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AN OPTIMAL SELECTION OF A LEADER

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ABSTRACT

There are n periods to go and n options appear one by one at the head of each period. Each time an option appears, we observe the value attached to the option and then decide, based on the observed value, either to stop(accept the largest value observed so far) or continue observing. Once an option is chosen or the final time is reached, the selection procedure terminates. Satisfaction level is assumed to be 1 or 0, depending on whether the chosen value remains the largest. If satisfaction level q(q=1 or 0) lasts for t units of time, reward qt is earned. Thus we are led to the problem of finding a procedure that will maximize the expected reward. The models considered here are distinguished according to the two aspects: (1) Value attached to the option(FI case and NI case), (2) Selection procedure(recall case and no recall case). Thus there are four possible combinations of models. In each of these models, we derive the optimal strategy and the maximal expected reward.

In this paper, we present some stochastic models of optimal selection which can be cast in the following form. A number of options or opportunities appear sequentially over a fixed finite horizon. Each time an option appears, we observe the value (random variable attached to the option) and, based on the observed value, decide either to select or reject the option. The value may be the effectiveness number of the option. If some option is selected or final time is reached, the selection procedure terminates. An option is referred to as a <u>leader</u> if its value attains a new high. When the value only represents the rank of the option relative to those preceding it, relatively best option is called a leader. When a leader appears, we say that a <u>lead time</u> starts. That is, each leader initiates its lead time. It ends when a new leader arrives, whereas it lasts until the final time if no further leader appears in the remaining periods.

It is sometimes natural to assume that the chosen option gives complete satisfaction (satisfaction level 1) as long as it remains a leader but the satisfaction level decreases each time a new option appears which is better than the current option. If satisfaction level q ($0 \leq q \leq 1$) lasts for t units of time, reward qt is earned. This setting seems to be appropriate to some cases of technological choice problem, job search problem and marriage problem.

In this paper we confine ourselves to the simplest case in which the satisfaction level is 1 or 0 depending on whether the current option is a leader or not. It then turns out that maximizing the expected reward is equivalent to maximizing the expected lead time. Thus we are led to the problem of finding a procedure that selects an option (leader) whose expected lead time will be maximized.

The models can be distinguished according to the following three aspects.

(1) Value attached to the option

The typical problems concerning the value are the ' full-information problem (FI problem),in which the value is sampled from a known continuous distribution F, and the ' no-information problem (NI problem),in which the value is only the rank of the option relative to those preceding it. In the FI problem we can assume without loss of generality that the underlying distribution F is uniform on [0,1] .

(2) Selection procedure

Our problem is considered both in the case where once an option is rejected, it may not be recalled later (no recall case) and in the case where such recall of previous option is allowed (recall case). It should be noted that since, in the recall case, we evidently recall the latest leader (including the present option), our aim is to maximize the expected residual lead time of the chosen option, where residual lead time is defined as (lead time) - (time duration elapsed without being chosen). It may be possible that we come to the final time without choosing an option, in which case the lead time is of course zero.

(3) Arrival process of the option

Two arrival processes considered here are regular arrival(equal-paced arrival) where there are n periods and n options appear one by one at the head of each period, and Poisson arrival where options appear in the interval [0,T] according to a Poisson process which is independent of the values of the options.

As each aspects has two cases, FI or NI, recall or no recall, and regular or Poisson, there are eight possible combinations of models. However, due to the limit to the number of pages for a paper, we only consider the regular arrival case and treat four possible combinations in Sections 1 and 2. Our problems deal with optimization of time, but they have the similar structure as that of the so called best choice problem concerning three aspects stated above (see , e.g., Gilbert and Mosteller 1966, Sakaguchi 1973, Cowan and Zabczyk 1978, Bojdecki 1978, Yang 1974, Smith and Deely 1975, Petruccelli 1982, and Tamaki 1986).

1. FI Problem with Regular Arrival

There are n periods to go and n options appear one by one at the head of each period. Let X_j, j=1,2,...,n, be the value attached to the j-th option and assume that $X_1, X_2, \ldots, X_n$ are random sample from a uniform distribution on $[0,1]$.

