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A Note on the Modified Zero-Fourth-Order Cumulant Approximation

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The modified zero-fourth-order cumulant approximation (MZCA) has been developed by Tatsumi and his collaborators with extensive applications to various turbulent fields.[1)] In the present note the necessary and sufficient condition for the validity of the method is given.

For illustration, we first consider the following exactly solvable model:

$$\dot{q} + ibq = 0 \quad , \tag{1}$$

where b is a Gaussian random variable with zero mean and take constant value in any one realization. Namely

$$q = \exp(-ibt)$$

and its average is obtained as

$$\langle q \rangle = \exp(-\sigma^2 t^2/4) \tag{2}$$

in which $\sigma^2/2 = \langle b^2 \rangle$, and any moment can be calculated exactly.

If eq.(1) is solved by closure, one has

$$\langle \dot{q} \rangle + i\langle bq \rangle = 0 \quad , \tag{3}$$

$$\frac{d}{dt}\langle bq \rangle + i\langle b^2 q \rangle = 0 \quad , \tag{4}$$

while $\langle b^2 q \rangle = \langle b^2 q \rangle_c + \langle b^2 \rangle \langle q \rangle$. The formal solution of eq.(4) is given by

$$\langle bq \rangle = -\int_0^t G(t-t')i\langle b^2 q(t') \rangle dt'$$

where G(t) is the Green's function equal to 1. Following the procedure of the modified zero-fourth order cumulant approximation, one may solve (4) as

$$\langle bq \rangle = -\int_0^t G(t-t')dt' \; i\langle b^2 \rangle \langle q(t) \rangle$$

$$= -it\langle b^2 \rangle \langle q \rangle \quad . \tag{5}$$

Introducing eq.(5) into eq.(3) we obtain the exact solution (2). On the other hand $\langle bq \rangle$ and $\langle b^2 q \rangle$ can be calculated exactly, which are expressed as

$$\langle bq \rangle = -it\langle b^2 \rangle \langle q \rangle \quad , \tag{6}$$

$$\langle b^2 q \rangle = -it\langle b^2 \rangle \langle bq \rangle + \langle b^2 \rangle \langle q \rangle \quad . \tag{7}$$

Equation (6) demonstrates that eq.(5) given by the MZCA yields the exact solution for $\langle bq \rangle$, while eq.(7) shows that $\langle b^2 q \rangle_c = -it\langle b^2 \rangle \langle bq \rangle$. Substituting eqs.(6) and (7) into eqs.(3) and

(4) we have

$$\frac{d}{dt}\langle q\rangle + \langle b^2\rangle t\langle q\rangle = 0 \quad ,$$

$$\frac{d}{dt}\langle bq\rangle + \langle b^2\rangle t\langle bq\rangle = -i\langle b^2\rangle\langle q\rangle \quad ,$$

hence

$$-i\langle b^2 q\rangle_c = [\langle \dot{q}\rangle/\langle q\rangle]\langle bq\rangle \quad . \qquad (8)$$

In other words, in eq.(4) written as

$$\frac{d}{dt}\langle bq\rangle + i\langle b^2 q\rangle_c = -i\langle b^2\rangle\langle q\rangle \quad , \qquad (4)'$$

if (8) is valid, it takes the form

$$\frac{d}{dt}\langle bq\rangle - [\langle \dot{q}\rangle/\langle q\rangle]\langle bq\rangle = -i\langle b^2\rangle\langle q\rangle \quad ,$$

consequently, $\langle q\rangle$ is a solution of the homogeneous equation so that the Green's function $G(t-t')$ becomes $\langle q(t)\rangle/\langle q(t')\rangle$ to give

$$\begin{aligned}\langle bq\rangle &= \int_0^t G(t-t')[-i\langle b^2\rangle\langle q(t')\rangle]dt' \\ &= \int_0^t \langle q(t)\rangle/\langle q(t')\rangle[-i\langle b^2\rangle\langle q(t')\rangle]dt' \\ &= -it\langle b^2\rangle\langle q\rangle\end{aligned}$$

that is the result obtained by the procedure of the MACA. On the other hand, if the solution of eq.(4)' is given by eq.(5), substituting eq.(5) for $\langle bq\rangle$ in eq.(4)' yields eq.(8). Hence for eq.(1), eq.(8) is the necessary and the sufficient condition for the exact validity of the MZCA. Here the two points should be marked.

