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Discrete Dirichlet Potentials on an Infinite Network

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Introduction

As a continuation to the previous papers [3] and [8], we shall study some analogies of discrete Dirichlet potentials on an infinite network and Dirichlet potentials on a Riemannian manifold. In order to investigate the behavior of discrete Dirichlet potentials at an ideal boundary point of the network, we introduce discrete analogues of the Royden boundary and the harmonic boudary of the network. We shall show in § 5 that a discrete Dirichlet function of order p ($1 < p < \infty$) is a Dirichlet potential of order p if and only if it vanishes on the p-harmonic boundary. Another chracterization of discrete Dirichlet potentials was proved by the author [8] with the aid of extremal length of a family of infinite paths on the network. In this proof, we did not use the concept of the Royden boundary. We shall give another proof of this characterization by using p-harmonic boundary. These results have counter parts in [2] in case p = 2.

§ 1. Preliminaries

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a strictly positive function on Y. Assume that the quartet N = {X, Y, K, r} is an infinite network, i.e., the graph $G = \{X, Y, K\}$ is connected, locally finite and has no self-loop. More precisely we assume the following conditions:

(1.1) The range of K is $\{-1, 0, 1\}$;

(1.2) For each $y \in Y$, $e(y) = \{x \in X; K(x, y) \neq 0\}$ (the end of y) consists of exactly two nodes x, x_2 and $K(x_1, y)K(x_2, y) = -1$;

(1.3) For each $x \in X$, $Y(x) = \{y \in Y; K(x, y) \neq 0\}$ is a nonempty finite set;

(1.4) For any $x, x' \in X$, there are $x_1, \cdots, x_n \in X$ and $y_1, \cdots, y_{n+1} \in Y$ such that $e(y_j) = \{x_{j-1}, x_j\}$, $j = 1, \cdots, n+1$ with $x_0 = x$ and $x_{n+1} = x'$.

For notation and terminology, we mainly follow [3] and [8].

Let $L(X)$ be the set of all real functions on X and $L_0(X)$ be the set all $u \in L(X)$ with finite support. For $u \in L(X)$, its discrete derivative $du \in L(Y)$ and its discrete Dirichlet integral $D_p(u)$ of order p $(1 < p < \infty)$ are defined by

$$du(y) = -r(y)^{-1}\sum_{x\in X} K(x, y)u(x),$$

$$D_p(u) = \sum_{y\in Y} r(y)|du(y)|^p.$$

Denote by $D^{(p)}(N)$ the set of all $u \in L(X)$ with finite Dirichlet integral of order p. It is easily seen that $D^{(p)}(N)$ is a reflexive Banach space with the norm $\|u\|_p = [D_p(u) + |u(b)|^p]^{1/p}$, where b is a fixed element of X. Let $D_0^{(p)}(N)$ be the closure of $L_0(X)$ in $D^{(p)}(N)$ with respect to this norm. Note that $D_0^{(p)}(N)$ is independent of the choice of b. We call an element of $D_0^{(p)}(N)$ a Dirichlet potential of order p.

Let $\varphi_p(t)$ be the real function on the real line R defined by $\varphi_p(t) = |t|^{p-1}\mathrm{sign}(t)$. For each $w \in L(Y)$, let us define $\varphi_p(w) \in$

$L(Y)$ by $\varphi_p(w)(y) = \varphi_p(w(y))$ for $y \in Y$.

For each $u \in L(X)$, the p-Laplacian $\Delta_p u \in L(X)$ of u is defined by

$$\Delta_p u(x) = \sum_{y\in Y} K(x, y)\varphi_p(du(y)).$$

We say that u is p-harmonic on a subset A of X if $\Delta_p u(x) = 0$ on A. Denote by $HD^{(p)}(N)$ the set of all $u \in D^{(p)}(N)$ which is p-harmonic on X. It should be noted that $HD^{(p)}(N)$ is not a linear space if $p \neq 2$.

For a set S of real functions on X, denote by BS the subset of S which consists of bounded functions. We shall use the notation $BD^{(p)}(N)$, $BD_0^{(p)}(N)$ and $BHD^{(p)}(N)$ in this sense.

§ 2. Royden boundary

For $u \in BD^{(p)}(N)$, we consider the norm $\|u\|_B$ defined by

$$\|u\|_B = [D_p(u)]^{1/p} + \sup\{|u(x)|;\ x \in X\}.$$

It is easily seen that $BD^{(p)}(N)$ is a normed space with respect to this norm and that $BD^{(p)}(N)$ is closed under max. and min. operations, i.e., if $u, v \in BD^{(p)}(N)$, then $\max(u, v)$ and $\min(u, v)$ belong to $BD^{(p)}(N)$.