Let $L_k=\max(X_1, X_2,\ldots,X_k)$, then in the recall case we are said at time k, to be in state (k,x) if $L_k=x$. If we select the leader in this state, its expected residual lead time is

$$m_k(x) = \sum_{j=k+1}^{n} (j-k)x^{j-k-1}(1-x)+(n-k+1)x^{n-k}$$

$$=\sum_{j=0}^{n-k} x^j, \quad 1\leq k\leq n, \tag{1.1}$$

which clearly holds regardless of when the leader has appeared. (1.1) follows since the probability that the next earliest leader appears in the j-th period is $x^{j-k-1}(1-x)$ and the probability that no further leader appears in the remaining periods is x^{n-k}.

In the no recall case, naturally all decisions will take place only at the occurrence of a leader. Therefore in the no recall case we are said to be in state (k,x) when $L_k=X_k=x$. It is easy to see that if the leader is selected, its expected lead time is also given by (1.1).

1.1 Recall case

Here previously rejected option (leader) can be freely recalled at a later time. Let $V_k(x)$ be the maximal expected residual lead time starting from (k,x), then we obtain, using (1.1), the following equation from the principle of optimality

$$V_k(x) = \max \left\{ m_k(x),\ xV_{k+1}(x)+\int_x^1 V_{k+1}(y)dy \right\}, \qquad (1.2)$$

$$1 \leq k < n, \quad V_n(x) \equiv 1.$$

The second term in RHS of (1.2) follows since when we decide to take at least one more observation state makes transition from (k,x) into $(k+1,x)$ with probability x, and into $(k+1,y)$ with transition density 1, $x < y \leq 1$.

Let

$$G = \bigcup_{k=1}^{n-1} \left\{(k,x):\ m_k(x) \geq x m_{k+1}(x)+\int_x^1 m_{k+1}(y)dy\right\} \cup \left\{(n,x):\ 0 \leq x \leq 1\right\} \qquad (1.3)$$

be the set of states for which, stopping (making a selection) is at least as good as continuing exactly one more transition and then stopping. Now define the one stage look ahead (OLA) policy as one that stops as soon as the state enters G. It is known that roughly speaking if G is <u>closed</u>, i.e., Pr{state leaves G at some finite future time | state belongs now to G$\}$ = 0, and realizable, i.e., Pr$\{$state eventually enters G$\}$=1, then the OLA policy is optimal. The OLA approach plays an important role not only in Sections 1.1 and 1.2, but also in Sections 2.1 and 2.2. For a rigorous proof, see for example Cowan and Zabczyk (1978) and Bojdecki (1978).

We return to (1.3). Simple calculation leads to

$$x m_{k+1}(x) + \int_x^1 m_{k+1}(y)dy = \sum_{j=1}^{n-k} x^j + \sum_{j=1}^{n-k} (1-x^j)/j.$$

Hence, G in (1.3) can be written as

$$G = \bigcup_{k=1}^{n-1} \{(k,x):\ \sum_{j=1}^{n-k}(1-x^j)/j \leq 1\} \cup \{(n,x):\ 0 \leq x \leq 1\}\ . \qquad (1.4)$$

Since $\sum_{j=1}^{n-k}(1-x^j)/j$ is strictly decreasing in x, from $\sum_{j=1}^{n-k} 1/j$ at x=0 to 0 at x=1, the inequality in (1.4) is equivalent to $x \geq s_{n-k}$, where s_k, $k \geq 1$, is the unique root, in $0 \leq x \leq 1$, of the equation

$$\sum_{j=1}^{k} (1-x^j)/j = 1. \qquad (1.5)$$

Therefore, if we let $s_0=0$, we can rewrite G as

$$G = \bigcup_{k=1}^{n} \{(k,x):\ s_{n-k} \leq x \leq 1\} \qquad (1.6)$$

and have the following result.

