数理解析研究所講究録
第 614 巻 1987 年 92-96

Berman' principle and its application to Peano like functions

京大教養　河野　敬雄 (Norio Kôno)

§1. Introduction. P.Lévy had first introduced the Brownian local time and using it he investigated the various properties of Brownian motion paths, many of them are very much curious and delicate from the view point of classical real analysis (for more details see [10]). Although the local time is very much familiar and very much useful in the theory of stochastic processes, the notion itself is purely analytic one. S.M.Berman first developed the local time of real functions by Fourier analytic method to investigate the properties of the real functions and applied it to non Markovian stochastic processes. It is he who claimed that regularity of local time implies irregularity of the original function and vice versa. We shall call this "Berman's principle". In fact it is very difficult to imagine a real continuous deterministic function (not sample functions of a stochastic process)which has a space-time continuous local time. Only example so far is found in [12].

In §2 we will summarize some properties of real functions having local time mainly due to S.M.Berman ([1]-[6]) and D.Geman - J.Horowitz ([7]-[9]) and in §3 we will apply them to Peano like functions which we called a self-affine function ([12]).

§2. Definition and lemmas for local time.

Let f be a Borel measurable real function defined on a probability space ([0,1],dt). Then we can define a probability measure μ on the real line as the distribution of f,i.e.

$$\mu(E) = |\{0 \leq t \leq 1;\ f(t) \in E\}| \quad \text{for } \forall E \in B(R) \text{(Borel } \sigma\text{-field of R)},$$

where $|A|$ denotes the Lebesgue measure of $A \in B([0,1])$.

Now we impose a basic assumption.

<u>Assumption</u>. μ is absolutely continuous with respect to Lebesgue measure, i.e. there exists a Borel measurable function $\alpha(x)$ such that

$$\mu(E) = \int_E \alpha(x)dx .$$

Under this assumption any induced measure $\mu(T,\cdot)$ defined by restricting the function f on a subset (T,dt) ,$T\in B([0,1])$ also have densities $\alpha(T,x)$ with respect to Lebesgue measure, so

$$\int_T g(f(t))\, dt = \int_{-\infty}^{+\infty} g(x)\alpha(T,x)dx .$$

holds for any bounded measurable functions g.

Denoting $\alpha([0,t],x)$ by $\alpha(t,x)$ we can chose a nice version for it as follows.

<u>Lemma 1</u>.[7]

(i) $\alpha(t,x)$ is right continous and non-decreasing in t,

(ii) $\alpha(t,x)$ is $B([0,1])\times B(R)$ measurable,

(iii)almost every x,the support of the measure $\alpha(dt,x)$ is carried by $\{t;f(t)=x\}$,

(iv) almost every t, for any $\varepsilon > 0$

$\alpha([t,t+\varepsilon),f(t)) > 0$ and $\alpha((t-\varepsilon,t],f(t)) > 0$.

If f has a local time satisfing some conditions, we can obtain something about the behavior of f (bad behaviors in the classical sense).

<u>Lemma 2</u>.[6,8] If $\alpha(t,x)$ is continuous in t for a.e. x, then

(i) for a.e.t, $\forall K>0$,

$$\lim_{\varepsilon\to 0} |\{ t-\varepsilon\leqq s\leqq t+\varepsilon ;\ |f(t) - f(s)|\leqq K|t-s|\}|/\varepsilon = 0$$

i.e. $$\operatorname{ap-lim}_{s\to t} |f(t) - f(s)|/|t-s| = +\infty .$$

(This is much stronger than $\limsup_{s\to t} |f(t) - f(s)|/|t-s| = +\infty$.)

(ii) for a.e.t, $L_t = \{0\leqq s\leqq t;\ f(s) = f(t)\}$ is uncountable.

(iii)for a.e.t, t is not a point of increase for f.

(If there exists $\delta>0$ such that $f(s)\leqq f(t)\leqq f(s')$ holds for any $t-\delta\leqq s\leqq t\leqq s'\leqq t+\delta$,then t is called a point of increase.)

Lemma 3.[4] If $\alpha(t,x)$ is jointly continuous in (t,x), then

(i) $\forall t$ and $\forall K>0$, $\lim_{\varepsilon\to 0} |\{s; |s-t|<\varepsilon, |f(t)-f(s)|\leqq K|t-s|\}| = 0$,

(ii) $\{x;$ the number of $\{t; f(t)=x\}$ is countable$\}$ is nowhere dense in the range of f.