Theorem 2.1. $BD^{(p)}(N)$ is a Banach algebra with respect to the norm $\|u\|_B$.

Proof. Let $u, v \in BD^{(p)}(N)$ and put $\alpha = \sup\{|u(x)|;\ x \in X\}$ and $\beta = \sup\{|v(x)|;\ x \in X\}$. Then $|duv(y)| \le \beta|du(y)| + \alpha|dv(y)|$,

$$\begin{aligned} D_p(uv) &\le \sum_{y\in Y} r(y)[\beta|du(y)| + \alpha|dv(y)|]^p \\ &\le \{[D_p(\beta u)]^{1/p} + [D_p(\alpha v)]^{1/p}\}^p \\ &= \{\beta[D_p(u)]^{1/p} + \alpha[D_p(v)]^{1/p}\}^p \end{aligned}$$

by the Minkowski's inequality. Thus we have

$$\begin{aligned} \|uv\|_B &\le [D_p(uv)]^{1/p} + \alpha\beta \\ &\le \beta[D_p(u)]^{1/p} + \alpha[D_p(v)]^{1/p} + \alpha\beta \le \|u\|_B\|v\|_B. \end{aligned}$$

Therefore $BD^{(p)}(N)$ is a normed algebra.

If $\{u_n\}$ is a Cauchy sequence in $BD^{(p)}(N)$ with respect to $\| \|_B$, then it is a Cauchy sequence in $D^{(p)}(N)$ with respect to $\| \|_p$. Thus there exists $u \in D^{(p)}(N)$ such that $\|u - u_n\|_p \to 0$ as $n \to \infty$. Since $\{u_n\}$ converges uniformly on X, we see that $u \in BD^{(p)}(N)$ and $\|u - u_n\|_B \to 0$ as $n \to \infty$. Therefore $BD^{(p)}(N)$ is a Banach algebra with respect to the norm $\|u\|_B$.

By a compactification of X which is regarded as a locally compact Hausdorff space with respect to the discrete topology, we mean a compact Hausdorff space X^* containing X as a dense open subset. There is a unique (up to a homeomorphism) compactification X^* of X such that every $f \in BD^{(p)}(N)$ can be continuously extended to X^* and the class of extended functions separates points of X^* (cf.[4; § 8]). This compactification is called the Royden p-compactification of N and $\Gamma = \Gamma^{(p)} = X^* - X$ is called the p-Royden boundary of N. The extension of $f \in BD^{(p)}(N)$ to X^* is denoted by f again.

Put $\mathcal{D}(\Gamma) = \{f|_\Gamma;\ f \in BD^{(p)}(N)\}$. By the Stone-Weierstrass theorem, $\mathcal{D}(\Gamma)$ is dense in the set $C(\Gamma)$ of all continuous functions on Γ with respect to the sup-norm.

Next, we introduce the Royden p-harmonic boundary:

$$\Gamma_h = \Gamma_h^{(p)} = \{x \in \Gamma;\ f(x) = 0 \text{ for all } f \in BD_0^{(p)}(N)\}.$$

Note that Γ_h is a compact subset of Γ.

§ 3. Decomposition theorems

We proved in [8]

Theorem 3.1. Let N be of hyperbolic type of order p. Then every $u \in D^{(p)}(N)$ can be decomposed uniquely in the form: $u = v + h$ with $v \in D_0^{(p)}(N)$ and $h \in HD^{(p)}(N)$.

In order to obtain a similar decomposition of $BD^{(p)}(N)$, we need the following type of Clarkson's inequality (cf.[8; Lemma 2.2]):

Lemma 3.1. Let $u, v \in D^{(p)}(N)$. Then the following inequalities hold:

(3.1) $D_p(u + v) + D_p(u - v) \leq 2^{p-1}[D_p(u) + D_p(v)]$ in case $2 \leq p$;

(3.2) $[D_p(u + v)]^{1/(p-1)} + [D_p(u - v)]^{1/(p-1)}$

$$\leq 2[D_p(u) + D_p(v)]^{1/(p-1)} \text{ in case } 1 \leq p \leq 2.$$

Let $u \in D^{(p)}(N)$ and A be a nonempty proper subset of X. Then the following extremum problem has a unique solution:

(3.3) Find $\beta(A) = \inf\{D_p(f);\ f \in L(X),\ f = u \text{ on } X - A\}$.

In fact, let $\{f_n\}$ be a sequence in $L(X)$ such that $D_p(f_n) \to \beta(A)$ as $n \to \infty$. Then we see by Lemma 3.1 that $\{u_n\}$ is a Cauchy sequence in $D^{(p)}(N)$ with respect to the norm $\| \ \|_p$. Thus the existence of an optimal solution is established. The uniqueness of an optimal solution follows from [8; Lemma 2.1].