THEOREM 1.1. The optimal policy stops as soon as the state enters G in (1.6). And the expected residual lead time E(RLT) is given by

$$E(RLT) = \sum_{j=1}^{n}\frac{1}{j} + \sum_{j=2}^{n}\sum_{k=n-j+1}^{n-1}\frac{1}{j}s_k^j . \tag{1.7}$$

Proof. s_k is evidently non-decreasing in k. This monotonicity property of $\{s_k\}$, combined with the fact that our process " goes to the right and upwards ", immediately establishes that G is closed and realizable. See Appendix 1 for calculation of E(RLT).

It is of interest to see the following limiting relation hold

LEMMA 1.2. Let $a_k=k(1-s_k)/s_k$, then $\lim_{k\to\infty} a_k = a^*$ (≈ 1.3450), where a^* is the unique root a in $(0,\infty)$ of the equation

$$\int_0^1 \left(\frac{1-e^{-au}}{u}\right)du = 1. \tag{1.8}$$

Proof. By definition, s_k satisfies

$$\sum_{j=1}^{k}(1-s_k^j)/j = 1 . \tag{1.9}$$

Thus writing (1.9) as

$$\sum_{j=1}^{k}\frac{k}{j}\left\{1-\left(\left(1+\frac{a_k}{k}\right)^{-k}\right)^{\frac{j}{k}}\right\}\frac{1}{k} = 1$$

and putting $j/k=u$ and then letting $k\to\infty$ immediately yield (1.8).

1.2. No recall case

Here, previously rejected option cannot be recalled at a later time and all decisions must be made immediately after a leader has appeared. Let $V_k(x)$ be the maximal expected lead time starting from state (k,x), then we have the following equation

$$V_k(x) = \max\left\{ m_k(x),\ \sum_{j=k+1}^{n} x^{j-k-1}\int_x^1 V_j(y)dy \right\}, \qquad (1.10)$$

$$1 \leq k < n, \quad V_n(x) \equiv 1.$$

The second term in RHS of (1.10) follows since, when we take at least one more observation , state makes transition from (k,x) into (j,y) with transition density x^{j-k-1}, for $k<j\leq n$, $x<y\leq 1$. Let

$$M_k(x) = \sum_{j=k+1}^{n} x^{j-k-1}\int_x^1 m_j(y)dy,$$

then the OLA stopping region is given by from (1.10)

$$G = \bigcup_{k=1}^{n-1} \{(k,x):\ m_k(x) \geq M_k(x)\} \cup \{(n,x):\ 0 \leq x \leq 1\} . \qquad (1.11)$$

Straightforward calculation leads to

$$M_k(x) = \sum_{j=k+1}^{n} x^{j-k-1}\int_x^1 \left(\sum_{i=0}^{n-j} y^i\right)dy$$

$$= \sum_{j=k+1}^{n} x^{j-k-1}\sum_{i=0}^{n-j} (1-x^{i+1})/(i+1) \qquad (1.12)$$

$$= \sum_{i=1}^{n-k} \left(\sum_{j=k+1}^{n-i+1} x^{j-k-1}\right)(1-x^i)/i$$

$$= (1-x)^{-1} \sum_{i=1}^{n-k} (1-x^i)(1-x^{n-k+1-i})/i. \qquad (1.13)$$

Hence, from (1.13) and $m_k(x)=(1-x^{n-k+1})/(1-x)$, the inequality $m_k(x)\ M_k(x)$ turns out to be equivalent to

$$\mathcal{S}_{n-k}(x) \leqq 1,$$

where

$$\mathcal{S}_k(x) = \sum_{i=1}^{k} (1-x^i)(1-x^{k+1-i})/i(1-x^{k+1}), \quad \text{for } k \geq 1. \qquad (1.14)$$

$\mathcal{S}_k(x)$ is strictly decreasing in $x \in (0,1)$. To show this, define for $1 \leq i \leq k$,

$$\mathcal{S}_{k\ i}(x) = (1-x^i)(1-x^{k+1-i})/(1-x^{k+1}). \qquad (1.15)$$

Then since both $1-x^i$ and $(1-x^{k+1-i})/(1-x^{k+1})$ are strictly decreasing, $\mathcal{S}_{k\ i}(x)$ is also strictly decreasing and so is $\mathcal{S}_k(x)$ through the form of

$$\mathcal{S}_k(x) = \sum_{i=1}^{k} \frac{\mathcal{S}_{k\ i}(x)}{i} \qquad (1.16)$$

