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On the Homology Groups of the Mapping Class Groups of Orientable Surfaces with Twisted Coefficients

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1. Introduction. Let Σ_g be a closed orientable surface of genus g and let $M_g = \pi_0 \mathrm{Diff}_+ \Sigma_g$ be its mapping class group. Also let $M_{g,*}$ and $M_{g,1}$ respectively be the mapping class groups of Σ_g relative to the base point $* \in \Sigma_g$ and an embedded disc $D^2 \subset \Sigma_g$. It is known that these groups are perfect for all $g \geq 3$ (see [2,3]) and Harer determined the second homology group of them in his fundamental paper [2]. The purpose of the present note is to announse our results on the homology groups of them with coefficients in the first homology group $H_1(\Sigma_g, \mathbb{Z})$ of Σ_g on which the mapping class groups act naturally.

2. Low dimensional homologies. First we consider the first homology. The results of our previous paper [7] imply

Theorem 1. (i) $H_1(M_g; H_1(\Sigma_g, \mathbb{Z})) \cong \mathbb{Z}/2g-2 \qquad (g \geq 2)$.

(ii) $H_1(M_{g,1}; H_1(\Sigma_g, \mathbb{Z})) \cong H_1(M_{g,*}; H_1(\Sigma_g, \mathbb{Z})) \cong \mathbb{Z} \qquad (g \geq 2)$.

These groups are detected by the crossed homomorphism $f: M_{g,*} \times H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{Z}$ defined in [7]. Next the second homology group is given by the following Theorem which is one of our main results.

Theorem 2. (i) $H_2(M;H_1(\Sigma_g,\mathbb{Z})) = 0$ for all $g \geq 12$, where M stands for any of M_g, $M_{g,*}$ or $M_{g,1}$.
(ii) $H_2(M;H_1(\Sigma_g,\mathbb{Q})) = 0$ for all $g \geq 9$, where M is the same as above.

Corollary 3. $H^2(M_g;H^1(\Sigma_g,\mathbb{Z})) \cong \mathbb{Z}/2g-2$ $(g \geq 9)$.

The group $H^2(M_g;H^1(\Sigma_g,\mathbb{Z}))$ has the following geometric meaning. Choose a generator $\theta \in H^2(M_g;H^1(\Sigma_g,\mathbb{Z}))$. To any oriented differentiable Σ_g-bundle $\pi: E \to X$, we have associated in [8] a family of Jacobian manifolds $\pi': J' \to X$, which is a flat T^{2g}-bundle over X with structure group $H_1(\Sigma_g,\mathbb{Z}/2g-2) \rtimes Sp(2g,\mathbb{Z})$, and a fibrewise embedding $j': E \to J'$ which induces an isomorphism on the first integral homology on each fibre (topological version of Earle's embedding theorem [1]). We have

Proposition 4. (compare with [1], §8). Let $\pi: E \to X$ be an oriented Σ_g-bundle. Then the associated family of Jacobian manifolds $\pi': J' \to X$ has a cross-section if and only if $h^*(\theta)$ vanishes in $H^2(\pi_1(X);H^1(\Sigma_g,\mathbb{Z}))$ where $h: \pi_1(X) \to M_g$ is the holonomy homomorphism of the given Σ_g-bundle and $\pi_1(X)$ acts on $H^1(\Sigma_g,\mathbb{Z})$ naturally.

Corollary 5. The natural homomorphism $\pi: M_{g,*} \to M_g$ induces an isomorphism $H_3(M_{g,*},\mathbb{Z}) \cong H_3(M_g,\mathbb{Z})$ for all $g \geq 10$.

(It is easy to show that the homomorphism $H_3(M_{g,*},\mathbb{Z}) \to H_3(M_g,\mathbb{Z})$ is surjective for all $g \geq 3$).

3. Outline of the proof of Theorem 2. The proof of Theorem 2 is based on Harer's method [2] of computing the second homology group of the mapping class groups which is in turn based on the paper [5] of Hatcher and Thurston. As in [2], let X_2 be the (slightly modified) Hatcher-Thurston complex of the compact surface $\Sigma_g - \mathring{D}^2$ with one boundary component. It is simply connected and the mapping class group $M_{g,1}$ acts naturally on it cellularly. Harer defines an $M_{g,1}$-subcomplex $Y_2 \subset X_2$, which is still simply connected and the number of two-cells in its $M_{g,1}$-orbit is reduced drastically to six. Then he adds two types of three-cells to Y_2 to obtain Y_3 and he uses the standard technique of spectral sequences to deduce his result mentioned above.

We start with Harer's complex Y_3 (with a slight modification of the definition of one of the three-cells because the boundary of his original three-cell is not contained in Y_2). We add five more types of three-cells to Y_3 to obtain Y_3' and then compute the standard spectral sequence which converges to $H_*(Y_3' \times_M K; H_1(\Sigma_g, \mathbb{Z}))$ where K is a contructible $M_{g,1}$-complex. We first construct enough cycles whose homology classes generate $H_2(Y_3' \times_M K; H_1(\Sigma_g, \mathbb{Z}))$ and then prove that these cycles are all homologous to zero in $H_2(M_{g,1}; H_1(\Sigma_g, \mathbb{Z}))$. The necessary computations for that are very complicated and lengthy compared with the corresponding ones in the case of constant coefficients. The condition $g \geq 12$ in the statement of Theorem 2 reflects this situation. Details will be given in [9].