By the standard variational technique, we can prove

Lemma 3.2. The optimal solution u^A of problem (3.3) has the following properties:

(3.4) $u^A = u$ on $X - A$ and $D_p(u^A) = \beta(A)$.

(3.5) $\Delta_p u^A(x) = 0$ on A.

(3.6) If $|u(x)| \leq c$ on X, then $|u^A(x)| \leq c$ on X.

(3.7) If $u \geq 0$ on X, then $u^A \geq 0$ on X.

We have

Lemma 3.3. Let $u \in D^{(p)}(N)$ and $\{N_n\}$ ($N_n = \langle X_n, Y_n \rangle$) be an exhaustion of N. Denote by u_n^* the optimal solution of problem (3.3) with $A = X_n$. Then $D_p(u_n^* - u_m^*) \to 0$ as $n, m \to \infty$.

Proof. Put $\beta_n = \beta(X_n)$. Since $u_n^* = u$ on $X - X_m$ for $m \ge n$, we have $0 \le \beta_m \le \beta_n \le D_p(u) < \infty$. Thus $\{\beta_n\}$ converges. In case $p \ge 2$, we have by (3.1)

$$\beta_n \le D_p((u_n^* + u_m^*)/2) \le D_p((u_n^* + u_m^*)/2) + D_p((u_n^* - u_m^*)/2)$$
$$\le 2^{p-1}[D_p(u_n^*/2) + D_p(u_m^*/2)] = 2^{-1}(\beta_n + \beta_m).$$

In case $1 < p \le 2$, we have by (3.2)

$$\beta_n^{1/(p-1)} \le [D_p((u_n^* + u_m^*)/2)]^{1/(p-1)}$$
$$\le [D_p((u_n^* + u_m^*)/2)]^{1/(p-1)} + [D_p((u_n^* - u_m^*)/2]^{1/(p-1)}$$
$$\le 2[D_p(u_n^*/2) + D_p(u_m^*/2)]^{1/(p-1)} = 2^{-1}(\beta_n + \beta_m)^{1/(p-1)}.$$

It follows that $D_p(u_n^* - u_m^*) \to 0$ as $n, m \to \infty$.

We shall prove

Theorem 3.2. Let N be of hyperbolic type of order p. Then every $u \in BD^{(p)}(N)$ can be decomposed uniquely in the form: $u = v + h$ with $v \in BD_0^{(p)}(N)$ and $h \in BHD^{(p)}(N)$.

Proof. Let u_n^* be the function defined in Lemma 3.3 and put $c = \sup\{|u(x)|;\ x \in X\}$. Then $|u_n^*| \le c$ on X by Lemma 3.2. By choosing a subsequence if necessary, we may assume that $\{u_n^*(b)\}$ converges. Then $\{u_n^*\}$ is a Cauchy sequence in $D^{(p)}(N)$ with respect to $\| \ \|_p$ by Lemma 3.3. There exists $h \in D^{(p)}(N)$ such that $\|u_n^* - h\|_p \to 0$ as $n \to \infty$. We see easily that $h \in BHD^{(p)}(N)$. Put $v_n = u - u_n^*$ and $v = u - h$. Then $v_n \in L_0(X)$ and $\|v - v_n\|_p = \|u_n^* - h\|_p \to 0$ as $n \to \infty$, so that $v \in BD_0^{(p)}(N)$. This is a desired decomposition. The uniqueness of the decomposition follows from Theorem 3.1.

We shall prove the following weak minimum principle:

Theorem 3.3. Let $u = v + h$ with $v \in BD_0^{(p)}(N)$ and $h \in BHD^{(p)}(N)$. If $u \ge 0$ on X, then $h \ge 0$ on X.

Proof. Note that $u \in BD^{(p)}(N)$. Let u_n^* be the function defined in Lemma 3.3. Then $u_n^* \geq 0$ on X by Lemma 3.2. By the same reasoning as in the proof of Theorem 3.2, we see that there exist $\tilde{h} \in BHD^{p)}(N)$ and $\tilde{v} \in BD_0^{(p)}(N)$ such that $u = \tilde{u} + \tilde{h}$ and $\tilde{h} \geq 0$ on X. By the uniqueness of the decomposition, we conclude that $h = \tilde{h} \geq 0$ on X.

