

Bessel capacity of symmetric generalized Cantor sets

Kaoru HATANO (Shimane Univ.)

秦野 薫

§1. Introduction

In [10] M. Ohtsuka obtained a necessary and sufficient condition for a symmetric generalized Cantor set to be of zero  $\alpha$  (or logarithmic)-capacity. In the non-linear potential theory metric properties of sets of zero Bessel capacity were investigated and the Bessel capacity of Cantor sets of special type was estimated in [8; §7]. In order to explain their results, let us recall the definitions of Bessel capacity and symmetric generalized Cantor sets.

Let  $g_\alpha = g_\alpha(x)$  be the Bessel kernel of order  $\alpha$ ,  $0 < \alpha < \infty$ , on the  $n$ -dimensional Euclidean space  $R^n$  ( $n \geq 1$ ), whose Fourier transform is  $(1 + |\xi|^2)^{-\alpha/2}$ , and  $h_\alpha = h_\alpha(x) = |x|^{\alpha-n}$  be the Riesz kernel,  $0 < \alpha < n$ . The Bessel capacity  $B_{\alpha,p}$  is defined as follows: For a set  $A \subset R^n$ ,

$$B_{\alpha,p}(A) = \inf \int f(x)^p dx,$$

the infimum being taken over all functions  $f \in L_p^+$  such that

$$g_\alpha * f(x) \geq 1 \text{ for all } x \in A.$$

For  $R_{\alpha,p}(A)$ , we just replace  $g_\alpha$  by  $h_\alpha$ . We shall always assume that  $1 < p < \infty$  and  $0 < \alpha p \leq n$ .

Let  $\{k_j\}_{j=1}^{\infty}$  be a sequence of integers and  $\{\ell_j\}_{j=0}^{\infty}$  be a sequence of positive numbers such that  $k_j \geq 2$  and  $k_{j+1}\ell_{j+1} < \ell_j$  ( $j \geq 0$ ). Let  $\delta_{j+1} = (\ell_j - k_{j+1}\ell_{j+1}) / (k_{j+1} - 1)$  ( $j = 0, 1, \dots$ ). Let  $I$  be a closed interval of length  $\ell_0$  in  $R^1$ . In the first step, we remove from  $I$   $(k_1 - 1)$  open intervals each of the same length  $\delta_1$  so that  $k_1$  closed intervals  $I_i^{(1)}$  ( $i = 1, \dots, k_1$ ) each of length  $\ell_1$  remain. Set  $E^{(1)} = \bigcup_{i=1}^{k_1} I_i^{(1)}$ . Next in the second step, we remove from each  $I_i^{(1)}$   $(k_2 - 1)$  open intervals each of the same length  $\delta_2$  so that  $k_2$  closed intervals  $I_{i,j}^{(2)}$  ( $j = 1, \dots, k_2$ ) each of length  $\ell_2$  remain. We set  $E^{(2)} = \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} I_{i,j}^{(2)}$ . We continue this process and obtain  $E^{(j)}$ ,  $j \geq 1$ . We define  $E = \bigcap_{j=1}^{\infty} E_n^{(j)}$ , where the set  $E_n^{(j)} = E^{(j)} \times \dots \times E^{(j)}$  is the product set of  $n$   $E^{(j)}$ 's in  $R^n$ . We call the set  $E$  the  $n$ -dimensional symmetric generalized Cantor set constructed by the system  $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$ .

The Cantor set  $E$  considered by Maz'ya and Khavin [8] is the one constructed as above with  $k_j = 2$  for all  $j \geq 1$ . For such a Cantor set  $E$ , they proved the following theorem.

**Theorem A.** If  $\alpha p < n$ , then

$$B_{\alpha,p}(E) = 0 \text{ is equivalent to } \sum_{j=1}^{\infty} 2^{-jn/(p-1)} \ell_j^{(\alpha p - n)/(p-1)} = \infty$$

and if  $\alpha p = n$ , then

$$B_{\alpha,p}(E) = 0 \text{ is equivalent to } \sum_{j=1}^{\infty} 2^{-jn/(p-1)} (-\log \ell_j) = \infty.$$

In [5] we obtain upper and lower estimates for the Bessel capacity of symmetric generalized Cantor sets. Namely, we have