The monotonicity property of $\mathcal{S}_k(x)$ and the fact that $\mathcal{S}_k(0)=\sum_{i=1}^{k} 1/i (\geq 1)$ and $\mathcal{S}_k(1)=0$ assure that $\mathcal{S}_k(x)=1$ has the unique root t_k in $[0,1)$.
G in (1.11) now becomes

$$G = \bigcup_{k=1}^{n} \{(k,x):\ t_{n-k} \leq x \leq 1\}\ , \qquad (1.17)$$

where t_0 is interpreted as zero.

THEOREM 1.3. The optimal policy stops as soon as the state enters G in (1.17). And the expected lead time E(LT) is given by

$$E(LT) = c_n + \sum_{j=1}^{n} \frac{1}{j} \sum_{k=n-j}^{n-1} (c_{n-j} - c_{k-n+j} - 1) t_k^j , \tag{1.18}$$

where

$$c_i = \begin{cases} \sum_{j=1}^{i} 1/j, & \text{if } i \geq 1 \\ 0, & \text{if } i=0 \end{cases} . \tag{1.19}$$

Proof. To prove that G is closed, it suffices to show that t_k is non-decreasing in k. We show this by establishing $\mathcal{f}_k(x) \geq \mathcal{f}_{k-1}(x)$. But this can be proved by showing that

$$\mathcal{f}_{k\,i}(x) - \mathcal{f}_{k-1\,i}(x) \geq 0 \quad \text{for } 1 \leq i \leq k-1, \tag{1.20}$$

because from (1.16)

$$\mathcal{f}_k(x) - \mathcal{f}_{k-1}(x) = \frac{1}{k} \mathcal{f}_{k\,k}(x) + \sum_{i=1}^{k-1} \frac{1}{i} (\mathcal{f}_{k\,i}(x) - \mathcal{f}_{k-1\,i}(x)).$$

(1.20) follows from (1.15) since

$$\mathcal{f}_{k\,i}(x) - \mathcal{f}_{k-1\,i}(x)$$

$$= (1-x^i)\{(1-x^k)(1-x^{k+1-i}) - (1-x^{k+1})(1-x^{k-i})\}/(1-x^k)(1-x^{k+1})$$

$$= x^{k-i}(1-x)(1-x^i)^2/(1-x^k)(1-x^{k+1})$$

$$\geq 0.$$

Realizability is self evident. For calculation of E(LT), see Appendix 2.

As in the recall case, we have the following limiting relation.

LEMMA 1.4. Let $a_k = k(1-t_k)/t_k$, then $\lim_{k\to\infty} a_k = a^{**}$ (≈ 2.1198), where a^{**} is the unique root a in $(0,\infty)$ of the equation

$$\int_0^1 \frac{(1-e^{-au})(1-e^{-a(1-u)})}{(1-e^{-a})u}\, du = 1. \tag{1.21}$$

Proof. From the definition of t_k and (1.14), we have

$$(1-t_k^{k+1})^{-1} \sum_{j=1}^{k} (1-t_k^{j})(1-t_k^{k+1-j})/j = 1. \tag{1.22}$$

Thus writing (1.22) as

$$\frac{\sum_{j=1}^{k} \frac{k}{j} \left\{1-\left(\left(1+\frac{a_k}{k}\right)^{-k}\right)^{\frac{j}{k}}\right\} \left\{1-\left(1+\frac{a_k}{k}\right)^{-(k+1)}\left(\left(1+\frac{a_k}{k}\right)^{k}\right)^{\frac{j}{k}}\right\} \frac{1}{k}}{\left\{1-\left(1+\frac{a_k}{k}\right)^{-(k+1)}\right\}} = 1$$

and putting $j/k=u$ and then letting $k\to\infty$ yields (1.21).