Here we observe that $BD_0^{(p)}(N)$ plays the role of the class $M_\Delta(R)$ in [6] or $\mathcal{D}_{DB,\Delta}(U)$ in [4]. In fact, we have

Lemma 3.4. For every $u \in BD_0^{(p)}(N)$, there exists a sequence $\{f_n\}$ in $L_0(X)$ such that $\{f_n\}$ is uniformly bounded and $\|u - f_n\|_p \to 0$ as $n \to \infty$.

Proof. In the proof of Theorem 3.2, we take $f_n = u - u_n^*$. Since $u \in BD_0^{(p)}(N)$ and $\|u_n^* - h\|_p \to 0$ as $n \to \infty$, we see that $h = 0$ and $\|u_n^*\|_p \to 0$ as $n \to \infty$. Therefore $\|u - f_n\|_p \to 0$ as $n \to \infty$.

§ 4. A limit theorem on $D_0^{(p)}(N)$

In order to obtain a fundamental limit property of sequences in $D_0^{(p)}(N)$, we begin with some definitions. Let q be a positive number such that $1/p + 1/q = 1$. Denote by $L_q(Y; r)$ the set of all $w \in L(Y)$ such that $H_q(w) = \sum_{y \in Y} r(y)|w(y)|^q < \infty$. For $w_1, w_2 \in L(Y)$, we define $\langle w_1, w_2 \rangle$ by

$$\langle w_1, w_2 \rangle = \sum_{y \in Y} r(y) w_1(y) w_2(y)$$

whenever the sum is well-defined.

We shal prove

Lemma 4.1. Let N be of hyperbolic type of order p and $a \in X$ and consider the following extremum problem:

(4.1) $d_p(a, \infty) = \inf\{D_p(v);\ v \in D_0^{(p)}(N) \text{ and } v(a) = 1\}$.

Then there exists $\omega_a \in L_q(Y;\ r)$ such that

(4.2) $\langle du, \omega_a \rangle = u(a)d_p(a, \infty)$ for every $u \in D_0^{(p)}(N)$.

Proof. Since N is of hyperbolic type of order p, $d_p(a, \infty) > 0$. By the standard variational technique, we see that there exists a unique $u_a \in D_0^{(p)}(N)$ such that $u_a(a) = 1$, $d_p(a, \infty) = D_p(u_a)$ and

$$\sum_{y \in Y} r(y)[\varphi_p(du_a(y))]dv(y) = 0$$

for every $v \in D_0^{(p)}(N)$ with $v(a) = 0$. Put $\omega_a(y) = \varphi_p(du_a(y))$. Then

$$H_q(\omega_a) = \sum_{y \in Y} r(y)|\omega_a(y)|^q = \sum_{y \in Y} r(y)|du_a(y)|^{(p-1)q} = D_p(u_a) < \infty,$$

so that $\omega_a \in L_q(Y;\ r)$. For any $u \in D_0^{(p)}(N)$, $u - u(a)u_a$ belongs to $D_0^{(p)}(N)$, and hence

$$\sum_{y \in Y} r(y)\omega_a(y)[du(y)] = u(a)\sum_{y \in Y} r(y)\omega_a(y)[du_a(y)] = u(a)D_p(u_a).$$

Namely ω_a satisfies our requirement.

Lemma 4.2. Let N be of hyperbolic type of order p. Then $dD_0^{(p)}(N) = \{du;\ u \in D_0^{(p)}(N)\}$ is a closed subset of $L_p(Y;\ r)$ with respect to the norm $[H_p(w)]^{1/p}$.

Proof. Let $u_n \in D_0^{(p)}(N)$, $w \in L_p(Y;\ r)$ and $H_p(du_n - w) \to 0$ as $n \to \infty$. Since $D_p(u_n) = H_p(du_n)$, we see that $\{D_p(u_n)\}$ is bounded. We show that $\{u_n(b)\}$ is bounded. Supposing the contrary, we may assume that $u_n(b) \to \infty$ as $n \to \infty$. Put $u_n' = u_n/u_n(b)$. Then $D_p(u_n') = D_p(u_n)/|u_n(b)|^p \to 0$ as $n \to \infty$ and $u_n'(b) = 1$. Thus

$$\|u_n' - 1\|_p = [D_p(u_n')]^{1/p} \to 0$$

as $n \to \infty$. Therefore $1 \in D_0^{(p)}(N)$, which contradicts the assumption that N is of hyperbolic type of order p (cf. [7]). By choosing a subsequence if necessary, we may assume that $\{u_n(b)\}$ converges. We have

$$\|u_n - u_m\|_p = [D_p(u_n - u_m) + |u_n(b) - u_m(b)|^p]^{1/p}$$

$$= [H_p(du_n - du_m) + |u_n(b) - u_m(b)|^p]^{1/p} \to 0$$

as n, m → ∞. Since $D_0^{(p)}(N)$ is a Banach space, there exists $u \in D_0^{(p)}(N)$ such that $\|u_n - u\|_p \to 0$ as $n \to \infty$. We have

$$[H_p(w - du)]^{1/p} \le [H_p(w - du_n)]^{1/p} + [H_p(du - du_n)]^{1/p}$$
$$\le [H_p(w - du_n)]^{1/p} + \|u - u_n\|_p \to 0$$

as $n \to \infty$, so that $w = du \in dD_0^{(p)}(N)$.