(i) In view of eq.(3), the MZCA is exact, if and only if the cumulant $\langle b^2 q\rangle_c$ is given by eq.(8) as a certain combination of $\langle q\rangle$ and $\langle bq\rangle$.

(ii) Under eq.(8), eq.(4)' becomes a differential equation of resonance type so that the solution becomes secular type.

It is also noticed that the crucial point of the above argument on the MZCA is in solving eq.(4). In this regard, in order to extend the result to general case it will be of help to show the following theorem for the equation

$$(\frac{d}{dt} + \nu)X + gX = f \quad . \tag{9}$$

Theorem:

if X is solved as

$$X = \int_0^t \exp[-\nu(t-t')]dt' \, f(t) \tag{10}$$

then

$$g = -i\dot{f}/f \quad , \tag{11}$$

conversely, if (11) is valid, we get (10). The proof is straightforward.

We now consider the Navier-Stokes equation. Using the same notation as was used by Tatsumi in Ref.1), we have the cumulants hierarchy for the homogeneous turbulence:

$$[\frac{\partial}{\partial t} + \nu(k^2+k'^2)]C^{(2)}_{pq}(k,\ k',\ t)$$

$$= i \sum_{[k,k']}^{[p,q]} k_\alpha \Delta_{p\beta}(k) \int C^{(3)}_{\alpha q\beta}(k-h,\ k',\ h,\ t)dh \quad ,$$

$$(k + k' = 0) \tag{12}$$

$$[\frac{\partial}{\partial t} + \nu(k^2+k'^2+k''^2)]C^{(3)}_{pqr}(k,\ k',\ k'',\ t)$$

$$= i \sum_{[k,k',k'']}^{[p,q,r]} k_\alpha \Delta_{p\beta}(k) [\frac{1}{2} \int C^{(4)}_{\alpha qr\beta}(k-h,\ k',\ k'',\ h,\ t)dh$$

$$- C^{(2)}_{\alpha q}(-k',\ k',\ t)C^{(2)}_{r\beta}(k'',\ -k'',\ t)] \quad ,$$

$$(k + k' + k'' = 0) \quad . \tag{13}$$

Here $C^{(2)}_{pq}$ $C^{(3)}_{pqr}$ are the second and third cumulants respectively, the suffices refere to components of velocity vectors, k, k' and k" are the 3d wave-number vectors and the summations are taken over all simultaneous permutations of (p, q, r) and (k, k', k") while

$$\Delta_{p\beta}(k) = \delta_{p\beta} - k_p k_\beta / k^2 \quad .$$

Introduce $f_{pqr}(k,\ k',\ k'',\ t)$ and $g_{pqr}(k,\ k',\ k'',\ t)$ by

$$f_{pqr}(k,\ k',\ k'',\ t)$$

$$= -i \sum_{[k,k',k'']}^{[p,q,r]} k_\alpha \Delta_{p\beta} C^{(2)}_{\alpha q}(-k',\ k',\ t)C^{(2)}_{r\beta}(k'',\ -k'',\ t) \quad ,$$

$$-g_{pqr}(k,\ k',\ k'',\ t)C^{(3)}_{pqr}(k,\ k',\ k'',\ t)$$

$$= -i \sum_{[k,k',k'']}^{[p,q,r]} k_\alpha \Delta_{p\beta} \int C^{(4)}_{\alpha q r \beta}(k-h,\ k',\ k'',\ h,\ t)dh \quad .$$

Then eq.(13) is written as

$$[\frac{\partial}{\partial t} + \nu(k^2+k'^2+k''^2)]C^{(3)} + gC^{(3)} = f \qquad (13)'$$

for p, q, r and k, k', k" with k + k' + k" = 0, which takes the same form as eq.(9); therefore, if and only if

$$g_{pqr}(k,\ k',\ k'',\ t) = -\dot{f}_{pqr}(k,\ k',\ k'',\ t)/f_{pqr}(k,\ k',\ k'',\ t) \quad ,$$

$C^{(3)}$ is given by the MZCA, that is,

$$C^{(3)}_{pqr} = \int_0^t \exp[-\nu(k^2+k'^2+k''^2)(t-t')]dt' f_{pqr}(k,\ k',\ k'',\ t) \quad .$$

Reference

1) T. Tatsumi: Advances in Applied Mechanics, ed, N.C. Freeman (Academic Press, New York, 1980), Vol.20, pp.39-133.