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AN INTEGRAL REPRESENTATION THEOREM FOR THE HELMHOLTZ EQUATION

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§1. The purpose of this paper is to show the integral representation for positive solutions of the Helmholtz equation $(\Delta - I)f = 0$ on $(0,\infty)^n \times R^{N-n}$ by a passage to the theory of the heat equation. In the case $n = 0$, it is well-known (see for example [2] and [3]) that every positive solution has an integral representation

$$(1) \qquad f(X) = \int_{S^{N-1}} \exp(\langle X,A\rangle)d\mu(A)$$

where μ is a positive measure on the sphere. We give here a new proof of this fact as an illustration of our method. Let $f > 0$ be a solution of $\Delta f = f$ on R^N. Then the function $u(X,t) = e^t f(X)$ satisfies $\Delta u = \frac{\partial u}{\partial t}$ on $R^N \times R$. Hence by the integral representation theorem for positive solutions of the heat equation ([1, p.374]) there is a positive measure μ on R^N such that

$$u(X,t) = \int_{R^N} \exp(\langle X,A\rangle + t\|A\|^2)d\mu(A).$$

Since $0 = (\Delta - I)^2 f(X) = \int (\|A\|^2 - 1)^2 \exp(\langle X,A\rangle + t(\|A\|^2 - 1))d\mu(A)$, we have $\mathrm{supp}(\mu) \subset S^{N-1}$, so that (1) is obtained.

In section 2 we describe our main theorem for general $n \geqq 1$. After giving the integral representation theorems for the heat equation in section 3, we prove the theorem in section 4. Finally we make a remark about the minimal Martin boundary at infinity with respect to the Helmholtz equation.

§2. Given integers N and n with $1 \leqq n \leqq N$, let $D = (0,\infty)^n \times R^{N-n} = \{X = (x_1, x_2, \ldots, x_N) \in R^N;\ x_i > 0$ for $i = 1,2,\ldots,n\}$. The Green function of the Helmholtz equation $\Delta - I$ on D is given by

$$G(X,Y) = \int_0^\infty e^{-t}\left[\prod_{i=1}^{n}\{w(x_i-y_i,t) - w(x_i+y_i,t)\}\prod_{i=n+1}^{N} w(x_i-y_i,t)\right]dt$$

where $w(x,t) = (4\pi t)^{-1/2}\exp(-x^2/4t)$ if $t > 0$, and $= 0$ if $t \leqq 0$.

Now, for each $A = (a_1, \ldots, a_N) \in \partial D$, we define

$$H_1(X,A) = \left(\prod_{i\in\tau(A)} \frac{\partial}{\partial y_i}\right)G(X,Y)\Big|_{Y=A}$$

where $\tau(A) = \{i;\ 1 \leqq i \leqq n$ and $a_i = 0\}$.

For every subset $\Sigma \subset \{1,2\ldots,n\}$, we put $\Sigma_1 = \{1,2,\ldots,n\} - \Sigma$ and $S_\Sigma = \{A \in S^{N-1};\ a_i = 0$ for any $i \in \Sigma$ and $a_i > 0$ for any $i \in \Sigma_1\}$. For each $A \in \overline{D} \cap S^{N-1}$, we also define

$$H_2(X,A) = \prod_{i\in\Sigma} x_i \prod_{i\in\Sigma_1} \sinh(a_i x_i) \prod_{i=n+1}^{N} \exp(a_i x_i) \quad \text{if } A \in S_\Sigma.$$

Observe that $H_j(\cdot,A)$, $j = 1, 2$, are positive solutions of the Helmholtz equation on D.

We now state the theorem in this paper.

Theorem. For every positive solution f of the Helmholtz equation on $D = (0,\infty)^n \times R^{N-n}$, there are unique Borel measures μ_1 on ∂D and μ_2 on $\overline{D} \cap S^{N-1}$ such that

$$f(X) = \int H_1(X,A)d\mu_1(A) + \int H_2(X,A)d\mu_2(A). \tag{2}$$

Furthermore if f is continuous on $\overline{D}$ then $d\mu_1(A) = f(A)d\sigma(A)$, where $d\sigma(.)$ is the surface measure on ∂D.

§3. In this section we give integral representation theorems for the heat equation. Following [1], a solution of the heat equation will be said to be parabolic.

