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Subordinate families to convolution kernels of logarithmic type

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§ 1. Let X be a locally compact abelian group with countable basis, $M_K(X)$ the usual topological vector space of all real Radon measures on X with compact support and let $M_K^o(X) = \{\mu \in M_K(X);\ \int d\mu = 0\}$. A real convolution kernel N on X means a real Radon measure on X. We say that N is a convolution kernel of logarithmic type (resp. a Hunt convolution kernel) if for any $\mu \in M_K^o(X)$ (resp. $\mu \in M_K(X)$), $N*\mu$ is of form

(1.1) $$N*\mu = \int_o^\infty \alpha_t * \mu dt,$$

where $(\alpha_t)_{t\geq o}$ is a semi-transient convolution semi-group (resp. a transient convolution semi-group) on X. In this case, $(\alpha_t)_{t\geq o}$ is uniquely determined (see [1] and [6]) and called the convolution semi-group of N. Evidently Hunt convolution kernels are always convolution kernels of logarithmic type.

Let $H_b(R^+)$ be the set of all Hunt convolution kernels on R supported by $R^+ = \{t \in R;\ t \geq 0\}$ whose convolution semi-groups are sub-Markovian, where R denotes the 1-dimensional Euclidean space.

For any Hunt convolution kernel $N = \int_o^\infty \alpha_t dt$ on X and for any $\kappa \in H_b(R^+)$, $\int \alpha_t d\kappa(t)$ defines a convolution kernel on X and is a Hunt convolution kernel (see [3]). Put

(1.2) $$H(N;\ X) = \{N_{(\kappa)} = \int \alpha_t d\kappa(t);\ \kappa \in H_b(R^+)\}.$$

We call H(N; X) the subordinate family to N. To determine Hunt convolution kernels as a kind of the perturbation of N, H(N; X) plays a fundamental role.

Let N be a convolution kernel of logarithmic type, $(\alpha_t)_{t\geq o}$ the convolution semi-group of N and let $\kappa \in H_b(R^+)$. It was a question if there exists a convolution kernel $N_{(\kappa)}$ of logarithmic type on X satisfying

$$(1.3) \qquad N_{(\kappa)}*\mu = \lim_{a\to\infty} \int_o^a \alpha_t*\mu d\kappa(t) \quad (= \int \alpha_t*\mu d\kappa(t))$$

for every $\mu \in M_K^o(X)$ (see [5]). In the previous paper [8], we gave an affirmative solution:

Theorem. Let N be a convolution kernel of logarithmic type on X and $(\alpha_t)_{t\geq o}$ the convolution semi-group of N. Then for any $\kappa \in H_b(R^+)$, there exists a convolution kernel $N_{(\kappa)}$ of logarithmic type on X satisfying (1.3) for every $\mu \in M_K^o(X)$.

The purpose of this note is to give a sketch of the proof and to discuss an application of this theorem.

§ 2. Let $C_K(X)$ denote the usual topological vector space of all finite continuous functions with compact support, $C_b(X)$ the usual Banach space of all bounded continuous functions on X and M(X) the topological vector space of all real Radon measures on X with the vague topology. We denote by $C_K^+(X)$, $C_b^+(X)$, $M_K^+(X)$ and by $M^+(X)$ their subsets of all non-negative elements, respectively.

A family $(\alpha_t)_{t\geq o}$ in $M^+(X)$ is called a convolution semi-group on X if α_o = the unit measure ε at the origin 0 of X, $\alpha_t*\alpha_s = \alpha_{t+s}$ for all $t, s \in R^+$ and if $R^+ \ni t \to \alpha_t \in M(X)$ is continuous.

We say that:

$(\alpha_t)_{t\geq o}$ is sub-Markovian (resp. Markovian) if for any $t \geq 0$, $\int d\alpha_t \leq 1$ (resp. $\int d\alpha_t = 1$).

$(\alpha_t)_{t\geq o}$ is transient if for any $f \in C_K^+(X)$, $\int_o^\infty \int f d\alpha_t dt < \infty$.

$(\alpha_t)_{t\geq o}$ is recurrent if it is not transient.

$(\alpha_t)_{t\geq o}$ is semi-transient if for any $p > 0$, the convolution semi-group $(\exp(-pt)\alpha_t)_{t\geq o}$ is transient and if for any $\mu \in M_K^o(X)$, $(\int_o^t \alpha_s * \mu ds)_{t\geq o}$ is bounded in $M(X)$, where $\int_o^t \alpha_s * \mu ds$ is the real Radon measure on X defined by $C_K(X) \ni f \to \int_o^t \int f d\alpha_s * \mu ds$.