Theorem 4.1. Let $\{u_n\}$ be a sequence in $D_0^{(p)}(N)$ which converges pointwise to $u \in L(X)$. If $\{D_p(u_n)\}$ is bounded, then $u \in D_0^{(p)}(N)$.

Proof. No proof is necessary in case N is of parabolic type of order p, so we assume that N is of hyperbolic type of order p. Put $w_n = du_n$. Since $\{H_p(w_n)\}$ is bounded, we can find a weakly convergent subsequence of $\{w_n\}$ in $L_p(Y; r)$. Denote it again by $\{w_n\}$ and let w be the limit. Since $dD_0^{(p)}(N)$ is convex and strongly closed by Lemma 4.2, we see that $dD_0^{(p)}(N)$ is weakly closed. Thus we can find $g \in D_0^{(p)}(N)$ such that $w = dg$ and $\langle dv_n, w' \rangle \to \langle dg, w' \rangle$ as $n \to \infty$ for every $w' \in L_q(Y; r)$. For $a \in X$, we have $\omega_a \in L_q(Y; r)$ by Lemma 4.1, so that

$$\langle du_n, \omega_a \rangle = u_n(a)d_p(a, \infty) \text{ and } \langle dg, \omega_a \rangle = g(a)d_p(a, \infty).$$

It follows that $u(a) = g(a)$ for every $a \in X$. Thus $u \in D_0^{(p)}(N)$.

We say that a function T from R to R is a normal contraction of R if $T0 = 0$ and $|Ts_1 - Ts_2| \le |s_1 - s_2|$ for every $s_1, s_2 \in R$. We define $Tu \in L(X)$ for $u \in L(X)$ by $(Tu)(x) = Tu(x)$.

We have

Theorem 4.2. Let T be a normal contraction of R. If $u \in D_0^{(p)}(N)$, then $Tu \in D_0^{(p)}(N)$.

Proof. It suffices to consider the case where N is of hyperbolic type of order p. There exists a sequence $\{f_n\}$ in $L_0(X)$

such that $\|f_n - u\|_p \to 0$ as $n \to \infty$. Put $v_n = Tf_n$ and $v = Tu$. Then $v_n \in L_0(X)$, $\{D_p(v_n)\}$ is bounded and

$$|v_n(x) - v(x)| = |Tf_n(x) - Tu(x)| \le |f_n(x) - u(x)| \to 0$$

as $n \to \infty$. Thus $Tu = v \in D_0^{(p)}(N)$ by Theorem 4.1.

Lemma 4.3. Let F be a closed subset of X^* such that $F \cap \Gamma_h = \phi$. Then there exists $f \in BD_0^{(p)}(N)$ such that $f = 1$ on F and $0 \le f \le 1$ on X^*.

Proof. Let $x \in F$. Since $x \notin \Gamma_h$, there exists $f_x \in BD_0^{(p)}(N)$ such that $f_x(x) \ne 0$. We may assume that $f_x \ge 0$ on X and $f_x(x) > 0$ by Theorem 4.2. Since f_x is continuous on X^*, there is an open neighborhood V_x of x in X^* such that $f_x > 0$ on V_x. Since F is compact, we can choose a finite number of points $x_1, \cdots, x_n \in F$ such that $V_{x_1} \cup \cdots \cup V_{x_n} \supset F$. Put $g = \sum_{i=1}^n f_{x_i}$ and $\alpha = \inf\{g(x); x \in F\}$. Then $g \in BD_0^{(p)}(N)$ and $\alpha > 0$. Hence $f = \min(1, g/\alpha)$ is the required function in view of Theorem 4.2.

We have

Theorem 4.3. N is of parabolic type of order p if and only if $\Gamma_h^{(p)} = \phi$.

Proof. If N is of parabolic type of order p, then $1 \in BD_0^{(p)}(N)$, so that $\Gamma_h = \phi$. Assume that $\Gamma_h = \phi$. Taking $F = X^*$ in Lemma 4.3, we see that $1 \in BD_0^{(p)}(N)$. Thus N is of parabolic type of order p.