For $x, t \in R$ and $a \geqq 0$, we put

$$k(x,t,a) = \begin{cases} w(x-a,t) - w(x+a,t) & \text{if } a > 0 \\ \frac{x}{t} w(x,t) \quad (= 0 \text{ if } t = 0) & \text{if } a = 0 \end{cases}$$

and

$$k^*(x,t,a) = \begin{cases} \sinh(ax)\exp(ta^2) & \text{if } a > 0 \\ x & \text{if } a = 0. \end{cases}$$

Let $D = (0,\infty)^n \times R^{N-n}$ as before. For each $(X,t) \in D\times(-\infty,\infty)$ and $(A,s) \in \overline{D}\times[-\infty,\infty)$ we define

$$K((X,t),(A,s)) = \begin{cases} \prod_{i=1}^{n} k(x_i,t-s,a_i) \prod_{i=n+1}^{N} w(x_i-a_i,t-s) & \text{if } s \in R. \\ \prod_{i=1}^{n} k^*(x_i,t,a_i) \prod_{i=n+1}^{N} \exp(a_i x_i + t a_i^2) & \text{if } s = -\infty. \end{cases}$$

The following was proved in [4, Theorems 2.2 and 3.4] in the case $n = 1$ (see also [5]), and a similar proof can be carried out for arbitrary $n \geqq 1$ so that we have

Proposition 1. For every positive parabolic function u on $D\times(0,\infty)$, there is a unique Borel measure μ on $\partial(D\times(0,\infty))$ such that

$$u(X,t) = \int K((X,t),(A,s))d\mu(A,s).$$

In particular if u is continuous on $\overline{D}\times[0,\infty)$ then $d\mu(A,s) = u(A,s)d\sigma(A,s)$, where $d\sigma(.,.)$ is the surface measure of $\partial(D\times(0,\infty))$, and if u is continuous on $\overline{D}\times(0,\infty)$ then $d\mu(A,s) = u(A,s)d\sigma(A)ds$ on $\partial D\times(0,\infty)$, where $d\sigma(.)$ is the surface measure on ∂D.

By the Appell transform the integral representation on $D\times(-\infty,0)$ was given in [4, Theorem 4.1] (in the case $n = 1$). Since this method is available for arbitrary $n \geqq 1$, we also see

Proposition 2. For every positive parabolic function u on $D\times(-\infty,0)$ there is a unique Borel measure μ on $\partial D\times(-\infty,0)\cup\overline{D}\times\{-\infty\}$ such that

(3) $$u(X,t) = \int K((X,t),(A,s))d\mu(A,s).$$

In particular if u is continuous on $\overline{D}\times(-\infty,0)$ then $d\mu(A,s) = u(A,s)d\sigma(A)ds$ on $\partial D\times(-\infty,0)$.

We remark here that the second assertion is deduced from the last assertion in Proposition 1 by applying the Appell transform.

Before returning to the Helmholtz equation, we make an observation on the Martin boundary of $D\times(-\infty,0)$ with respect to the heat equation. (For details, we refer to [1, p.262-383]). Let $A_1 = (a_1,a_2,\dots,a_N)$ with $a_i = 1$, $1 \leqq i \leqq n$ and $= 0$, $n+1 \leqq i \leqq N$. Then $((A_1,0),D\times(-\infty,0))$ is a Martin point set pair ([1, p.359]). By the same manner as in [1, p.374-375, in the case $N = n = 1$] we see that the Martin boundary $\partial^M(D\times(-\infty,0))$ for this pair is $\partial D\times(-\infty,0)\cup\overline{D}\times\{-\infty\}\cup\{0_\infty\}$ and the Martin kernel is given by

$$K^*((X,t),(A,s)) = \frac{K((X,t),(A,s))}{K((A_1,0),(A,s))}$$

for $(A,s) \in \partial D\times(-\infty,0)\cup\overline{D}\times\{-\infty\}$ and $K^*((X,t),0_\infty) = 0$. In the Martin topology, $(Y,r) \in D\times(-\infty,0)$ tends to $(A,s) \in \partial D\times(-\infty,0)$ if and only if $(Y,r) \to (A,s)$, (Y,r) tends to $(A,-\infty) \in \overline{D}\times\{-\infty\}$ if and only if $r \to -\infty$ and $Y/-r \to A$, and (Y,r) tends to 0_∞ if and only if $r \to 0$ or $\|Y\|/(1-r) \to \infty$. Thus, on $\partial D\times(-\infty,0)$ the Martin topology coincides with the Euclidean topology. Similarly to [1, p.367], we also see that 0_∞ is the only non-minimal Martin boundary point. If u is positive parabolic on $D\times(-\infty,0)$ and ${}^{pf}u(A_1,0) < \infty$ (the parabolic fine limit at