**Theorem.** Let  $E$  be the  $n$ -dimensional symmetric generalized

Cantor set constructed by the system  $[\{k_j\}_{j=1}^{\infty}, \{\ell_j\}_{j=0}^{\infty}]$  with  $\ell_0 \leq 1$ . If  $\alpha p < n$ , then

$$C^{-1} \{ \ell_0^{(\alpha p - n)/(p-1)} + \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-n/(p-1)} \ell_j^{(\alpha p - n)/(p-1)} \}^{1-p} \\ \leq B_{\alpha, p}(E) \leq C \{ \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-n/(p-1)} \ell_j^{(\alpha p - n)/(p-1)} \}^{1-p}$$

and if  $\alpha p = n$ , then

$$C^{-1} \{ 1 + (-\log \ell_0) + \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-n/(p-1)} (-\log \ell_j) \}^{1-p} \\ \leq B_{\alpha, p}(E) \leq C \{ \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-n/(p-1)} (-\log \ell_j) \}^{1-p},$$

where the number  $C (\geq 1)$  depends only on  $n$ ,  $p$  and  $\alpha$ .

Remark. (i) If the condition  $\ell_0 \leq 1$  is dropped, then the assertion is still valid for the case  $\alpha p < n$  but in this case the constant  $C$  also depends on  $\ell_0$ . (ii) In case  $\alpha p < n$ , since  $R_{\alpha, p}(A)$  is comparable to  $B_{\alpha, p}(A)$  whenever  $\text{diam } A \leq r_0 (< \infty)$  and  $R_{\alpha, p}(rA) = r^{n-\alpha p} R_{\alpha, p}(A)$ , where  $rA = \{rx; x \in A\}$  for  $r > 0$ , the above result holds replaced  $B_{\alpha, p}$  by  $R_{\alpha, p}$  for all  $\ell_0 (> 0)$ . (iii) In case  $p = 2$ , this theorem is a refinement of Ohtsuka's result in [10], because if  $0 < 2\alpha < n$ , then  $R_{\alpha, 2}$  is comparable to  $C_{2\alpha}$ , where  $C_{2\alpha}$  denotes the Riesz capacity corresponding to the Riesz kernel  $|x|^{2\alpha-n}$  and if  $2\alpha = n$ , then  $B_{\alpha, 2}(A)$  is comparable to the logarithmic capacity of  $A$ , provided  $\text{diam } A \leq r_0 (< 1)$ . Clearly, Theorem A is a corollary of this theorem.

In §2 and §3 we shall give an outline of the proof of our theorem. As an application of our estimates in §4 we construct

a set which belongs to the  $(\beta, q)$ -fine topology  $\tau_{\beta, q}$  but not to the  $(\alpha, p)$ -fine topology  $\tau_{\alpha, p}$ , provided either  $0 < \beta q < \alpha p < n$  or  $0 < \beta q = \alpha p < n$  and  $q > p$  or  $0 < \beta q < \alpha p = n$  or  $\beta q = \alpha p = n$  and  $q > p$ , and give its brief proof. (Inclusion relations among these fine topologies have been obtained in [3; Theorem B].)

## §2. The upper estimate

In this section we obtain the upper estimate. In the sequel, for simplicity, let  $a = 1/(p-1)$  and  $d = n - \alpha p$ . We use the following theorem obtained by Maz'ya and Khavin which is a generalization of a Carleson's theorem ([4; §1V, Theorem 2]).

Theorem B ([8; Theorem 7.3]). Let  $A$  be a Borel set in  $R^n$  with diameter  $\leq n^{1/2}$  and for  $r > 0$ , let  $\mathcal{A}(r)$  be the minimum number of closed balls with radii  $\leq r$  which cover  $A$ . Then

$$B_{\alpha, p}(A) \leq C \left( \int_0^{n^{1/2}} (r^d \mathcal{A}(r))^{-a} r^{-1} dr \right)^{1-p},$$

where  $C$  depends only on  $n$ ,  $p$  and  $\alpha$ .

Remark. In this theorem we can replace the above  $\mathcal{A}(r)$  by a measurable function  $\tilde{\mathcal{A}}(r)$ , where  $A$  is covered by at most  $\tilde{\mathcal{A}}(r)$  union of closed balls with radii  $\leq r$ .

