数理解析研究所講究録
第 607 巻 1987 年 1-4

Problems in Finite Extremal Set Theory

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We will discuss three groups of problems in finite extremal set theory in this synoptic. Henceforth, we assume everything finite unless otherwise stated.

1. A partially ordered set P is ranked if there exists a function $r: P \to \{0, 1, 2, \ldots\}$ such that $r(x) = 0$ for minimal elements x in P and $r(y) = r(x) + 1$ if y covers x in P. We call $r(x)$ the rank of x. Let P_k denote the set of all rank k elements. P is said to be Sperner if $\max_k |P_k| = \max\{|A| : A \text{ is an antichain in } P\}$. The common value is called the Sperner number. An order-filter $F = \langle a_1, a_2, \ldots, a_k \rangle$ generated by $a_1, a_2, \ldots, a_k$ is the set of elements b above some a_i. We are now only interested in the case when $r(a_1) = r(a_2) = \ldots = r(a_k) = t$.

Let B^n denote the Boolean algebra of order n, which consists of all 2^n subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. In B^n, if $r(a_1) = \ldots = r(a_k) = 1$, then we denote $\langle a_1, \ldots, a_k \rangle$ by $C(n,k)$.

Lih [4] generalizes Sperner's classical result to show

Theorem 1. $C(n,k)$ is Sperner and its Sperner number is

$$\binom{n}{\lceil n/2 \rceil} - \binom{n-k}{\lceil n/2 \rceil}.$$

A ranked partially ordered set P has the LYM property if every antichain A in P satisfies the inequality:

$$\sum_{x \in A} (1/|P_{r(x)}|) \leq 1.$$

LYM is stronger than Sperner. Griggs [2] strengthens Lih's results to show, among other things, the following theorem.

Theorem 2. $C(n,k)$ is LYM and the maximum-sized antichains in $C(n,k)$ are

1. $C_{\lceil 1/2n \rceil}$,
2. $C_{1/2(n-1)}$, for odd n and $i \geqq 1/2(n+3)$, and
3. $C_{1/2(n+2)}$, for even n and $k = 1$.

Lih [4] also gives the conjecture that if $F = \langle a_1, \ldots, a_k \rangle \subset B^n$ and all a_i's are of a fixed rank t, then F is Sperner.

Griggs [2] shows that this conjecture is false. However, the most sweeping counterexamples are given by Zha [5], which shows that Lih's conjecture is false for every $t \geqq 4$. Nevertheless, Zhu [6] establishes the truth when $t = 2$ and n is odd.

Problem 1. Is Lih's conjecture true when (i) $t = 3$, n odd, and (ii) $t = 2, 3$ and n even?

Zha [5] proves several positive partial results. For instance, the conjecture holds if $t = 2$, n even, and $a_i \cap a_j \neq \phi$ for any i and j.

Problem 2. Characterize those F's which make Lih's conjecture ture when $t \geqq 4$.

2. Let P be a partially ordered set, not necessarily ranked. A subset $S \subset P$ is called a cutset if every maximal chain has nonempty intersection with S.

Problem 3. Relate maximum and minimum sizes of a minimal cutset, in the sense of set inclusion, with other parameters of P.

The most concrete example is to let P be B^n. Here the minimum is trivial, which is 1. However, it seem rather difficult to answer the following.

Problem 4. Find the maximum size of a minimal cutset in B^n.

We originally conjectured that the answer was 2^{n-1}. The minimal cutset attaining this number consists of all subsets containing either 1 or 2, but not both.

Recently the following counterexample of 33 elements was found for n = 6.

S = 5, 6, 12, 14, 24, 35, 36, 45, 46, 123, 125, 126, 135, 136, 145, 146, 235, 236, 245, 246, 345, 346, 1234, 1256, 1345, 1346, 1456, 2345, 2346, 2356, 2456, 12456, 13456.

3. Covering a polygon with the minimum number of rectangles is a computationally difficult problem. Its practical applications include the creation of a mask for etching an integrated circuit.

We assume polygons and rectangles are aligned with the coordinate axes, and are finite subsets of unit squares in a grid, with integer vertices. A rectangular cover for a polygon R is a collection of rectangles contained within R, whose union is R. A minimum cover is one with the minimum number of rectangles.

Chvátal originally conjectured that the number of rectangles in a minimum cover of R is equal to the maximum number of squares in R with no two in a common rectangle. This is false. The strongest positive result is that the duality holds when the polygon is vertically convex. This is done by Györi [3], who reduced this duality to a duality concerning intervals on the real line. Franzblau and Kleitman [1] reproves Györi's results by an algorithmic argument, which considers only intervals with integer endpoints. This prompts us to formulate similar problems for sets.

Let S and G be families of nonempty subsets of X. We say that G generates S if every element of S is the union of some elements of G. Trivially, S generates S. The interesting question is how small can a generating set of S be? On the other hand, if $S_1, S_2, \ldots, S_m$ is a sequence of elements of S such that $S_k \setminus \bigcup\{ S_j : j = 1, \ldots, k-1 \} \neq \phi$ for $k = 2, 3, \ldots, m$, then the sequence is called an increasing sequence. Obviously this length is smaller than the size of a generating set.

Problem 5. Characterize S such that the minimum size of a generating set is equal to the length of a longest increasing sequence in S.

Without a full characterization, interesting sufficient conditions for S are nice to know. Franzblau and Kleitman's result can be regarded as the case when every element of S is of the form $\{ i, i+1, \ldots, i+j \}$.

References

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