdots = 0$, we obtain the Matukuma's equation (M).

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