Now, let  $A = E$ . Then  $\mathcal{A}(r) \leq (k_1 \dots k_{j+1})^n$  for  $t_{j+1} \leq r < t_j$  ( $j = 0, 1, \dots$ ), where  $t_j = n^{1/2} \ell_j / 2$ , because  $E_n^{(j+1)}$  can be covered by  $(k_1 \dots k_{j+1})^n$  closed balls with radii  $t_{j+1}$ .

In the case where  $\alpha p < n$ , by Theorem B we can obtain

$$B_{\alpha, p}(E) \leq C \left\{ \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-an} \ell_j^{-ad} \right\}^{1-p},$$

where the constant  $C$  depends only on  $n$ ,  $p$  and  $\alpha$ , and in the sequel the symbol  $C$  stands for a constant  $\geq 1$ , whose value may vary from a line to the next. In the case where  $\alpha p = n$ , similarly we obtain

$$B_{\alpha,p}(E) \leq C \left\{ \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-\alpha n} (-\log l_j) \right\}^{1-p}.$$

Thus the estimate from the above is proved.

### §3. The lower estimate

To obtain the lower estimate, we may assume that  $\sum_{j=1}^{\infty} (k_1 \dots k_j)^{-\alpha n} l_j^{-\alpha d} < \infty$  for  $\alpha p < n$  and  $\sum_{j=1}^{\infty} (k_1 \dots k_j)^{-\alpha n} (-\log l_j) < \infty$  for  $\alpha p = n$ . For a Borel set  $A$  in  $R^n$ , we consider another capacity  $\tilde{b}_{\alpha,p}$  defined by

$$\tilde{b}_{\alpha,p}(A) = \sup \nu(R^n),$$

where the supremum is taken over all non-negative measures  $\nu$  such that  $\nu(R^n \setminus A) = 0$  and  $\int W_{\alpha,p}^{\nu}(x) d\nu(x) \leq 1$ . Here  $B(x, r)$  denotes the open ball with center at  $x$  and radius  $r$  and

$$W_{\alpha,p}^{\nu}(x) = \int_0^1 \{r^{-d} \nu(B(x, r))\}^{\alpha} r^{-1} dr.$$

Then it follows from [6; Theorem 1] (and also see [1] and [2]) that there exists a positive number  $C (\geq 1)$  such that

$$(1) \quad C^{-1} \tilde{b}_{\alpha,p}^p(A) \leq B_{\alpha,p}(A) \leq C \tilde{b}_{\alpha,p}^p(A)$$

for every Borel set  $A \subset R^n$ .

The following lemma can be proved by using Fatou's lemma and [7; Introduction, Corollary 1 of Lemma 0.1].

Lemma. If non-negative measures  $\mu_j$  converge vaguely to  $\mu$  as  $j \rightarrow \infty$ , then for every  $x \in \mathbb{R}^n$

$$\liminf_{j \rightarrow \infty} W_{\alpha, p}^{\mu_j}(x) \geq W_{\alpha, p}^{\mu}(x).$$

Let  $\mu_j = (k_1 \dots k_j)^{-n} \ell_j^{-n} \chi_{E_n^{(j)}} dx$  on  $\mathbb{R}^n$  for  $j = 1, 2, \dots$ ,

where  $\chi_A$  denotes the characteristic function of  $A$  and  $dx$  means the  $n$ -dimensional Lebesgue measure. Then  $\mu_j(\mathbb{R}^n) = 1$  and for  $x \in E_n^{(j)}$  we obtain

$$(2) \quad \mu_j(B(x, r)) \leq \begin{cases} C(k_1 \dots k_j)^{-n} \ell_j^{-n} r^n, & 0 < r \leq \ell_j, \\ C(k_1 \dots k_q)^{-n} s^n, & r_{q, s} \leq r < r_{q, s+1} \end{cases}$$

$$(1 \leq s \leq k_q - 1, 1 \leq q \leq j),$$

where  $r_{q, s} = s\ell_q + (s-1)\delta_q$ , since for  $r_{q, s} \leq r < r_{q, s+1}$  ( $1 \leq s \leq k_q - 1$  and  $1 \leq q \leq j$ ), the number of cubes composing the set  $E_n^{(j)}$  which meet  $B(x, r)$  is at most  $(6s)^n (k_{q+1} \dots k_j)^n$ .