For a convolution semi-group $(\alpha_t)_{t\geq o}$ on X, we put

$$(2.1) \qquad \Gamma((\alpha_t)) = \overline{\bigcup_{t\geq o} \operatorname{supp}(\alpha_t)}.$$

Then $\Gamma((\alpha_t))$ is a semi-group in X. Let $(\alpha_t)_{t\geq o}$ be a convolution semi-group on X such that for any $p > 0$, $(\exp(-pt)\alpha_t)_{t\geq o}$ is transient Put $N_p = \int_o^\infty \exp(-pt)\alpha_t dt$ for every $p > 0$, and $(N_p)_{p>o}$ is called the resolvent defined by $(\alpha_t)_{t\geq o}$. For any $p > 0$ and $q > 0$, the resolvent equation

$$(2.2) \qquad N_p - N_q = (q - p)N_p * N_q$$

holds.

By using the resolvent defined by a convolution semi-group, we obtain the following

Lemma 1. Let $(\alpha_t)_{t\geq o}$ be a semi-transient convolution semi-group on X. If $(\alpha_t)_{t\geq o}$ is recurrent, then $\Gamma((\alpha_t)) = X$ and $(\alpha_t)_{t\geq o}$ is Markovian.

Combining the above lemma and one of the main results in [5], we have the following

Proposition 1. Let $(\alpha_t)_{t\geq o}$ be a recurrent convolution semi-group on X. Then $(\alpha_t)_{t \geq o}$ is semi-transient if and only if $\Gamma((\alpha_t)) = X$, $(\alpha_t)_{t \geq o}$ is Markovian and for any $p > 0$, N_p is non-singular with respect to a fixed Haar measure ξ on X, where $(N_p)_{p>o}$ is the resolvent defined by $(\alpha_t)_{t \geq o}$. In this case, $(N_{p,s})_{p>o}$ is bounded in $M(X)$, where $N_{p,s}$ is the singular part of N_p with respect to ξ.

Proposition 2. Let $(\alpha_t)_{t \geq o}$ be a convolution semi-group on X. Then $(\alpha_t)_{t \geq o}$ is the convolution semi-group of a convolution kernel of logarithmic type if and only if $(\alpha_t)_{t \geq o}$ is semi-transient.

This follows from the above lemma and another main theorem in [5].

Let N be a positive convolution kernel on X and $\mu \in M^+(X)$ with $N*\mu \in M^+(X)$. For an open set ω in X, a non-negative Radon measure μ' on X is called an inner N-balayaged measure of μ to ω if $\mathrm{supp}(\mu') \subset \overline{\omega}$, $N*\mu' \leq N*\mu$ in X, $N*\mu' = N*\mu$ in ω and if for any $\nu \in M^+(X)$ with $N*\nu \in M^+(X)$, $N*\nu \geq N*\mu'$ in X whenever $N*\nu \geq N*\mu$ in ω.

Lemma 2 (see, for example, [4]). Let N be a Hunt convolution kernel on X and $\mu \in M^+(X)$ with $N*\mu \in M^+(X)$. Then for any open set ω in X, there exists a uniquely determined inner N-balayaged measure μ'_ω of μ to ω.

Lemma 3 (see [4]). Let $(\alpha_t)_{t \geq o}$ be a convolution semi-group on X and assume that for any $p > 0$, $(\exp(-pt)\alpha_t)_{t \geq o}$ is transient. Let $(N_p)_{p>o}$ be the resolvent defined by $(\alpha_t)_{t \geq o}$. For any $p > 0$, any open set ω in X and for $\mu \in M^+(X)$ with $N*\mu \in M^+(X)$, we denote by $\mu'_{p,\omega}$ the inner N_p-balayaged measure of μ to ω. Then we have:

(1) The mapping $X \ni x \to \varepsilon'_{x,p,\omega} \in M(X)$ is Borel measurable, that is, for any $f \in C_K(X)$, the function $\int f d\varepsilon'_{x,p,\omega}$ of x is Borel measurable on X, where ε_x denotes the unit measure at x and where $\varepsilon'_{x,p,\omega} = (\varepsilon_x)'_{p,\omega}$.

