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ON TIME CHANGE OF SYMMETRIC MARKOV PROCESSES

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1. Introduction

Let X be a locally compact separable metric space and m be an everywhere dense positive Radon measure on X. Let $(\mathcal{E},\mathcal{F})$ be an irreducible regular Dirichlet space on $L^2(X;m)$ and $M = (\Omega,\mathcal{B},X_t,P_x)$ be its corresponding m-symmetric Markov process. We shall suppose that we are given a positive Radon measure μ charging not a set of zero capacity. Then there exists a positive continuous additive functional (PCAF) $(A(t))_{t\geq 0}$ with associated smooth measure μ. Let $Y_t = X(A^{-1}(t))$ and $(\mathcal{E}^\mu,\mathcal{F}^\mu)$ be the Dirichlet space of the μ-symmetric time changed Markov process $M^\mu = (\Omega,\mathcal{B},Y_t,P_x)$. The purpose of this note is to characterize the extended Dirichlet space $(\mathcal{E}^\mu,\mathcal{F}^\mu_e)$ of $(\mathcal{E}^\mu,\mathcal{F}^\mu)$.

If $(\mathcal{E},\mathcal{F})$ is transient, then its extended Dirichlet space $(\mathcal{E},\mathcal{F}_e)$ is a Hilbert space continuously embedded in an $L^2(X;gdm)$ for some strictly positive m-integrable function g. If M is recurrent in the sense of Harris, then $1 \in \mathcal{F}_e$ and $\mathcal{E}(1,1) = 0$. We shall identify $\mathcal{F}_e$ and the quotient space of $\mathcal{F}_e$ by constant functions and define $\mathcal{E}$ naturally on it, then $(\mathcal{E},\mathcal{F}_e)$ is a Hilbert space continuously embedded in an $L^1(X;gdm)$ for some positive integrable function g.

Let Y be the support of A, γ be the restriction operator to Y, $\mathcal{F}_{X-Y} = \{u \in \mathcal{F}_e\ ;\ u = 0$ q.e. on $Y\}$ and $\mathcal{F}_e = \mathcal{F}_{X-Y} + \mathcal{H}^Y$ be the orthogonal decomposition. Then the main result is, for a suitable choise of the version, $\mathcal{F}_e^\mu = \gamma\mathcal{H}^Y$ and $\mathcal{E}^\mu(\gamma u,\gamma u) = \mathcal{E}(u,u)$ for all $u \in \mathcal{H}^Y$. In the transient case, this result is proved in [3] and [7]. In the recurrent case, the proof in [7] is insufficient so that we shall restrict our attention in the recurrent case. In this note, we shall only summalize the results, the detailed proof will be appeared elsewhere.

2. Results

In the followings we shall suppose that M is recurrent in the sense of Harris, that is, $\int_0^\infty f(x_t)dt = \infty$ P_x - a.s. for all $f \geqq 0$ such that $\int f(x)dm(x) > 0$, for each $x \in X$. Sufficient conditions for this is given in [4].

Let μ be the positive Radon measure satisfying the conditions of section 1, then there corresponds a PCAF $(A(t))_{t\geqq 0}$ in the sense

$$(2.1)\qquad \langle\mu,f\rangle = \lim_{t\to 0} \frac{1}{t} E_m[\int_0^t f(X_s)dA(s)].$$

Let Y_μ be the support of μ and Y be the fine support of $(A(t))$, that is, Y_μ is the smallest closed set outside of which μ vanishes and $Y = \{x;\ P_x[A_t>0 \text{ for all } t>0] = 1\}$. Then, by [3; Lemma 5.5.1], $Y \subset Y_\mu$ and $\mu(Y_\mu - Y) = 0$. Moreover, by a similar method to [6;§5], we have

<u>Lemma 1.</u> M^{μ} is a μ-symmetric normal strong Markov process on Y.

Define $V_{tA}^{pq}f$ and $V_{At}^{qp}f$ by $V_{tA}^{pq}f(x) = E_x[\int_0^{\infty} e^{-pt-qA_t} f(X_t)dA_t]$ and $V_{At}^{qp}f = E_x[\int_0^{\infty} e^{-pt-qA_t} f(X_t)dt]$.

Let $\mathcal{D}$ be the class of functions defined by

(2.2) $$\mathcal{D} = \{V_{tA}^{pq}f; p,q>0, f\in C_0(X)\}\cup\{V_{At}^{qp}f; p,q>0, f\in C_0(X)\},$$

then we have

<u>Lemma 2.</u> (c.f. [6;§5]).

(i) $\mathcal{D} \subset \mathcal{F}$ and $\gamma\mathcal{D} \subset \mathcal{F}^{\mu}$.