First, we assume  $\alpha p < n$  and estimate  $W_{\alpha, p}^{\mu_j}$  on  $E$ . For  $x \in E$ , by virtue of (2) we have

$$(3) \quad W_{\alpha, p}^{\mu_j}(x) \leq C\{\ell_0^{-\alpha d} + \sum_{q=1}^{\infty} (k_1 \dots k_q)^{-\alpha n} \ell_q^{-\alpha d}\}.$$

Note that by our assumption the right side of (3) is convergent. From the sequence  $\{\mu_j\}$  we can extract a subsequence which converges vaguely to some measure  $\mu$  with support in  $E$  and  $\mu(\mathbb{R}^n) = 1$ . Hence by the Lemma

$$(4) \quad W_{\alpha, p}^{\mu}(x) \leq C\{\ell_0^{-\alpha d} + \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-\alpha n} \ell_j^{-\alpha d}\}$$

for every  $x \in E$ . Since  $\int W_{\alpha,p}^{c\mu}(x) d(c\mu)(x) = c^{ap} \int W_{\alpha,p}^{\mu}(x) d\mu(x)$  for  $c > 0$ , it follows from (4) that

$$\tilde{b}_{\alpha,p}(E) \geq C^{-1} \{l_0^{-ad} + \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-an} l_j^{-ad}\}^{(1-p)/p}.$$

Thus on account of (1) we obtain the desired lower estimate in case  $\alpha p < n$ .

Next, we assume that  $\alpha p = n$ . For  $x \in E$  we can obtain

$$W_{\alpha,p}^{\mu_j}(x) \leq C \{1 + (-\log l_0) + \sum_{q=1}^{\infty} (k_1 \dots k_q)^{-an} (-\log l_q)\}.$$

Hence by an argument similar to the above, we can prove the desired result. Thus the lower estimate is obtained.

Remark. We can obtain an integral estimate of the Bessel capacity of symmetric generalized Cantor sets. Let  $A(r) = (k_1 \dots k_{j+1})^n$  for  $t_{j+1} \leq r < t_j$  ( $j = 0, 1, \dots$ ) (for the definition of  $t_j$  see §2). Then we easily prove

$$\begin{aligned} C^{-1} \int_0^{n^{1/2}} (r^d A(r))^{-a} r^{-1} dr &\leq \sum_{j=1}^{\infty} (k_1 \dots k_j)^{-na} l_j^{-ad} \\ &\leq C \int_0^{n^{1/2}} (r^d A(r))^{-a} r^{-1} dr \end{aligned}$$

and hence

$$\begin{aligned} C^{-1} \{l_0^{-ad} + \int_0^{n^{1/2}} (r^d A(r))^{-a} r^{-1} dr\}^{1-p} &\leq B_{\alpha,p}(E) \\ &\leq C \left\{ \int_0^{n^{1/2}} (r^d A(r))^{-a} r^{-1} dr \right\}^{1-p}. \end{aligned}$$

#### §4. Application

Following N. G. Meyers [9], we shall say that a set  $E$  is  $(\alpha, p)$ -thin at  $x \in \mathbb{R}^n$  if

$$\int_0^1 \{r^{-d} B_{\alpha, p}(E \cap B(x, r))\}^a r^{-1} dr < \infty.$$

We define the  $(\alpha, p)$ -fine topology  $\tilde{\tau}_{\alpha, p}$  (see, e.g. [3]) to be the collection of all sets  $H \subset \mathbb{R}^n$  such that  $\mathbb{R}^n \setminus H$  is  $(\alpha, p)$ -thin at every point of  $H$ . In this section we construct sets stated in the introduction by using the estimate of Bessel capacity of Cantor sets.

**Proposition.** Assume that (i)  $0 < \beta q < \alpha p < n$  or (ii)  $0 < \beta q = \alpha p < n$  and  $q > p$  or (iii)  $0 < \beta q < \alpha p = n$  or (iv)  $\beta q = \alpha p = n$  and  $q > p$ . Then there exists a generalized Cantor set  $E$  such that  $(\mathbb{R}^n \setminus E) \cup \{x_0\} \in \tilde{\tau}_{\beta, q} \setminus \tilde{\tau}_{\alpha, p}$ , where  $x_0 \in E$ .