(2) For any $\mu \in M^+(X)$ with $N_p*\mu \in M^+(X)$, we have

$$\mu'_{p,\omega} = \int \varepsilon'_{x,p,\omega} d\mu(x), \tag{2.3}$$

that is, $\int f d\mu'_{p,\omega} = \int\int f d\varepsilon'_{x,p,\omega} d\mu(x)$ for every $f \in C_K(X)$.

(3) For any $p > q > 0$ and any $\mu \in M^+(X)$ with $N_q*\mu \in M^+(X)$, we have

$$\mu'_{q,\omega} = \mu'_{p,\omega} + (p - q)(N_p*(\mu - \mu'_{p,\omega}))'_{q,\omega}. \tag{2.4}$$

Let $(\alpha_t)_{t\geq o}$ be a Markovian convolution semi-group on X and $(N_p)_{p>o}$ the resolvent defined by $(\alpha_t)_{t\geq o}$. For any $p > 0$, any $\mu \in M_K^+(X)$ and for any open set ω in X, the positive mass principle of N_p [1] gives $\int d\mu'_{p,\omega} \leq \int d\mu$. Since $(\mu'_{p,\omega})_{p>o}$ increases as p decreases, $\lim_{p\to o} \mu'_{p,\omega}$ exists in $M^+(X)$. Putting $\mu'_\omega = \lim_{p\to o} \mu'_{p,\omega}$, we call it the inner balayaged measure of μ to ω with respect to $(\alpha_t)_{t\geq o}$. Furthermore, for any $\mu \in M^+(X)$ with $\int d\mu < \infty$, it can be defined by the usual limit process. Lemma 3 gives the following

Lemma 4. Let $(\alpha_t)_{t\geq o}$ be a Markovian convolution semi-group on X, $(N_p)_{p>o}$ the resolvent defined by $(\alpha_t)_{t\geq o}$ and ω an open set in X. Then for any $\mu \in M^+(X)$ with $\int d\mu < \infty$ and any $p > 0$,

$$\mu'_\omega = \int \varepsilon'_{x,\omega} d\mu(x), \tag{2.5}$$

where $\varepsilon'_{x,\omega} = (\varepsilon_x)'_\omega$ and

$$\mu'_\omega = \mu'_{p,\omega} + p(N_p*\mu - N_p*\mu'_{p,\omega})'_\omega. \tag{2.6}$$

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1) This means that for any $\mu, \nu \in M_K^+(X)$, $N_p*\mu \leq N_p*\nu$ on X implies $\int d\mu \leq \int d\nu$.

To prove our main theorem, the following lemma will plays an essential role.

Lemma 5. Let $(\alpha_t)_{t\geq o}$ be a semi-transient and recurrent convolution semi-group on X. Then for any $t_o > 0$ and any $p > 0$, $\mathrm{supp}(\sum_{n=o}^{\infty} \exp(-pnt_o)\alpha_{nt_o}) = X$ and $\sum_{n=o}^{\infty} \exp(-pnt_o)\alpha_{nt_o}$ is non-singular with respect to ξ.

For $\mu \in M^+(X)$, $\mathrm{supp}(\mu)$ denotes the support of μ.

It suffices to show our conclusion in the case of $t_o = 1$. Let $(N_p)_{p>o}$ be the resolvent defined by $(\alpha_t)_{t\geq o}$. Put $N'_p = \sum_{n=o}^{\infty} \exp(-pn)\alpha_n$ and $N''_p = \int_o^1 \exp(-pt)\alpha_t dt$. Then

$$(2.7) \qquad N'_p * N''_p = N_p.$$

By (2.7), the recurrence of $(\alpha_t)_{t\geq o}$ and the Choquet-Deny theorem ([2]), we obtain that $\mathrm{supp}(N'_p) = X$ for every $p > 0$.