$(A_1,0)$, cf. [1, p.359]), then there is a unique Borel measure μ^* on $\partial^M(D\times(-\infty,0))$ with $\int d\mu^* = {}^{Pf}u(A_1,0)$ such that

$$(4) \qquad u(X,t) = \int K^*((X,t),(A,s))d\mu^*(A,s).$$

§4. In this section we give a proof of the theorem. Now, let $f > 0$ be a solution of $\Delta f = f$ on $D = (0,\infty)^n \times R^{N-n}$.

We first assume that f is continuous on $\overline{D}$. Then the function $u(X,t) = e^t f(X)$ is continuous on $\overline{D}\times(-\infty,0)$ and parabolic on $D\times(-\infty,0)$. By Proposition 2, there is a Borel measure μ_2 on $\overline{D}$ (from now on we identify $\overline{D}\times\{-\infty\}$ with $\overline{D}$) such that

$$(5) \quad u(X,t) = \iint_{-\infty}^{0} K((X,t),(A,s))e^s f(A)dsd\sigma(A) + \int K((X,t),(A,-\infty))d\mu_2(A).$$

An elementary culculation shows that for each $A \in \partial D$

$$(6) \qquad e^{-t}\int_{-\infty}^{0} K((X,t),(A,s))e^s ds = \int_0^\infty e^{-t}K((X,t),(A,0))dt = H_1(X,A),$$

which also implies that $e^{-t}\int K((X,t),(A,-\infty))d\mu_2(A)$ is independent of t and is a solution of $\Delta f = f$. It follows that $\mathrm{supp}(\mu_2) \subset \overline{D} \cap S^{N-1}$, for

$$(7) \qquad \begin{aligned} 0 &= (\Delta - I)^2 \int_{\overline{D}} K((X,t),(A,-\infty))d\mu_2(A) \\ &= \int_{\overline{D}} (\|A\|^2-1)^2 K((X,t),(A,-\infty))d\mu_2(A). \end{aligned}$$

Since for each $A \in \overline{D} \cap S^{N-1}$

(8) $$\lim_{t\uparrow 0} K((X,t),(A,-\infty)) = H_2(X,A) \text{ (increasingly)},$$

we have the second part of the Theorem by letting $t \uparrow 0$ in (5).

In the general case, we put

$$f_m(X) = f(x_1+1/m, x_2+1/m, \ldots, x_n+1/m, x_{n+1}, \ldots, x_N)$$

and $u_m(X,t) = e^t f_m(X)$ for each $m \geqq 1$. Then f_m is continuous on $\overline{D}$ and satisfies the Helmholtz equation on D. Hence by (4) and the above proof, there exists a Borel measure $\mu_{2,m}$ on $\overline{D} \cap S^{N-1}$ such that

$$e^t f_m(X) = \int K^*((X,t),(A,s))d\mu^*_{1,m}(A,s) + \int K^*((X,t),(A,-\infty))d\mu^*_{2,m}(A)$$

(9)

$$= \iint K((X,t),(A,s))e^s f_m(A)dsd\sigma(A) + \int K((X,t),(A,-\infty))d\mu_{2,m}(A),$$

where $\mu^*_{1,m} = K((A_1,0),(A,s))e^s f_m(A)dsd\sigma(A)$ and $\mu^*_{2,m} = K((A_1,0),(A,-\infty))d\mu_{2,m}(A)$. Since ${}^{pf}u_m(A_1,0) = \lim_{t\uparrow 0} e^t f_m(A_1) = f_m(A_1)$ is bounded in m, $(\mu^*_{i,m})_{m=1}^{\infty}$ $(i = 1, 2)$ is a vaguely bounded sequence of positive measures on the Martin boundary $\partial^M(D\times(-\infty,0))$, so that we may assume that this has a vague limit μ^*_i $(i = 1, 2)$. Then we see that $\mathrm{supp}(\mu^*_2) \subset \overline{D} \cap S^{N-1}$ and

(10) $$\lim_{m\to\infty} \int K((X,t),(A,-\infty))d\mu_{2,m}(A) = \int K^*((X,t),(A,-\infty))d\mu_2(A).$$