§ 5. Duality theorem

We have

Theorem 5.1. (Minimum principle) Assume that N is of hyperbolic type of order p. If $u \in BHD^{(p)}(N)$ and $u \geq 0$ on Γ_h, then $u \geq 0$ on X.

Proof. For any $\varepsilon > 0$, put $F_\varepsilon = \{x \in X^*; u(x) + \varepsilon \leq 0\}$. Then F_ε is a closed set in X^* and $F_\varepsilon \cap \Gamma_h = \phi$. By Lemma 4.3, there exists $f \in BD_0^{(p)}(N)$ such that $f = 1$ on F_ε and $0 \leq f \leq 1$ on X^*. Since u is bounded, $u \geq - c$ on X^* for some constant $c \geq 0$. We have $u + \varepsilon + cf \geq 0$ on X^*. Since $u + \varepsilon \in BHD^{(p)}(N)$ and $cf \in BD_0^{(p)}(N)$, we see by Theorem 3.3 that $u + \varepsilon \geq 0$ on X. Since ε is arbitrary, it follows that $u \geq 0$ on X.

The reasoning of this proof is the same as in [4; Theorem 8.2].

Now we shall prove the following duality theorem:

Theorem 5.2. Assume that N is of hyperbolic type of order p. Then $BD_0^{(p)}(N) = \{u \in BD^{(p)}(N); u = 0 \text{ on } \Gamma_h^{(p)}\}$.

Proof. Let $\mathscr{D}_1 = \{u \in BD^{(p)}(N); u = 0 \text{ on } \Gamma_h\}$. By the definition of Γ_h, $BD_0^{(p)}(N) \subset \mathscr{D}_1$. Let $u \in \mathscr{D}_1$ and $u = v + h$ with $v \in BD_0^{(p)}(N)$ and $h \in BHD^{(p)}(N)$. We have $h = u - v = 0$ on Γ_h. It follows from Theorem 5.1 that $h = 0$ on X, so that $u = v \in BD_0^{(p)}(N)$.

In order to obtain a similar characterization of $D_0^{(p)}(N)$, we recall the following lemma(cf.[7; Lemma 3.1]):

Lemma 5.2. Let $u \in D^{(p)}(N)$ be non-negative. Then $\|u - \min(u, n)\|_p \to 0$ as $n \to \infty$.

For $u \in D^{(p)}(N) \cap L^+(X)$, $u^{(n)} = \min(u, n) \in BD^{(p)}(N)$ and $u^{(n)}$ is continuous on X^*, so that we define u(x) for $x \in \Gamma$ by the limit of $u^{(n)}(x)$ as $n \to \infty$. In case $u \in D^{(p)}(N)$ is of any sign, we define u(x) for $x \in \Gamma$ by $u^+(x) - u^-(x)$, where $u^+ = \max(u, 0)$ and $u^- = \max(- u, 0)$ are the positive part and the negative part of u respectively.

As a generalization of Theorem 5.2, we have

Theorem 5.3. Let N be of hyperbolic type of order p. Then $D_0^{(p)}(N) = \{u \in D^{(p)}(N);\ u = 0 \text{ on } \Gamma_h\}$.

Proof. Let $u \in D^{(p)}(N)$. By considering the positive part and the negative part of u separately, we may restrict our attention to the case where u is non-negative. For a positive integer n, we put $u^{(n)} = \min(u, n)$. Assume that $u \in D_0^{(p)}(N)$. Then $u^{(n)} \in BD_0^{(p)}(N)$ by Theorem 4.2, so that $u_n = 0$ on Γ_h by Theorem 5.2. Thus $u = 0$ on Γ_h. Next we assume that $u = 0$ on Γ_h. Then $u^{(n)} \in BD^{(p)}(N)$ and $u^{(n)} = 0$ on Γ_h, so that $u^{(n)} \in BD_0^{(p)}(N)$ by Theorem 5.2. Thus $u \in D_0^{(p)}(N)$ by Lemma 5.2.

For later use, we shall prove

Lemma 5.3. Let N be of hyperbolic type of order p and F be a nonempty compact set in $\Gamma - \Gamma_h$. Then there exists $v \in D^{(p)}(N)$ such that $v = \infty$ on F and $v = 0$ on Γ_h.