Let $\omega \neq \phi$ be an open set in X. By Lemma 4 and the Choquet-Deny theorem ([2]), we obtain that

$$(2.8) \qquad \lim_{p\to o} N'_p * (\varepsilon - \varepsilon''_\omega) = -a_\omega \xi \quad \text{in} \quad \omega$$

with some constant $a_\omega \geq 0$, where ε''_ω is the inner balayaged measure of ε to ω with respect to the convolution semi-group $(\beta_t)_{t\geq o}$ which defines the resolvent $(\frac{1}{p+1}N'_{\log(p+1)})_{p>o}$. We remark here that $(\beta_t)_{t\geq o}$ is also Markovian and recurrent. Furthermore, putting $\eta_\omega = \lim_{p\to o} N'_p * (\varepsilon - \varepsilon''_\omega)$, we obtain that

$$(2.9) \qquad \sum_{n=o}^{\infty} \alpha_n * (\varepsilon - \varepsilon''_\omega) = \eta_\omega$$

and

and that

$$(2.10) \qquad N*(\varepsilon - \varepsilon''_\omega) = N''_o*\eta_\omega,$$

where N is the convolution kernel of logarithmic type on X whose convolution semi-group is equal to $(\alpha_t)_{t\geq o}$. Assume that there exists $p > 0$ such that N'_p is singular with respect to ξ. Then for any $n \geq 1$, $\alpha_n*(\varepsilon - \varepsilon''_\omega)$ is singular with respect to ξ. By (2.8) and (2.9), we have $a_\omega = 0$, so that $\eta_\omega \geq 0$ in X. Therefore (2.10) gives $N*(\varepsilon - \varepsilon''_\omega) \geq 0$ in X. Since ω is arbitrary and $\int d\varepsilon''_\omega = 1$, we have $N \geq N*\varepsilon_x$ in X for every $x \in X$, so that for any $x \in X$, $N = N*\varepsilon_x$, which is a contradiction. Thus N'_p is non-singular with respect to ξ for every $p > 0$.

Lemma 5 gives the following

Proposition 3. Let $(\alpha_t)_{t\geq o}$ be a semi-transient and recurrent convolution semi-group on X. Then there exists $t_o > 0$ such that for any $t \geq t_o$, $\mathrm{supp}(\alpha_t) = X$ and α_t is non-singular with respect to ξ.

In the same manner as in Proposition 32 in [7], we have the following

Lemma 6. Let $(\alpha_t)_{t\geq o}$ be a semi-transient and recurrent convolution semi-group on X and let $\kappa \in H_b(R^+)$. Denote by $(\gamma_t)_{t\geq o}$ the convolution semi-group of κ and by $(\kappa_p)_{p>o}$ the resolvent defined by $(\gamma_t)_{t\geq o}$. Assume that the convolution semi-group $(\int\alpha_s d\gamma_t(s))_{t\geq o}$ is semi-transient and recurrent. Let $N_{(\kappa)}$ be the convolution kernel of logarithmic type on X whose convolution semi-group is equal to $(\int\alpha_s d\gamma(s))_{t\geq o}$. Then for any $\mu \in M_K^o(X)$,

$$(2.11) \qquad N_{(\kappa)}*\mu = \lim_{p\to o} N_{(\kappa_p)}*\mu = \lim_{t\to\infty} \int_o^t \alpha_s*\mu d\kappa(s) \quad (=\int\alpha_t*\mu d\kappa(t)).$$

§ 3. Let us give a sketch of the proof of our theorem. We may assume that $(\alpha_t)_{t\geq o}$ is semi-transient and recurrent. Let κ be in $H_b(R^+)$, $(\gamma_t)_{t\geq o}$ the convolution semi-group of κ and $(\kappa_p)_{p>o}$ the resolvent defined by $(\gamma_t)_{t\geq o}$. It suffices to show our conclusion in the case that $(\int\alpha_s d\gamma_t(s))_{t\geq o}$ is recurrent. Hence it suffices to show that $(\int\alpha_s d\gamma_t(s))_{t\geq o}$ is semi-transient (see Lemma 6). By the usual method, it suffices to assume that κ is elementary, that is, $\kappa = c(\varepsilon + \sum_{n=1}^{\infty}(\sigma)^n)$, where c is a positive constant, $\sigma \in M^+(R)$ with $\mathrm{supp}(\sigma) \subset R^+$ and with $\int d\sigma \leq 1$, $(\sigma)^1 = \sigma$ and where $(\sigma)^n = (\sigma)^{n-1}*\sigma$ for every $n \geq 1$. By the recurrence of $(\int\alpha_s d\gamma_t(s))_{t\geq o}$, $\int d\sigma = 1$ and $\sigma((0, \infty)) > 0$. Hence there exists $t_o > 0$ such that denoting by σ_o the restriction of σ to $[t_o, \infty)$, we have $\sigma_o \neq 0$, $\int d\sigma_o < 1$ and $\mathrm{supp}(\sigma_o) \ni t_o$. Put

$$(3.1) \qquad \kappa_{o,p} = \frac{c}{pc+1}(\varepsilon + \sum_{n=1}^{\infty}(\frac{1}{pc+1}\sigma_o)^n)$$

for every $p > 0$. Since $\kappa_p \geq \kappa_{o,p}$, it suffices to show that $N_{(\kappa_{o,p})}$ is non-singular with respect to ξ (see Proposition 1). Since for any $n \geq 1$,

$$(3.2) \qquad \mathrm{supp}((\int\alpha_t d\sigma_o(t))^n) \supset \mathrm{supp}(\alpha_{nt_o}),$$