Remark (1). Another expression for $M_k(x)$ is given by

$$M_k(x) = \sum_{i=0}^{n-k} (c_{n-k-i}-c_i)x^i , \tag{1.23}$$

since (1.12) is written as

$$\sum_{i=0}^{n-k-1} \left(\sum_{j=0}^{n-k-1-i} 1/(j+1) \right) x^i - \sum_{i=1}^{n-k} \left(\sum_{j=0}^{i-1} 1/(j+1) \right) x^i .$$

Therefore from (1.1) and (1.23), t_k, $k \geq 1$, can be defined as a unique root x in $[0,1)$ of the equation

$$\sum_{i=0}^{k} (c_{k-i} - c_i - 1)x^i = 0.$$

2. NI Problem with Regular Arrival

n options appear one at a time in random order, with all permutations equally likely. As the option appear in order, we observe the rank of each relative to those preceding it. As noted previously, in the NI problem, an option is referred to as a leader if it is best among those that have already appeared.

In the recall case, we are said to be in state k when we have just observed the first k options. If we select the leader among those, its expected residual lead time is

$$m_k = \sum_{j=k+1}^{n} (j-k) \frac{k}{j(j-1)} + (n-k+1) \frac{k}{n}$$

$$= k \sum_{j=k}^{n} \frac{1}{j} , \quad 1 \leq k \leq n. \tag{2.1}$$

This follows since the probability that the next earliest leader appears in the j-th period is $k/j(j-1)$ and the probability that no further leader appears in the remaining periods is k/n. In the no recall case, we are said to be in state k if we have observed the first k options and found that the k-th option is a leader. The expected lead time when we select the leader is also given by (2.1).

2.1. Recall case

Let V_k be the maximal expected residual lead time starting from state k, then we evidently have

$$V_k = \max(m_k, V_{k+1}), \qquad 1 \leq k < n, \quad V_n = 1$$

This suggests that the OLA stopping region is given by

$$G = \bigcup_{k=1}^{n-1} \{k: m_k \geq m_{k+1}\} \cup n = \bigcup_{k=1}^{n-1} \{k: \sum_{j=k+1}^{n} 1/j \leq 1\} \cup n \, . \tag{2.2}$$

We have the following theorem.

THEOREM 2.1. Let s* be the smallest integer s such that

$$\sum_{j=s+1}^{n} 1/j \leq 1, \tag{2.3}$$

then the optimal policy observes the first s* options and then recalls the leader among those The expected residual lead time E(RLT) is given by

$$E(RLT) = m_{s*} \ , \tag{2.4}$$

and the limiting relations are given by

$$\lim_{n\to\infty} s^*/n = \lim_{n\to\infty} E(RLT)/n = e^{-1} \ . \tag{2.5}$$

Proof. Since G in (2.2) is written as $G=\{k: s^* \leq k \leq n\}$ and since k does not decrease as time progresses, G proves to be closed and realizable. (2.4) and (2.5) are straightforward from (2.1) and (2.3).

Remark (1). Note that s* attains the peak of the unimodal function m_k.

2.2. No recall case

Let V_k be the maximal expected lead time starting from state k, then we immediately have

$$V_k = \max\left\{m_k, \sum_{j=k+1}^{n} (k/j(j-1))V_j\right\}, \quad 1\leq k<n, \quad V_n=1. \tag{2.6}$$

The OLA stopping region is given by from (2.6)

$$G = \bigcup_{k=1}^{n-1}\left\{k: m_k \geq k \sum_{j=k+1}^{n} m_j/j(j-1)\right\} \cup n = \bigcup_{k=1}^{n-1}\{k: \gamma_k \geq 0\} \cup n, \tag{2.7}$$

where

$$\gamma_k = \sum_{i=k}^{n} \frac{1}{i}\left(1 - \sum_{j=i+1}^{n} \frac{1}{j}\right), \quad 1\leq k<n. \tag{2.8}$$

We now have the following theorem.