Proof. By Lemma 4.3, there exists $u \in BD_0^{(p)}(N)$ such that $u = 1$ on F and $0 \le u \le 1$ on X^*. Let u_n^* be the function defined in Lemma 3.2. Then $u_n^* = 1$ on F. By the same reasoning as in the proof of Lemma 3.4, we may assume that $\|u_n^*\|_p \to 0$ as $n \to \infty$. By choosing subsequences if necessary, we may assume that

$$u_n^*(b) < 2^{-np} \text{ and } D_p(u_n^*) < 2^{-np}.$$

Put $v_n = \sum_{k=1}^{n} u_k^*$ and $v = \sum_{n=1}^{\infty} u_n^*$. Since $v_n \in BD^{(p)}(N)$ and $v_n = 0$ on Γ_h, we see by Theorem 5.2 that $v_n \in BD_0^{(p)}(N)$. We have

$$\|v_n - v\|_p = \|\textstyle\sum_{k=n+1}^{\infty} u_k^*\|_p \le \sum_{k=n+1}^{\infty} \|u_k^*\|_p < \sum_{k=n+1}^{\infty} 2^{-k} < 2^{-n} \to 0$$

as $n \to \infty$, so that $v \in D_0^{(p)}(N)$. Hence $v = 0$ on Γ_h by Theorem 5.3. On the other hand, we have $v \ge v_n = n$ on F. Thus $v = \infty$ on F.

§ 6. Another characterization of Dirichlet potentials

For a family Λ of paths in N, its extremal length $\lambda_p(\Lambda)$ of order p is defined by

$$\lambda_p(\Lambda)^{-1} = \inf\{H_p(W);\ W \in E(\Lambda)\},$$

where $E(\Lambda)$ is the set of all $W \in L^+(Y)$ such that $\sum_P r(y)W(y) \geq 1$ for all $P \in \Lambda$. The notion of the extremal length of a network was first introduced by Duffin [1]. We say as in [5] that a property holds for p-almost every path of Λ if it does for the member of Λ except for those belonging to a subfamily with infinite extremal length of order p.

Denote by $P_{a,\infty}(N)$ the set of all paths from $a \in X$ to the ideal boundary of N and by $P_\infty(N)$ the union of $P_{a,\infty}(N)$ for all $a \in X$. We call an element of $P_\infty(N)$ an infinite path.

We proved in [3] that every Dirichlet function u(x) of order p has a limit as x tends to the ideal boundary of N along p-almost every infinite path $P \in P_\infty(N)$. We denote this limit simply by u(P).

We proved in [3] and [8]

Theorem 6.1. Let $u \in D^{(p)}(N)$. Then $u \in D_0^{(p)}(N)$ if and only if $u(P) = 0$ for p-almost every $P \in P_\infty(N)$.

In the proof of this theorem, we used the following

Lemma 6.1. Let $h \in HD^{(p)}(N)$ be nonconstant. Then there exist $x_0 \in X$ and an infinite subnetwork $N' = \langle X', Y' \rangle$ of N such that $x_0 \in X'$, $h(x) > h(x_0)$ on $X' - \{x_0\}$ and $\lambda_p(\Lambda') < \infty$, where Λ' is the set of all infinite paths in N which are contained in N'.

By Theorems 5.3 and 6.1, we have

Theorem 6.2. Let $u \in D^{(p)}(N)$. Then $u(x) = 0$ on $\Gamma_h^{(p)}$ if and only if $u(P) = 0$ for p-almost every $P \in P_\infty(N)$.

This result has a counterpart in [2] in case $p = 2$. In order to give another proof of Theorem 6.1 relying upon the Royden boundary as in [2], we introduce some notions. For every infinite path P, let e(P) the end part of P in the Royden compactification X^*, i.e., e(P) is the intersection of Γ and the closure of the set $C_X(P)$ of nodes of P in X^*. It is easy to verify that for $u \in D^{(p)}(N)$, u(P) exists and is equal to a constant c if and only if $u(x) = c$ on e(P). For a subset B of X^*, denote by $cl(B)$ the closure of B in X^*.

Theorem 6.3. Let Λ_0 be a subfamily of $P_\infty(N)$ such that $F = cl(\cup\{e(P); P \in \Lambda_0\})$ is disjoint from Γ_h. Then $\lambda_p(\Lambda_0) = \infty$.

Proof. In case N is of parabolic type of order p, $\lambda_p(\Lambda_0) \geq \lambda_p(P_\infty(N)) = \infty$ by [7]. Assume that N is of hyperbolic type of order p. By Lemma 5.3, there exists $v \in D^{(p)}(N)$ such that $v = \infty$ on F and $v = 0$ on Γ_h. For any $P \in \Lambda_0$ and $\varepsilon > 0$, we have $e(P) \subset F$ and $v(P) = \infty$, so that $\sum_P r(y)|d(\varepsilon v)(y)| = \infty$. Thus

$$\lambda_p(\Lambda_0)^{-1} \leq H_p(d(\varepsilon v)) = \varepsilon^p D_p(v).$$

Since ε is arbitrary, we have $\lambda_p(\Lambda_0) = \infty$.