Lemma 5 shows that $\mathrm{supp}(N_{(\kappa_{o,p})}) = X$ for every $p > 0$. Furthermore, Proposition 3 shows that $N_{(\kappa_{o,p})}$ is non-singular with respect to ξ. Thus $(\int\alpha_s d\gamma_t(s))_{t\geq o}$ is semi-transient, which proves our main theorem.

§ 4. Let N be a convolution kernel of logarithmic type on X and $(\alpha_t)_{t\geq o}$ the convolution semi-group of N. Put

$$H_1(N; X) = \{N_{(\kappa)} = \int\alpha_t d\kappa(t);\ \kappa \in H_b(R^+)\},$$

$$H_2(N;\ X) = \{(N_1)_{(\kappa)};\ N_1 \in H_1(N;\ X),\ \kappa \in H_b(R^+)\},$$

(4.1) $$\dots\dots\dots\dots\dots\dots,$$

$$H_n(N;\ X) = \{(N_{n-1})_{(\kappa)};\ N_{n-1} \in H_{n-1}(N;\ X), \kappa \in H_b(R^+)\},$$

$$\dots\dots\dots\dots\dots\dots.$$

Then for any $n \geq 1$, $H_n(N;\ X)$ is a family of convolution kernels of logarithmic type on X and $H_n(N;\ X) \subset H_{n+1}(N;\ X)$. Put

(4.2) $$H_\infty(N;\ X) = \overline{\bigcup_{n=1}^{\infty} H_n(N;\ X)},$$

where the closure is in the sense of the vague topology. We call $H_\infty(N;\ X)$ the generalized subordinate family to N.

Proposition 4. Let $M \in H_\infty(N;\ X)$. Then M is a convolution kernel of logarithmic type if and only if M is non-periodic, that is, for any $0 \neq x \in X$, $M*\varepsilon_x \neq M$.

Let M be a given convolution kernel on X. To get a conclusion that M is of logarithmic type, it is very useful to find out a certain convolution kernel N of logarithmic type satisfying $M \in H_\infty(N;\ X)$. For example, let R^n be the n-dimensional Euclidean space ($n \geq 1$) and put

$$N_o = \begin{cases} -|x|dx & \text{on } R^1 \\ (-\log|x|)dx & \text{on } R^2 \\ |x|^{2-n}dx & \text{on } R^n\ (n \geq 3) \end{cases}$$

where $|x|$ denotes the distance between x and 0 and where dx denote the Lebesgue measure. By discussing precisely $H_\infty(N_o;\ R^n)$, we can obtain several concrete conditions to be Hunt convolution kernels on R^n or convolution kernels of logarithmic type on R^n.

BIBLIOGRAPHY

[1]. C. Berg and G. Forst: Potential theory on locally compact abelian groups, Springer-Verlag, Berlin-Heidelberg-Newyork, 1975.

[2]. G. Choquet and J. Deny: Sur l'équation de convolution $\mu = \mu * \sigma$, C. R. Acad. Sci. Paris, 250 (1960), pp. 799 - 801.

[3]. M. Itô: Sur la famille sous-ordonnée au noyau de convolution de Hunt II, Nagoya Math. J., 53 (1974), 115 - 126.

[4]. —— : Sur les noyaux de convolution conditionnellement sous-médiens, ibid, 75 (1979), 1 - 39.

[5]. —— : Les noyau de convolution de type logarithmique, Théorie du potentiel, Proceeding Orsay 1983, Lecture Note in Math., 1096, 345 - 392, Springer-Verlag.

[6]. —— : Une caractérisation des noyaux de convolution réels de type logamithmique, Nagoya Math. J., 97 (1985), 1 - 49.

[7]. —— : Le principe semi-complet du maximum pour les noyaux de convolution réels, ibid, 101 (1986), 55 - 109.