THEOREM 2.2. Let t* be the smallest integer t such that

$$\gamma_t \geq 0, \tag{2.9}$$

then the optimal policy passes over the first t*-1 options and then chooses the earliest leader that appears. The expected lead time E(LT) is givrn by

$$E(LT) = (t^*-1) \sum_{j=t^*}^{n} \frac{1}{j-1} \sum_{k=j}^{n} \frac{1}{k}, \tag{2.10}$$

and the limiting relations are given by

$$\lim_{n\to\infty} t^*/n = e^{-2}, \qquad \lim_{n\to\infty} E(LT)/n = 2e^{-2} \,. \tag{2.11}$$

Proof. Since G in (2.7) is written as $G=\{k:t^*\leqq k\leqq n\}$ and since k does not decrease as time progresses, G proves to be closed and realizable. (2.10) comes from $E(LT)=V_1=(t^*-1)\sum_{j=t^*}^{n} m_j/j(j-1)$. (2.11) is straightforward from (2.9) and (2.10).

APPENDIX 1.

Let k be the time at which the maximum value observed so far initially exceeds the critical number, i.e., $k=\min\{j: L_j \geq \bar{s}_j\}$, where $\bar{s}_j = s_{n-j}$. Then the event $L_k=x$ for $k\geq 2$ can be distinguished into the following two cases (i) and (ii), depending on whether $L_k=L_{k-1}$ or not :

(i) $L_{k-1}=L_k$ ($> X_k$) and $\bar{s}_k < L_{k-1} < \bar{s}_{k-1}$,

(ii) $L_{k-1}<L_k$ ($=X_k$) and $L_{k-1} < \bar{s}_{k-1}$.

Therefore we have conditioning on $L_k=x$

$$
\begin{aligned}
E(RLT) &= E\{E(RLT \mid L_k=x)\} \\
&= \int_{\bar{s}_1}^{1} E(RLT \mid L_1=x)dx \\
&+ \sum_{k=2}^{n}\Big\{\int_{\bar{s}_k}^{\bar{s}_{k-1}} E(RLT \mid L_k=L_{k-1}=x)P(L_k=L_{k-1}=x)dx \\
&+ \int_{\bar{s}_k}^{\bar{s}_{k-1}}\int_{y}^{1} E(RLT \mid L_{k-1}=y, L_k=X_k=x)P(L_{k-1}=y, L_k=X_k=x)dxdy \\
&+ \int_{0}^{\bar{s}_k}\int_{\bar{s}_k}^{1} E(RLT \mid L_{k-1}=y, L_k=X_k=x)P(L_{k-1}=y, L_k=X_k=x)dxdy\Big\}, \quad (A.1)
\end{aligned}
$$

where the third and fourth terms correspond respectively to the cases $\bar{s}_k < L_{k-1} < \bar{s}_{k-1}$ and $L_{k-1} \leq \bar{s}_k$.

Substituting

$$P(L_k=L_{k-1}=x) = (k-1)x^{k-1}, \quad (A.2)$$

and

$$P(L_{k-1}=y, L_k=X_k=x) = (k-1)y^{k-2} \quad (A.3)$$

into the respective expressions in (A.1) and considering that, for $x \geq \bar{s}_1$, $E(RLT|L_1=x)=m_1(x)$ and for $x \geq \bar{s}_k$, $E(RLT|L_k=L_{k-1}=x)$ $=E(RLT|L_{k-1}=y, L_k=X_k=x)=m_k(x)$, we have

$$E(RLT)$$

$$= \int_{\bar{s}_1}^{1} m_1(x)dx + \sum_{k=2}^{n} \{ k\int_{\bar{s}_k}^{\bar{s}_{k-1}} x^{k-1} m_k(x)dx + \bar{s}_{k-1}^{k-1} \int_{\bar{s}_{k-1}}^{1} m_k(x)dx \}$$

$$= \int_{s_{n-1}}^{1} m_1(x)dx + \sum_{k=0}^{n-2} (n-k) \int_{s_k}^{s_{k+1}} x^{n-k-1} m_{n-k}(x)dx + \sum_{k=0}^{n-2} s_{k+1}^{n-k-1} \int_{s_{k+1}}^{1} m_{n-k}(x)dx \,. \quad (A.4)$$

The first term of (A.4) becomes

$$\sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} \frac{1}{j} s_{n-1}^{j} \,. \quad (A.5)$$