As a converse of this result, we have

Theorem 6.4. Let Λ_0 be a subfamily of $P_\infty(N)$ with $\lambda_p(\Lambda_0) = \infty$. Then the set $F = cl(\cup\{e(P); P \in P_\infty(N) - \Lambda_0\})$ contains Γ_h.

Proof. It suffices to consider the case where $\Gamma_h \neq \phi$. In this case N is of hyperbolic type of order p by Theorem 4.3, so that $\lambda_p(P_\infty(N) - \Lambda_0) < \infty$. It follows from Theorem 6.3 that $F \cap \Gamma_h \neq \phi$. Assume that there is a point $\alpha \in \Gamma_h - F$. We show that there exists $h \in HD^{(p)}(N)$ such that $h(\alpha) = 1$, $h = 0$ on $\Gamma_h \cap F$ and $0 < h < 1$ on X. In fact, we can find $u \in BD^{(p)}(N)$ such that $u(\alpha) = 1$, $u = 0$ on $\Gamma_h \cap F$

and $0 < u < 1$ on X, since $BD^{(p)}(N)$ separates points of X^*. By Theorem 3.2, u can be decomposed in the form: $u = v + h$ with $v \in BD_0^{(p)}(N)$ and $h \in BHD^{(p)}(N)$. Then h satisfies our requirement by Theorem 5.1. For this h, there exist $x_0 \in X$, an infinite subnetwork N' of N and a subfamily Λ' of infinite paths which satisfy the conditions in Lemma 6.1. Let us put $F_1 = Cl(\cup\{e(P); P \in \Lambda' - \Lambda_0\})$. Since $\lambda_p(\Lambda' - \Lambda_0) < \infty$, $\Gamma_h \cap F_1 \neq \phi$ by Theorem 6.3. By the relation $F_1 \cap \Gamma_h \subset F \cap \Gamma_h$, we obtain $h = 0$ on $F_1 \cap \Gamma_h$. On the other hand, we have $h(P) \geq h(x_0) > 0$ for all $P \in \Lambda'$, which implies $h \geq h(x_0)$ on F_1. This is a contradiction. Thus F contains Γ_h.

Proof of Theorem 6.1: No proof is necessary if N is of parabolic type of order p and we turn to the hyperbolic case. Let $u \in D^{(p)}(N)$. By considering the positive part and the negative part of u separately, we may assume that u is non-negative. Assume that $u(P) = 0$ for p-almost every $P \in \mathbf{P}_\infty(N)$. Let Λ_0 be the set of all $P \in \mathbf{P}_\infty(N)$ for which $u(P)$ does not exist or $u(P) \neq 0$. Then $\lambda_p(\Lambda_0) = \infty$, so the set $F = cl(\cup\{e(P); P \in \mathbf{P}_\infty(N) - \Lambda_0\})$ contains Γ_h by Theorem 6.4. If $u \in BD^{(p)}(N)$, then $u = 0$ on Γ_h. In fact, for $\alpha \in \Gamma_h$ there exists a generalized sequence $\{x_t\}$ in F which converges to α. Let $x_t \in e(P_t)$ with $P_t \in \mathbf{P}_\infty(N) - \Lambda_0$. Then $u(x_t) = u(P_t) = 0$. Since u is continuous on X^*, we conclude that $u(\alpha) = 0$. It follows from Theorem 5.2 that $u \in BD_0^{(p)}(N)$. If u is unbounded, then we see easily that $u^{(n)} = \min(u, n) \in BD_0^{(p)}(N)$. Since $\|u - u^{(n)}\|_p \to 0$ as $n \to \infty$, we conclude that $u \in D_0^{(p)}(N)$. For the proof of the sufficiency, assume that $u = 0$ on Γ_h. For each positive integer k, let Λ_k be the set of all $P \in \mathbf{P}_\infty(N)$ for which $u(P)$ exists and $u(P) \geq 1/k$ and put $F_k = cl(\cup\{e(P); P \in \Lambda_k\})$. Since $u = 0$ on Γ_h, we see that

F_k is disjoint from Γ_h, and hence $\lambda_p(\Lambda_k) = \infty$ by Theorem 6.3. Let Λ_0 be the set of all $P \in P_\infty(N)$ such that $u(P)$ does not exist and put $\Lambda_\infty = \cup_{n=0}^{\infty} \Lambda_k$. Then $\lambda_p(\Lambda_\infty) = \infty$ and $u(P) = 0$ for all $P \in P_\infty(N) - \Lambda_\infty$. Namely $u(P) = 0$ for p-almost every infinite path.

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