4. Non trivialities of higher homology groups. Harer's stability theorem [3] and Proposition 3-1 of [6] imply

Proposition 6. (i) The homology group $H_k(M_g;H_1(\Sigma_g,\mathbb{Q}))$ is independent of g in the range $g \geq 3(k+1)$.
(ii) For each prime number p, the homology group $H_k(M_g;H_1(\Sigma_g,\mathbb{Z}/p))$ is independent of g provided $g \geq 3(k+1)+1$ and p does not divide $2g-2$.

Remark 7. (i) In the above statements we understand all the homology groups to be abstract vector spaces over $\mathbb{Q}$ or $\mathbb{Z}/p$. There seems to be no canonical isomorphisms between them. One reason for this is the fact that the Gysin homomorphism (see below) is an unstable operation, namely it depends essentially on the genus.
(ii) The statement (i) in the above Proposition does not hold if we replace $H_1(\Sigma_g,\mathbb{Q})$ by $H_1(\Sigma_g,\mathbb{Z})$ (see Theorem 1, (i)).

Now we consider the cohomology group $H^*(M_g;H^1(\Sigma_g,\mathbb{Q}))$ instead of homology because it is more convenient for the statement of our non-triviality result. As in [6], let $e \in H^2(M_{g,*},\mathbb{Z})$ be the Euler class of the central extension $0 \to \mathbb{Z} \to M_{g,1} \to M_{g,*} \to 1$. We define a cohomology class $e_i \in H^{2i}(M_g,\mathbb{Z})$ by setting $e_i = \pi_*(e^{i+1})$ where $\pi_*\colon H^{2i+2}(M_{g,*},\mathbb{Z}) \to H^{2i}(M_g,\mathbb{Z})$ is the Gysin homomorphism induced from the projection $\pi\colon M_{g,*} \to M_g$. We call e_i the i-th characteristic class of oriented surface bundles. We also use the same letter e_i for the cohomology class $\pi^*(e_i) \in H^{2i}(M_{g,*},\mathbb{Z})$. Making an essential use of Harer's stability theorem [3], we have proved in [6]

Theorem 8. The homomorphism

$$\mathbb{Q}[e,e_1,e_2,\ldots] \longrightarrow H^*(M_{g,*},\mathbb{Q})$$

is injective up to degree $\frac{1}{3}g$.

Now as was shown in [6] (Proposition 3-1), the Hochschild-Serre spectral sequence $\{E_r^{p,q}, d_r\}$ for the rational cohomology group of the extension $1 \to \pi_1(\Sigma_g) \to M_{g,*} \to M_g \to 1$ collapses so that we have $E_\infty^{p,q} = E_2^{p,q} = H^p(M_g; H^q(\Sigma_g, \mathbb{Q}))$. Hence if we set

$$K_n(g) = \mathrm{Ker}\ (\pi_*: H_n(M_{g,*}, \mathbb{Q}) \to H_{n-2}(M_g, \mathbb{Q})),$$

then we have a short exact sequence

$$0 \to E_\infty^{n,0} = H^n(M_g, \mathbb{Q}) \xrightarrow{\pi^*} K_n(g) \xrightarrow{q} E_\infty^{n-1,1} = H^{n-1}(M_g; H^1(\Sigma_g, \mathbb{Q})) \to 0.$$

Now for each natural number i, the cohomology class

$$(2g-2)e^{i+1} + ee_i \in H^{2i+2}(M_{g,*}, \mathbb{Q})$$

is contained in $K_{2i+2}(g)$. Hence we can define an element $v_i \in H^{2i+1}(M_g; H^1(\Sigma_g, \mathbb{Q}))$ by

$$v_i = q((2g-2)e^{i+1} + ee_i).$$

The cup product of v_i with any element of $H^*(M_g, \mathbb{Q})$ belongs to $H^*(M_g; H^1(\Sigma_g, \mathbb{Q}))$ so that we have a homomorphism

$$\mathbb{Q}[e_1, e_2, \ldots]\langle v_1, v_2, \ldots\rangle \to H^*(M_g; H^1(\Sigma_g, \mathbb{Q})),$$

where the left hand side stands for the free $\mathbb{Q}[e_1, e_2, \ldots]$-module with basis $v_1, v_2, \ldots$. With these definitions and notations, we have the following non-triviality result.

Theorem 9. The homomorphism

$$\mathbb{Q}[e_1, e_2, \ldots]\langle v_1, v_2, \ldots\rangle \to H^*(M_g; H^1(\Sigma_g, \mathbb{Q}))$$

is injective up to degree $\frac{1}{3}g - 1$.

The result of Harer-Zagier [4] implies that the above homomorphism is far from being surjective. However it seems to be reasonable to make the following

Conjecture 10. The homomorphism in Theorem 9 is an isomorphism in the same range.

We can also formulate similar statements to Theorem 9 and Conjecture for the group $M_{g,*}$, but here we omit them.

Details will appear elsewhere.

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