The second term of (A.4) becomes

$$\sum_{k=0}^{n-2} \sum_{j=0}^{k} \frac{n-k}{n-k+j} (s_{k+1}^{n-k+j} - s_k^{n-k+j})$$

$$= \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \left(\frac{n-k}{n-j} + \frac{1}{n-j} \right) s_k^{n-j} - \sum_{j=0}^{n-2} \sum_{k=j}^{n-2} \frac{n-k}{n-j} s_k^{n-j}$$

$$= \sum_{j=2}^{n} \frac{1}{j} s_{n-1}^{j} - \sum_{k=1}^{n-2} s_k^{n-k} + \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{1}{j} s_k^{j} \,. \quad (A.6)$$

The third term of (A.4) becomes

$$\sum_{k=0}^{n-2} s_{k+1}^{n-k-1} \sum_{j=0}^{k} \frac{1}{j+1} (1-s_{k+1}^{j+1})$$

$$= \sum_{k=1}^{n-1} \left(\sum_{j=1}^{k} \frac{1}{j} \right) s_k^{n-k} - \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{1}{j-n+k} s_k^j . \qquad (A.7)$$

Hence, substituting (A.5)-(A.7) into (A.4) yields

$$E(RLT) = \sum_{j=1}^{n} \frac{1}{j} + \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{1}{j} s_k^j + \sum_{k=2}^{n-1} \left(\sum_{j=2}^{k} \frac{1}{j} \right) s_k^{n-k}$$

$$- \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{1}{j-n+k} s_k^j$$

$$= \sum_{j=1}^{n} \frac{1}{j} + \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{1}{j} s_k^j \quad ,$$

where the last equality follows since we have from (1.9)

$$\sum_{k=2}^{n-1} \left(\sum_{j=2}^{k} \frac{1}{j} \right) s_k^{n-k} = \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{1}{j-n+k} s_k^j .$$

APPENDIX 2.

As in Appendix 1, let k be the time at which the maximum value observed so far initially exceeds the critical number, i.e., $k=\min\{j: L_j \geq \bar{t}_j\}$, where $\bar{t}_j = t_{n-j}$. It is noted that (A.1) still holds for E(LT) with $\bar{s}_k$ replaced by $\bar{t}_k$. Now that hor $x \geq \bar{t}_1$, $E(LT|L_1=x)=m_1(x)$ and for $x \geq \bar{t}_k$, $E(LT|L_k=L_{k-1}=x)=M_k(x)$, $E(LT|L_{k-1}=y, L_k=X_k=x)=m_k(x)$, we have using (A.2) and (A.3)

E(LT)

$$= \int_{\bar{t}_1}^{1} m_1(x)dx + \sum_{k=2}^{n} \{ \int_{\bar{t}_k}^{\bar{t}_{k-1}} (k-1)x^{k-1} M_k(x)dx + \int_{\bar{t}_k}^{\bar{t}_{k-1}} x^{k-1} m_k(x)dx$$

$$+ \bar{t}_{k-1}^{k-1} \int_{\bar{t}_{k-1}}^{1} m_k(x)dx \}$$

$$= \int_{t_{n-1}}^{1} m_1(x)dx + \sum_{k=0}^{n-2} [\int_{t_k}^{t_{k+1}} \{ (n-k-1)M_{n-k}(x) + m_{n-k}(x) \} x^{n-k-1} dx$$

$$+ t_{k+1}^{n-k-1} \int_{t_{k+1}}^{1} m_{n-k}(x)dx] . \qquad (A.8)$$

By the way, we find from(1.1) and (1.22)

$$\int_{t_{n-1}}^{1} m_1(x)dx = c_n - \sum_{j=1}^{n} \frac{1}{j} t_{n-1}^{j} ,$$

$$\int_{t_{k+1}}^{1} m_{n-k}(x)dx = c_{k+1} - \sum_{j=1}^{k+1} \frac{1}{j} t_{k+1}^{j} ,$$

and

$$\int_{t_k}^{t_{k+1}} \{(n-k-1)M_{n-k}(x) + m_{n-k}(x)\} x^{n-k-1} dx$$

$$= \sum_{j=0}^{k} \frac{t_{k+1}^{n-k+j} - t_k^{n-k+j}}{n-k+j} + \sum_{j=0}^{k} \frac{n-k-1}{n-k+j} (c_{k-j} - c_j)(t_{k+1}^{n-k+j} - t_k^{n-k+j}) .$$

