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NOTES ON ZERO SETS OF CONVOLUTION POTENTIALS ON THE LINE

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[0] It is important in potential theory that every bounded potential $U\mu$ of a finite signed measure μ never vanishes everywhere unless $\mu = 0$. For an important class of potentials including Newton potentials and Riesz potentials, "everywhere" can be replaced with "everywhere outside a set of Lebesgue measure 0". In general we can not replace it with "everywhere outside an open set". Indeed there exist examples of Newton potentials $U\mu(x)$ $(= \int |x-y|^{2-d}\mu(dy))$, μ;a signed measure, such that $U\mu$ vanishes outside $\mathrm{supp}(\mu)$,but $\mu \neq 0$. On the other hand, we have the following:

(*) let u be a bounded Riesz potential $U\mu(x)$ $(= \int |x-y|^{\alpha-d}\mu(dy)$, $0<\alpha<2$ in case $d>2$, $0<\alpha<1$ in case $d=1$) of a finite signed measure μ . Then u never vanishes everywhere on an open set outside $\mathrm{supp}(\mu)$ unless $\mu=0$.

The fact (*) is a direct consequence from the anti-locality of the fractional power Δ^{α} of Laplacian, which was proved by Segal-Goodman [6] for a special class and later extended to a larger class of operators including Δ^{α} by M. Murata [4]. So it would be natural to consider the problem below:

(**) let U be a potential kernel of an operator of Lévy-Khintchin type without the diffusion term Can we characterize "a class of sets" for each signed measure μ so that $U\mu$ never vanishes everywhere on a set of "the class" unless $\mu = 0$?

Unfortunately the stage of our research is very far from the solution. So we shall restrict the object in this note to the simplest class of operators on the line, of the form: $Au(x) = \int_{R-\{0\}}[\, u(x+y) - u(x)\,]\, n(dy)$, where $n(dy)$ is the measure on $R-\{0\}$ such that $\int \{|y|/1+|y|\}\, n(dy) < \infty$. In §1 we shall give few elementary facts connected with the problem (**) for potentials of A. In §2 we shall consider the one-sided stable generator A_+, that is, the measure $n(dy)$ is $y^{1+\alpha}dy$ for $y>0$ and 0 for $y \leq 0$. For the potential kernel U of the one-sided stable generator A_+, if a bounded potetial $U\mu$ of a finite signed measure μ vanishes on an interval (a,b) outside $\mathrm{supp}(\mu)$, then the restriction of μ to $[b,\infty)$ is 0. This result follows from (*) by a simple observation, but we shall give an elementary proof in §2 without using the result (*). Our result has been proved by Y. Ishikawa [2] by a different method in case $U\mu$ belongs to C^2. In §3 we shall add a few remarks connected with the problem (**).

[1] Consider the Lévy-Khintchin operator A_+ of the form;

$$A_\pm u(x) = \int_{R-\{0\}}[\, u(x+y) - u(x)\,]\, n_\pm(dy) ,$$

where $n_\pm(dy)$ is a measure (called the Lévy measure) on $R-\{0\}$such that $\int (|y|/1+|y|)\, n_\pm(dy) < \infty$ and $\mathrm{supp}(n_\pm) \subset R^\pm$. We set

$$Au(x) = \int_{R-\{0\}}[\, u(x+y) - u(x)\,]\, \{n_+(dy) + n_-(dy)\}.$$

In case $n_-(dy) = n_+(-dy)$, we write A_S instead of A. We assume that there exist bounded continuous densities $p_t^\pm(0,x) \equiv p_t^\pm(-x)$, $p_t(0,x) \equiv p_t(-x)$ and $p_t^S(0,x) \equiv p_t^S(-x)$ relative to Lebesgue measure for the semi-groups of A_+, A and A_S respectively, such that

$$U_\pm(x) = \int_0^\infty p_t^\pm(-x)\,dt, \quad U(x) = \int_0^\infty p_t(-x)\,dt \quad \text{and} \quad U_S(x) = \int_0^\infty p_t^S(-x)\,dt$$

are locally integrable. Note that $U_+(x)=U_-(-x)$ if $n_+(dy)=n_-(-dy)$ and $U_S(x)=U_S(-x)$. For example, in case $n_+(dy)=y^{-1-\alpha}dy$ for $y>0$, $= 0$ for $y<0$, then $U_+(x) = \text{const.}|x|^{\alpha-1}$ for $x<0$, $= 0$ for $x\geq 0$ and $U_S(x) = \text{const.}|x|^{\alpha-1}$. We call these potential kernels the stable kernels and call $U_+*\mu$, $U*\mu$ and $U_S*\mu$ stable potentials. Hereafter we write $U\mu$ instead of $U*\mu$ for a signed measure μ . Put

$$M_U = \{\ \mu\ ;\ \text{a signed measure of compact support such that } U\mu \text{ is bounded}\ \}$$