To prove the proposition, we construct a Cantor set of zero  $B_{\beta, q}$ -capacity which is not  $(\alpha, p)$ -thin at each of its points. In case (i), (ii) or (iii) let  $k_j = 2$  for  $j \geq 1$  and let  $l_j = \{2^{-n(j+j_0)}(j+j_0)^{q-1}\}^{1/(n-\beta q)}$  for  $j \geq 0$ , where  $j_0$  is so chosen that  $2l_{j+1} < l_j$  ( $j \geq 0$ ) and  $l_0 \leq 1$ . Let  $E$  be a symmetric generalized Cantor set constructed by  $[\{k_j\}_{j=1}^{\infty}, \{l_j\}_{j=0}^{\infty}]$ . We have  $B_{\beta, q}(E) = 0$  by the Theorem, since  $\sum_{j=1}^{\infty} (2^{nj} l_j^{n-\beta q})^{-1/(q-1)} = \infty$ . Thus the set  $E$  is  $(\beta, q)$ -thin at every point. Next, we show that  $E$  is not  $(\alpha, p)$ -thin at each of its points. Since  $E \cap B(x, r)$  contains some symmetric generalized Cantor sets for every  $x \in E$  and  $r > 0$ , by using the lower estimate of our theorem we can prove that

$$\int_0^1 (r^{-d} B_{\alpha, p}(E \cap B(x, r)))^a r^{-1} dr = \infty.$$

The case (iv). Let  $k_j = 2$  for  $j \geq 1$  and let  $l_j = \exp\{-(j+j_0)^{-1} 2^{n(j+j_0)/(q-1)}\}$  for  $j \geq 0$ , where  $j_0 (\geq 0)$  is so chosen that  $2l_{j+1} < l_j$  for all  $j \geq 0$  and  $n^{1/2}l_0 < 1$ . Let  $E$  be a symmetric generalized Cantor set constructed by the system  $[\{k_j\}_{j=1}^{\infty}, \{l_j\}_{j=0}^{\infty}]$ . Then we can obtain  $B_{\beta,q}(E) = 0$  and for every  $x \in E$

$$\int_0^1 \{B_{\alpha,p}(E \cap B(x, r))\}^a r^{-1} dr = \infty.$$

Therefore in each case we have constructed a Cantor set with desired properties.

Finally, take a point  $x_0 \in E$  and set  $H = (R^n \setminus E) \cup \{x_0\}$ . Then  $H \in \tau_{\beta,q} \setminus \tau_{\alpha,p}$ , because  $B_{\beta,q}(E) = 0$ ,  $R^n \setminus H = E \setminus \{x_0\}$  and  $E \setminus \{x_0\}$  is not  $(\alpha,p)$ -thin at  $x_0$ .

Remark. For the present I can not construct a set contained in  $\tau_{\alpha,p} \setminus \tau_{\beta,q}$  in the case where  $0 < \beta q \leq \alpha p < n$  and  $(n - \beta q)/(q - 1) < (n - \alpha p)/(p - 1)$  by using symmetric generalized Cantor sets.

#### References

- [1] D. R. Adams and N. G. Meyers, Thinness and Wiener criteria for non-linear potentials, Indiana Univ. Math. J. 22(1972), 169 - 197.
- [2] D. R. Adams and N. G. Meyers, Bessel potentials. Inclusion relations among classes of exceptional sets, Indiana Univ. Math. J. 22(1973), 873 - 905.
- [3] D. R. Adams and L. I. Hedberg, Inclusion relations among

- fine topologies in non-linear potential theory, Indiana Univ. Math. J. 33(1984), 117 - 126.
- [4] L. Carleson, Selected problems on exceptional sets, Van Nostrand, New York, 1967.
- [5] K. Hatano, Bessel capacity of symmetric generalized Cantor sets, to appear in Hiroshima Math. J..
- [6] L. I. Hedberg and Th. H. Wolff, Thin sets in nonlinear potential theory, Ann. Inst. Fourier, Grenoble, 33-4 (1983), 161 - 187.
- [7] N. S. Landkof, Foundations of modern potential theory, Springer-Verlag, Berlin-Heidelberg-New York, 1972.
- [8] V. G. Maz'ya and V. P. Khavin, Non-linear potential theory, Russian Math. Surveys, 27(1972), 71 - 148.
- [9] N. G. Meyers, Continuity properties of potentials, Duke Math. J. 42(1975), 157 - 166.
- [10] M. Ohtsuka, Capacité d'ensembles de Cantor généralisés, Nagoya Math. J. 11(1957), 151 - 160.

Faculty of Education,  
Shimane University