(ii) $\mathcal{D}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and $\gamma\mathcal{D}$ is $\mathcal{E}_1^{\mu}$-dense in $\mathcal{F}^{\mu}$, where $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2(m)}$ and $\mathcal{E}_1^{\mu}(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2(\mu)}$.

(iii) If $u \in \mathcal{D}$ then $Hu(x) \equiv E_x[u(X_{\sigma})] \in \mathcal{F}$ and

(2.3) $$\mathcal{E}(Hu,Hu,) = \mathcal{E}^{\mu}(\gamma u,\gamma u),$$

where σ is the hitting time for Y.

Let $(\mathcal{E},\mathcal{F})$ be the extended Dirichlet space of $(\mathcal{E},\mathcal{F})$, that is, $u \in \mathcal{F}_e$ if there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. In this case $\mathcal{E}(u,u) \equiv \lim_{n\to\infty} \mathcal{E}(u_n,u_n)$. It is known that $1 \in \mathcal{F}_e$ and $\mathcal{E}(1,1) = 0$. If we identify $\mathcal{F}_e$ with the quotient space of $\mathcal{F}_e$ by the family of constant functions, then $(\mathcal{E},\mathcal{F}_e)$ is a Hilbert space and there exists an integrable function $q > 0$ and a linear functional $I(\cdot)$ such that

(2.4) $$\int|u(x) - I(u)|g(x)dm(x) \leq \mathcal{E}(u,u)^{1/2}$$

for all $u \in \mathcal{F}_e$. In particular, $\mathcal{F}_e$ is continuously embedded in $L^1(X{:}gdm)$. Any function of $\mathcal{F}_e$ has a quasi-continuous (q.c.) version. Hence we shall suppose that any function of $\mathcal{F}_e$ is q.c. Let $\mathcal{F}_e = \mathcal{F}_{X-Y} + \mathcal{H}^Y$ be the orthogonal decomposition in §1, then we have

<u>Lemma 3.</u> If $u \in \mathcal{F}_e$, then $Hu(x) \equiv E_x[u(X_\sigma)]$ is the orthogonal projection of u on $\mathcal{H}^Y$.

The main result of this note is the following

<u>Theorem 4.</u> $\mathcal{F}_e^\mu = \gamma\mathcal{H}^Y$ in the sense $\gamma\mathcal{H}^Y \subset \mathcal{F}_e^\mu$ and, conversely, for each $\phi \in \mathcal{F}_e^\mu$ there exists $u \in \mathcal{H}^Y$ such that $\gamma u = \phi$ μ-a.e. In this case,

$$\mathcal{E}(u,u) = \mathcal{E}(\phi,\phi).$$

Proof. We shall outline the proof. For the proof, we shall use the transient Dirichlet space $(\mathcal{E}^C, \mathcal{F}^C)$ on $L^2(X;dm)$ defined by $\mathcal{F}^C = \mathcal{F}$ and $\mathcal{E}^C(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(m_C)}$, where $m_C(\cdot) = m(\cdot \cap C)$ and C is a measurable set such that $0 < m(C) < \infty$. If $u \in \mathcal{H}^Y$, then, by Lemma 2, there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{D}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. Take the set C such that $\{u_n\}$ and u are bounded on C, then $\{u_n\}$ is an $\mathcal{E}^C$-Cauchy sequence. Hence, by [3;Theorem 3.1.4], it contains a subsequence $\{u_{n_k}\}$ which converges q.e. to u. This implies that $\lim_{k\to\infty} \gamma u_{n_k} = \gamma u$ μ-a.e. By Lemma 2, since $\{\gamma u_n\}$ is an $\mathcal{E}^\mu$-Cauchy sequence of functions of $\mathcal{F}^\mu$, we have $\gamma u \in \mathcal{F}_e^\mu$.

Suppose, conversely, that $\phi \in \mathcal{F}_e^{\mu}$. Then, by Lemma 2, there exists an $\mathcal{E}^{\mu}$-Cauchy sequence $\{\gamma u_n\}$ of functions of $\gamma\mathcal{D}$ such that $\lim_{n\to\infty} \gamma u_n = \phi$ μ-a.e. In this case, since $\{Hu_n\}$ is an $\mathcal{E}$-Cauchy sequence of functions of $\mathcal{H}^Y$, it converges to some $u \in \mathcal{H}^Y$ in $(\mathcal{E},\mathcal{F}_e)$. Using (2.4), we can see that there exists a subsequence $\{Hu_{n_k}\}$ which converges to u m-a.e. and hence q.e. Thus

$$\gamma u = \lim_{k\to\infty} \gamma Hu_{n_k} = \lim_{k\to\infty} \gamma u_{n_k} = \phi \qquad \mu\text{-a.e.}$$

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