We define M_{U_+} and M_{U_S} analogously. The following proposition is valid for a larger class of potentials than those mentioned above.

PROPOSITION 1.1 Assume that the maximum principle holds for U (that is, $\sup_{x\in R} U\lambda(x) = \sup_{x\in \mathrm{supp}(\lambda)} U\lambda(x)$ whenever λ is a (nonnegative) measure of compact support). Then, for each $\mu\in M_U$, $U\mu$ never vanishes everywhere on $\mathrm{supp}(\mu)$ unless $\mu = 0$.

Though the author could not find the published proof, the proposition would be known. The author's proof is based on delicate facts derived from the duality theory of Markov processes. So the proof is omitted here. The author does not know whether the condition "the maximum principle" is necessary or not.

Remark 1.1 The stable kernel satisfies the maximum principle. See Blumenthal-Gettor [1], p.264 (1.23).

Remark 1.2 In Prop. 1.1 we can not replace "supp(μ)" with "a proper open subset of the interior of supp(μ)" in general. Indeed, if we choose different open intervals I_1 and I_2 with the same center and let $U_S\mu_1$ and $U_S\mu_2$ be the equilibrium potentials for I_1 and I_2 respectively, then $U_S\mu_1 - U_S\mu_2 = 0$ on the smaller interval, but $(\mu_1 - \mu_2)(D) = 0$ for every open set D in a larger interval whenever $U_S(x) = \text{const.}|x|^{\alpha-1}$.

The following simple proposition would give us an insight into our problem (**). We suppose $n_+(dy) = n_-(-dy)$ in the proposition below.

PROPOSITION 1.2 For every $\mu \in M_{U_+}$ ($\mu \in M_{U_-}$), $U_+\mu$ (resp. $U_-\mu$) vanishes everywhere on the right infinite component (resp. the left infinite component) of the complement of supp(μ) , and the following statements are equivalent to each other: i) for each $\mu \in M_{U_+}$, $U_+\mu$ never vanishes everywhere on an open set which is left to supp(μ) unless $\mu = 0$; i)$'$ for each $\mu \in M_{U_+}$, if $U_+\mu$ vanishes everywhere on an interval (a,b) in the complement of supp(μ), then the restriction $\mu|_{[b,\infty)}$ of μ to $[b, \infty)$ is the zero measure; ii) for each $\mu \in M_{U_-}$, $U_-\mu$ never vanishes everywhere on an open set which is right to supp(μ) unless $\mu = 0$; ii)$'$ for each $\mu \in M_{U_-}$, if $U_-\mu$ vanishes everywhere on an interval (a,b) in the complement of supp(μ), then $\mu|_{(-\infty,a]} = 0$.

PROOF. In our case $U_+\mu = U_+ * \mu$ by definition and $U_+ = 0$ on R^+. So the first assertion is trivial. The equivalence between i) and ii) also follows easily, because $U_+\mu = 0$ on an open set Q left to supp(μ) if and only if $U_-\tilde{\mu} = 0$ on an open set $\tilde{Q}$ right to supp($\tilde{\mu}$), where $\tilde{\mu}(dy) = \mu(-dy)$ and $\tilde{Q} = -Q$. For the proof of the equivalence between i) and i)$'$, we have only to note that $U_+\mu(x) = U_+(\mu|_{[b,\infty)})(x)$ for $x \in (a,b)$.

Remark 1.3. It is open whether the statement i) (equivalently i)$'$, ii) and ii)$'$) is equivalent to the statement: iii) for each $\mu \in M_{U_S}$, $U_S\mu$ never vanishes on an open set in the complement of supp(μ).

