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P_k-Factorization of Complete Bipartite Graphs

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1. Introduction

Let P_k be a path on k points and $K_{m,n}$ be a complete bipartite graph with partite sets V_1 and V_2, where $|V_1|=m$ and $|V_2|=n$. A spanning subgraph F of $K_{m,n}$ is called a P_k-factor if each component of F is isomorphic to P_k. If $K_{m,n}$ is expressed as a line-disjoint sum of P_k-factors, then this sum is called a P_k-factorization of $K_{m,n}$.

In this paper, a necessary condition for the existence of a P_k-factorization of $K_{m,n}$ will be given. And it will be shown that the necessary condition is also sufficient when k is even.

2. P_k-Factor of $K_{m,n}$

With respect to a P_k-factor of $K_{m,n}$, we give the following theorem.

Theorem 1. A $K_{m,n}$ has a P_k-factor if and only if

(I) $m = n \equiv 0 \pmod{k/2}$ when k is even, and

(II) $m + n \equiv 0 \pmod{k}$, $(k-1)m \leqq (k+1)n$ and $(k-1)n \leqq (k+1)m$ when k is odd.

Proof. (Necessity) Suppose that $K_{m,n}$ has a P_k-factor F. Let t be the number of components of F. Then $t=(m+n)/k$. Each component is a path obtained by traversing V_1 and V_2. Thus when k is even, it holds that m=n=kt/2. Condition (I) is necessary. And when k is odd, let t_1 (t_2) be the number of components of F whose end points are in V_1 (V_2), respectively. Then it holds that $m=((k+1)t_1+(k-1)t_2)/2$ and $n=((k-1)t_1+(k+1)t_2)/2$. So we have $t_1=((k+1)m-(k-1)n)/2k$ and $t_2=((k+1)n-(k-1)m)/2k$. From $0\leqq t_1\leqq t$ and $0\leqq t_2\leqq t$, we must have $(k-1)m\leqq(k+1)n$ and $(k-1)n\leqq(k+1)m$. Condition (II) is necessary.

(Sufficiency) When k is even, put m=n=kt/2. Consider a Hamilton-path of $K_{n,n}$ and divide it into t paths of same length. Then they form a P_k-factor of $K_{n,n}$. When k is odd, for those parameters m and n satisfying (II), put $t_1=((k+1)m-(k-1)n)/2k$ and $t_2=((k+1)n-(k-1)m)/2k$ and $t=(m+n)/k$. Then t_1 and t_2 are integers such as $0\leqq t_1\leqq t$ and $0\leqq t_2\leqq t$. And it holds that $m=((k+1)t_1+(k-1)t_2)/2$ and $n=((k-1)t_1+(k+1)t_2)/2$. Using $(k+1)t_1/2$ points in V_1 and $(k-1)t_1/2$ points in V_2, consider t_1 P_k's whose end points are in V_1. Using remaining $(k-1)t_2/2$ points in V_1 and remaining $(k+1)t_2/2$ points in V_2, consider t_2 P_k's whose end points are in V_2. Then these t_1+t_2 P_k's are line-disjoint and they form a P_k-factor of $K_{m,n}$.

<u>Corollary 1</u>. A $K_{n,n}$ has a P_k-factor if and only if

(I)' $n\equiv 0 \pmod{k/2}$ when k is even, and

(II)' $n\equiv 0 \pmod{k}$ when k is odd.

3. P_k-Factorization of $K_{m,n}$

With respect to a P_k-factorization of $K_{m,n}$, we give the following theorem.

<u>Theorem 2</u>. If $K_{m,n}$ has a P_k-factorization, then it holds that

(I)" $m = n \equiv 0 \pmod{k(k-1)/2}$ when k is even, and

(II)" $m + n \equiv 0 \pmod{k}$, $(k-1)m \leqq (k+1)n$, $(k-1)n \leqq (k+1)m$ and $kmn/(k-1)(m+n)$ is an integer when k is odd.

Proof. Suppose that $K_{m,n}$ has a P_k-factorization. Let r be the number of P_k-foctors of $K_{m,n}$ and t be the number of components of each P_k-factor. Then $t=(m+n)/k$ and $r=kmn/(k-1)(m+n)$. Thus t and r are integers. By Theorem 1, it holds that $m=n\equiv 0 \pmod{k(k-1)/2}$ when k is even, and that $m+n=0 \pmod{k}$, $(k-1)m\leqq(k+1)n$, $(k-1)n\leqq(k+1)m$ and $kmn/(k-1)(m+n)$ is an integer when k is odd.

Corollary 2. If $K_{n,n}$ has a P_k-factorization, then it holds that

(I)"' $n \equiv 0 \pmod{k(k-1)/2}$ when k is even, and

(II)"' $n \equiv 0 \pmod{2k(k-1)}$ when k is odd.

We prepare the following extension theorem, which is very useful.

Theorem 3. If $K_{m,n}$ has a P_k-factorization, then $K_{sm,sn}$ has a P_k-factorization for every positive integer s.

Proof. If every subgraph $K_{1,1}$ of $K_{s,s}$ is replaced by $K_{m,n}$, then $K_{s,s}$ is replaced by $K_{sm,sn}$. Using $K_{1,1}$-factorization (1-factorization) of $K_{s,s}$, we can see that $K_{sm,sn}$ has a $K_{m,n}$-factorization. Using a P_k-factorization of $K_{m,n}$, we can easily construct a P_k-factorization of $K_{sm,sn}$. About a 1-factorization of $K_{s,s}$, see [1,2].