Applying these results to (A.8) gives

$$E(LT) = c_n - \sum_{j=1}^{n} \frac{1}{j} t_{n-1}^{j} + \sum_{k=0}^{n-2} \sum_{j=0}^{k} \frac{t_{k+1}^{n-k+j} - t_k^{n-k+j}}{n-k+j}$$

$$+\sum_{k=0}^{n-2}\sum_{j=0}^{k}\frac{n-k-1}{n-k+j}(c_{k-j}-c_j)(t_{k+1}^{n-k+j}-t_k^{n-k+j}) + \sum_{k=1}^{n-1}c_k t_k^{n-k}$$

$$-\sum_{j=2}^{n}\sum_{k=n-j+1}^{n-1}\frac{t_k^j}{j-n+k}\ . \tag{A.9}$$

The third term of (A.9) becomes

$$\sum_{k=0}^{n-2}\sum_{j=0}^{k}\frac{t_{k+1}^{n-k+j}}{n-k+j} - \sum_{k=0}^{n-2}\sum_{j=0}^{k}\frac{t_k^{n-k+j}}{n-k+j}$$

$$=\sum_{k=0}^{n-2}\frac{1}{n-k}\sum_{j=k+1}^{n-1}t_j^{n-k} - \sum_{k=0}^{n-2}\frac{1}{n-k}\sum_{j=k}^{n-2}t_j^{n-k}$$

$$=\sum_{k=0}^{n-2}\frac{t_{n-1}^{n-k}-t_k^{n-k}}{n-k}$$

$$=\sum_{j=1}^{n}\frac{t_{n-1}^j}{j} - \sum_{j=1}^{n}\frac{t_{n-j}^j}{j}\ . \tag{A.10}$$

The fourth term of (A.9) becomes

$$\sum_{j=2}^{n}\frac{1}{j}\sum_{k=n-j}^{n-2}(n-k-1)(c_{n-j}-c_{k-n+j})(t_{k+1}^j-t_k^j)\ . \tag{A.11}$$

Now the second summation with respect to k of (A.11) is simplified to

$$\sum_{k=n-j+1}^{n-1}(n-k)(c_{n-j}-c_{k-1-n+j})t_k^j - \sum_{k=n-j}^{n-2}(n-k-1)(c_{n-j}-c_{k-n+j})t_k^j$$

$$=\sum_{k=n-j+1}^{n-1}\{(n-k)(c_{k-n+j}-c_{k-1-n+j})+(c_{n-j}-c_{k-n+j})\}t_k^j - (j-1)c_{n-j}t_{n-j}^j$$

$$= \sum_{k=n-j+1}^{n-1} \{j(\frac{1}{j-n+k} - \frac{1}{j})+(c_{n-j}-c_{k-n+j})\} t_k^j - j(1 - \frac{1}{j})c_{n-j}t_{n-j}^j . \quad (A.12)$$

Substitute (A.12) into (A.11), then the fourth term of (A.9) reduces to

$$\sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} (\frac{1}{j-n+k} - \frac{1}{j})t_k^j + \sum_{j=2}^{n} \frac{1}{j} \sum_{k=n-j+1}^{n-1} (c_{n-j}-c_{k-n+j})t_k^j$$

$$- \sum_{j=2}^{n} c_{n-j}t_{n-j}^j + \sum_{j=2}^{n} \frac{1}{j} c_{n-j}t_{n-j}^j$$

$$= \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{t_k^j}{j-n+k} - \sum_{j=2}^{n} \sum_{k=n-j+1}^{n-1} \frac{t_k^j}{j}$$

$$+ \sum_{j=1}^{n} \frac{1}{j} \sum_{k=n-j}^{n-1} (c_{n-j}-c_{k-n+j})t_k^j - \sum_{k=1}^{n-1} c_k t_k^{n-k} . \quad (A.13)$$

Substituting (A.10) and (A.13) into (A.9) immediately yields (1.17).

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