[2] In this section we study the stable potentials. We use synbols $n_+(dy)$ and $U_+(x)$, but here $n_+(dy) = n_-(-dy) = y^{-1-\alpha}dy$ for $y>o$, $= 0$ for $y<0$ and $U_+(x) = U_-(-x) = \text{const.}|x|^{\alpha -1}$ for $x<0$, $= 0$ for $x\geq 0$. For the stable kernel, $U_S(x) = \text{const.}(U_+(x) + U_-(x))$ $= \text{const.}|x|^{\alpha-1}$. Now we shll prove

THEOREM. For $\mu \in M_{U_+}$ ($\mu \in M_{U_-}$), if $U_+\mu$ (resp. $U_-\mu$) vanishes on an open interval (a,b) in the complement of supp(μ), then the restriction $\mu|_{[b,\infty)}$ of μ to $[b,\infty)$ is the zero measure (resp. $\mu|_{(-\infty,a]}$ $= 0$).

PROOF. By Prop.1.2 it is sufficient to prove that $U_+\mu$ never vanishes on an open set which is left to supp(μ) unless $\mu = 0$. We introduce an operator $\bar{A}$:

$$\bar{A}u(x) = \int_{R-\{0\}}[u(x+y) - u(x)][n_+(dy) - n_-(dy)] .$$

Define the symbols $a(\xi)$, $\bar{a}(\xi)$ and $b(\xi)$ as follows:

$$a(\xi) = \int_{R-\{0\}} [e^{i\xi y}-1]n_-(dy),$$

$$\bar{a}(\xi) = \int_{R-\{0\}} [e^{i\xi y}-1][n_-(dy) - n_+(dy)] = 2i\text{Im}a(\xi),$$

$$b(\xi) = 2i\text{Im}a(\xi) \text{ for } \xi \geq 0, \quad = -2i\text{Im}a(\xi) \text{ for } \xi <0.$$

Then $A_+u(x) = (1/2\pi)\int_R e^{-ix\xi}a(\xi)\, \hat{u}(\xi)\, d\xi$ and $\bar{A}u(x) = (1/2\pi)\int_R e^{-ix\xi}\bar{a}(\xi)\, \hat{u}(\xi)\, d\xi$ for each smooth function u of compact support, where $\hat{u}(\xi) = \int_R e^{i\xi y}u(y)dy$. Set $Bu(x) = (1/2\pi)\int_R e^{-ix\xi}b(\xi)\, \hat{u}(\xi)\, d\xi$.