Using this theorem, we can obtain several results. When k is even, we have the following lemma.

Lemma 1. k is even and $m = n = k(k-1)/2$

$\Longrightarrow$ $K_{m,n}$ has a P_k-factorization.

Proof. The proof is shown by a construction algorithm. Let

$V_1 = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}\}$, where $m = n = k(k-1)/2$. Construct $k-1$ P_k's such as $P_k^{(i)} = v_{(i-1)a+1}^{(1)} v_{(i-1)b+1}^{(2)} v_{(i-1)a+2}^{(1)} v_{(i-1)b+2}^{(2)} \cdots v_{ia-1}^{(1)} v_{ib}^{(2)} v_{ia}^{(1)} v_{k(i)}^{(2)}$, where $a=k/2$, $b=k/2-1$ and $k(i)=((k/2-1)+1 \bmod k-1)+(k/2-1)(k-1)$. Then $F = P_k^{(1)} \cup P_k^{(2)} \cup \ldots \cup P_k^{(k-1)}$ is a P_k-factor. Increasing all point numbers of F in V_1 by $k-1$ (mod m) simultaneously $k/2$ times and increasing all point numbers of F in V_2 by $k-1$ (mod n) simultaneously $k/2$ times, we obtain $k^2/4$ P_k-factors. Then it can be easily checked that these P_k-factors are line-disjoint and that the sum of them is a P_k-factorization of $K_{m,n}$.

Applying Theorem 3 to Lemma 1 and considering Theorem 2, we have the following theorem.

Theorem 4. When k is even, a $K_{m,n}$ has a P_k-factorization if and only if $m = n \equiv 0 \pmod{k(k-1)/2}$.

When k is odd, we have the following lemmas.

Lemma 2. k is odd, $(k-1)m = (k+1)n$ and $kmn/(k-1)(m+n)$ is an integer

==> (i) $m + n \equiv 0 \pmod{k}$, and

(ii) $m = (k+1)s/2$, $n = (k-1)s/2$ when $k \equiv 3 \pmod{4}$,

$m = (k+1)s$, $n = (k-1)s$ when $k \equiv 1 \pmod{4}$,

where s is a positive integer.

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$m = (k-1)s$, $n = (k+1)s$ when $k \equiv 1 \pmod{4}$,

where s is a positive integer.

Lemma 2 and Lemma 3 can be easily checked. We have the following lemmas.

Lemma 4. $k \equiv 3 \pmod 4$, $m = (k-1)/2$, $n = (k+1)/2$

==> $K_{m,n}$ has a P_k-factorization.

Proof. The proof is shown by a simple construction algorithm. Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}\}$, where $m=(k-1)/2$ and $n=(k+1)/2$. Construct a P_k such as $P_k = v_1^{(2)} v_1^{(1)} v_2^{(2)} v_2^{(1)} \ldots v_{(k-1)/2}^{(2)} v_{(k-1)/2}^{(1)} v_{(k+1)/2}^{(2)}$. Then $F=P_k$ is a P_k-factor. Increasing all point numbers of F in V_2 by 2 (mod n) simultaneously n/2 times, we obtain n/2 P_k-factors. Then it can be easily checked that these P_k-factors are line-disjoint and that the sum of them is a P_k-factorization of $K_{m,n}$.

Lemma 5. $k \equiv 1 \pmod 4$, $m = k-1$, $n = k+1$

==> $K_{m,n}$ has a P_k-factorization.

Proof. The proof is shown by a simple construction algorithm. Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}\}$, where $m=k-1$ and $n=k+1$. Construct two P_k's such as $P_k^{(1)} = v_1^{(2)} v_1^{(1)} v_2^{(2)} v_2^{(1)} \ldots v_{(k-1)/2}^{(2)} v_{(k-1)/2}^{(1)} v_{(k+1)/2}^{(2)}$ and $P_k^{(2)} = v_{a+1}^{(2)} v_{b+1}^{(1)} v_{a+2}^{(2)} v_{b+2}^{(1)} \ldots v_{a+(k-1)/2}^{(2)} v_{b+(k-1)/2}^{(1)} v_{a+(k+1)/2}^{(2)}$, where $a=(k+1)/2$ and $b=(k-1)/2$. Then $F = P_k^{(1)} \cup P_k^{(2)}$ is a P_k-factor. Increasing all point numbers of F in V_2 by 2 (mod n) simultaneously n/2 times, we obtain n/2 P_k-factors. Then it can be easily checked that these P_k-factors are line-disjoint and that the sum of them is a P_k-factorization of $K_{m,n}$.

Applying Theorem 3 to Lemma 4 - Lemma 5 and considering Lemma 2 - Lemma 3, we have the following Theorems.

Theorem 5. k is odd, $(k-1)m = (k+1)n$ and $kmn / (k-1)(m+n)$ is an integer

==> $K_{m,n}$ has a P_k-factorization.

Theorem 6. k is odd, $(k-1)n = (k+1)m$ and $kmn / (k-1)(m+n)$ is an integer

==> $K_{m,n}$ has a P_k-factorization.

References

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