Now, we denote by $\mu^{**}_{1,1}$ and $\mu^{**}_{1,2}$ the restrictions of the measure μ^*_1

to $\partial D\times(-\infty,0)$ and to $\overline{D}$, respectively. We shall show that there is a measure μ_1 on D such that $\mu^{**}_{1,1}(A,s) = K((A_1,0),(A,s))e^s d\mu_1(A)ds$. Let ψ be an arbitrary continuous function on $\partial D\times(-\infty,0)$ with compact support and fix $-\infty < s_o < 0$. We can easily check that the function $\psi(A,s)e^s K((A_1,0),(A,s))/K((A_1,0),(A,s_o))$ in (A,s) is continuous and has compact support on $\partial D\times(-\infty,0)$ and that there is a constant $C = C(\psi,s_o) > 0$ such that $e^s K((A_1,0),(A,s)) \geqq CK((A_1,0),(A,s_o))$ on $\mathrm{supp}(\psi)$. Since

$$f_m(A_1) \geqq \int_{\partial D}\int_{-\infty}^{0} K((A_1,0),(A,s))e^s f_m(A)d\sigma(A)ds$$

$$\geqq C \iint_{\mathrm{supp}(\psi)} K((A_1,0),(A,s_o))f_m(A)d\sigma(A)ds,$$

we may assume that $K((A_1,0),(A,s_o))f_m(A)d\,\sigma(A)$ converges vaguely to a Borel maesure $\tilde{\mu}$ on ∂D as $m \to \infty$. Then

$$\int\psi d\mu^{**}_{1,1} = \lim_{m\to\infty} \int\psi d\mu^{*}_{1,m}$$

$$= \lim_{m\to\infty} \iint \frac{\psi(A,s)e^s K((A_1,0),(A,s))}{K((A_1,0),(A,s_o))} K((A_1,0),(A,s_o))f_m(A)d\sigma(A)ds$$

$$= \iint \psi(A,s)e^s K((A_1,0),(A,s))\frac{1}{K((A_1,0),(A,s_o))}d\tilde{\mu}(A)ds.$$

Therefore $d\mu^{**}_{1,1} = K((A_1,0),(A,s))e^s d\mu_1(A)ds$, where $d\mu_1(A) = (K((A_1,0),(A,s_o)))^{-1}d\tilde{\mu}(A)$.

Consequently, letting $m \to \infty$ in (9) and remarking (6) and (10), we have

$$e^t f(X) = \iint K((X,t),(A,s))e^s ds d\mu_1(A) + \int K^*((X,t),(A,-\infty))d(\mu^{**}_{1,2} + \mu^*_2)(A)$$

$$= \int e^t H_1(X,A)d\mu_1(A) + \int K((X,t),(A,-\infty))d\mu_2(A,s),$$

where $d\mu_2(A) = (K((A_1,0),(A,-\infty)))^{-1} d(\mu^{**}_{1,2}+\mu^*_2)(A)$. By the same manner as in (7) we see $\mathrm{supp}(\mu_2) \subset \overline{D} \cap S^{N-1}$. Hence, as a consequence of (8), the desired integral representation (2) follows by letting $t \uparrow 0$. Since the uniqueness of the representation measures follows from Proposition 2, we obtain our theorem.

§5. It is easily seen that our method is also available for the operator $\Delta - cI$ (c: real constant) on D. Remark that if $c < 0$ there is no positive solution. In the Martin boundary theoretic view point, our result explains that the minimal Martin boundary of D at infinity with respect to $\Delta - cI$ (i.e., the set of normalized minimal solutions which vanish at all finite boundary points) is homeomorphic to $c(S^{N-1} \cap \overline{D}) = \{cA,\ A \in S^{N-1} \cap \overline{D}\}$.

On the other hand, Landis & Nadirashvili [6] tells us that

$$\{f;\ \Delta f = 0 \text{ and } f > 0 \text{ in } D_E,\ f = 0 \text{ on } \partial D_E\}$$

is one dimensional, where $E \subset S^{N-1}$ is a domain with Lipschitz boundary and $D_E = \{X \in R^N;\ X \neq 0,\ X/\|X\| \in E\}$. By these observations it can be conjectured that the minimal Martin boundary of D_E at infinity with respect to $\Delta - cI$ would be homeomorphic to $c\overline{E}$, but we know no other example which reinforces this conjecture.

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