We extend the definition domain of A_+, $\bar{A}$ and B to a certain class of distributions as usual. (We do not mention the class precisely, because our proof does not enter the delicate situation about the domain.) Assume that $U_+\mu$ vanishes on an interval (a,b) which is left to $\mathrm{supp}(\mu)$ for $\mu \in M_{U_+}$. By translation we may assume that $b = 0$ and $\mathrm{supp}(\mu) \subset [0,\infty)$. Consider a function $\tilde{u}$ defined by $\tilde{u}(x) = 0$ for $x<0$ and $\tilde{u}(x) = U_+\mu(x)$ for $x \geq 0$. Since $A_+u(x) = 0$ on $(a,0)$ and $\mathrm{supp}(n_+) \subset [0,\infty)$, we see that $\bar{A}\tilde{u}(x) = 0$ on $(a,0)$. Set $F(x) = \bar{A}\tilde{u}(x) - B\tilde{u}(x)$ (in distribution sense). Then $F(x) = (2i/\pi)\int_{-\infty}^{0} e^{-ix\xi} \mathrm{Im}a(\xi)\,\hat{\tilde{u}}(\xi)\,d\xi$ (in distribution sense). The function $f(\xi) = \mathrm{Im}a(\xi)\hat{\tilde{u}}(\xi)$ is a tempered function (that is, f is continuous and $|f(\xi)| = O(|\xi|^{\beta})$ for some real β). Indeed f is bounded continuous. Further $f(0) = \mathrm{Im}a(0)\hat{\tilde{u}}(0) = 0$, and so $1_{(-\infty,0)}(\xi)f(\xi)$ is also a tempered function. Hence $\hat{V}(z) = \int_R e^{-iz\xi}1_{(-\infty,0)}(\xi)f(\xi)\,d\xi$ is holomorphic in $\{\mathrm{Im}\,z < 0\}$ and $\lim_{\varepsilon\downarrow 0} V(x-i\varepsilon) = \int_R e^{-ix\xi}1_{(-\infty,0)}(\xi)f(\xi)\,d\xi$ in the convergence of tempered distributions. Therefore F never vanishes on an open set unless F is identically zero. But $F(x) = 0$ on $(a,0)$. This is proved as follows. Note that $\mathrm{Im}a(\xi) = -\mathrm{Im}a(-\xi)$. Then we can easily show that $b(\xi) = \mathrm{const.}|\xi|^{\alpha} = \mathrm{const.}\ 2\mathrm{Re}a(\xi)$. Hence it holds that $Bu(x) = \mathrm{const.}(1/2\pi)\int_R e^{-ix\xi}2\mathrm{Re}a(\xi)\,\hat{\tilde{u}}(\xi)\,d\xi = \int_{R-\{0\}}[\tilde{u}(x+y) - \tilde{u}(x)]\,[n_+(dy) + n_-(dy)]$. Since $\tilde{u}(x) = 0$ for $x<0$, the last term equals to $A_+\tilde{u}(x)$ for $x<0$ and $A_+u(x) = A\tilde{u}(x) = 0$ for $x \in (a,0)$. As mentioned before, $\bar{A}\tilde{u}(x) = 0$ on $(a,0)$, we see that $F = 0$ on $(a,0)$. Consequently F is identically zero, and so $\mathrm{Im}a(\xi)\hat{\tilde{u}}(\xi) = 0$ for $\xi \in (-\infty,0)$. Therefore $\hat{\tilde{u}}(\xi) = 0$ for $\xi<0$. Since $\hat{u}(\xi)$ is bouded continuous and $\tilde{u}$ vanishes on $(-\infty,0)$, we can prove by the same procedure that $\tilde{u}$ is identically zero. As a result $U_+\mu(x) = 0$ for $x \in [0,\infty)$. Since $\mathrm{supp}(\mu) \subset [0,\infty)$, it follows from Prop. 1.1 that $\mu = 0$. The proof is finished.

Remark 2.1. Let $Au(x) = \int_{(-c,c)-\{0\}} [\, u(x+y) - u(x) \,]\, |y|^{-1-\alpha} dy$, where $0 < c < \infty$, and consider the potential kernel U corresponding to A. The analogous result (to Theorem) "If $U\mu$ vanishes on (a,b) outside $\mathrm{supp}(\mu)$, then $\mu|_{(a,a+c)} = 0$" would not be valid in general. Indeed, let u be a non-constant smooth function which vanishes near infinity. Then $u = Uf$ for $f \equiv -Au$. We could choose u so that u vanishes on $(-\infty,q)$, where $q = \inf(x;\ x \in \mathrm{supp}(f))$.

[3] In this section we shall add few remarks connected with the problem (**). The following problem is open as far as the author knows: is it true that each potential $U\mu$ never vanishes everywhere on the complement of $\mathrm{supp}(\mu)$ unless $\mu = 0$? . As mentioned in the introduction, it is true for Riesz potentials except Newtonian potentials. It is not true for Newtonian potentials. From the observation in Remark 2.1 it might be doubtful even for potentials corresponding to non-local operators. However it is interesting to determine the class of potentials for which the above statement is true. For most kernels U in potential theory there corresponds a operator A such that $AUf = -f$. So the problem below is closely connected with the problem (**). Consider the situation: We are given a function u on a domain D which belongs to the definition domain of a given operator A. Can we guess the zero set Z of u when we know $u = Au = 0$ on a subdomain Q ? If A has the anti-locality,

then $Z = D$. If the unique continuation theorem holds for A, then $\mathrm{Int}(\ x;\ Au(x) = 0\ ) \subset Z$. Segal-Goodman [6] and Murata [4] proved the anti-locality for the fractional power of Laplacian. The study of the unique continuation theorem has been well developed for differential operators. However, unfortunately, we do not know for what class of operators of Lévy-Khintchin type the anti-local property holds and the unique continuation theorem is valid. Of course we can not expect the result of the form above (i.e. $Z=D$ etc.) in general. The zero set would closely depend on the Lévy measure of the operator. One way to attack our problem would be in Sato's theory of Pseudo-differential operators [5]. Indeed Ishikawa's proof on the anti-locality for A_+ (one-sided stable generator) is based on the theory.

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