Lemma 4.[4,5] If $\alpha(t,x)$ satisfies Hölder condition of order $0<\beta<1$, i.e.there exists a constant $D>0$ such that

$$|\alpha(t,x) - \alpha(s,x)| \leqq D|t-s|^{\beta}$$

holds for any t,s and x, then

(i) $\forall t, \forall K>0$, and $\gamma>1-\beta$

$$\lim_{\varepsilon\to 0} |\{s; |t-s|<\varepsilon, |f(t)-f(s)|\leqq K|t-s|^{\gamma}\}|/\varepsilon = 0 ,$$

(ii) for any sub-interval $J \subseteq [0,1]$,

$$\max\{t\in J; f(t)\} - \min\{t\in J; f(t)\} \geqq |J|^{1-\beta}/D .$$

(This means that f is nowhere differentiable.)

§3. Definition and the properties of self-affine functions.

As the examples of irregular continuous functions, nowhere differentiable functions are well known. Such examples are Wierstrass functions,Peano functions (the x or y component of his surface filling continuous function) and Takagi function. In this section, we will generalize Peano functions which satisfy very nice scaling property. With a suggestion by Prof.Mandelbrot we call them self-affine function.

Definition.[12] A real function f defined on [0,1] is called a self-affine function with scale H if and only if there exist H and a positive integer r such that for any N = 1,2,..., and k = 0,1,...,r^N-1,

$$f(t) - f(t_{N,k}) = T_{N,k}\, r^{-NH} f(r^N(t-t_{N,k}))$$

holds, where $0\leqq t-t_{N,k} < r^{-N}$, $t_{N,k} = kr^{-N}$ and $T_{N,k} = 1$ or -1.

Theorem 1.(Representation theorem) A continuous function f is a self-affine function with scale $H>0$ such that $f(0) = 0$ and $f(1) = 1$, if and only if there exists a sequence of +1 or -1,say,$x(0),x(1),...,x(r-1)$ such that

$$\sum_{i=0}^{r-1} x(i) = r^H \quad \text{(necessarily } r \text{ and } r^H \text{ are integers)},$$

and f is expressed as follows:

$$f(t) = \sum_{n=1}^{\infty} y(t_{n-1})\, s(\varepsilon_n)\, r^{-nH} \quad ,$$

where $t = \sum_{k=1}^{\infty} \varepsilon_k r^{-k}$, $t_0 = 0$, $t_n = \sum_{k=1}^{n} \varepsilon_k r^{-k}$, $s(0) = 0$, $s(j) = \sum_{i=0}^{j-1} x(i)$,

$1 \leqq j \leqq r-1$, $y(t_0) = 1$ and $y(t_n) = \prod_{k=1}^{n} x(\varepsilon_k)$.

Theorem 2.[12] A non-constant continuous self-affine function with scale $1>H>0$ satisfies the Hölder continuity of order H at any point, but does not satisfy the Hölder continuity of order $H'>H$ at any point. In consequence, it is nowhere differentiable.

Theorem 3. For any rational number $0<H<1$, there exists a continuous self-affine function with scale H such that its local time $\alpha(t,x)$ exists and $\alpha(t,x)$ satisfies

(i) $\alpha(1,x) = 1$ for $0 \leqq x \leqq 1$,

$= 0$ for $x \notin [0,1]$ (i.e. f has the uniform distribution),

(ii) $\forall x$, $\alpha(t,x) - \alpha(s,x) \leqq 2r^{1-H} |t-s|^{1-H}$ $(s<t)$.

Taking account of Lemmas in §2, the above functions are very much irregular in classical sense but they have very nice properties in some sense.

Institute of Mathematics
Yoshida College,
Kyoto University,
Kyoto,Japan 606

References

[1] S.M.Berman; Local times and sample function properties of stationary Gaussian processes.Trans.Amer.Math.Soc. 137(1969),277-299.

[2] __________; Harmonic analysis of local times and sample functions of Gaussian processes.Trans.Amer.Math.Soc. 143(1969),269-281.

[3] __________; Occupation times of stationary Gaussian processes.J.Appl.Prob. 7(1970),721-733.

[4] __________; Gaussian processes with stationary increments:local times and sample function properties.Annal.Math.Stat.41(1970),1260-1272.

[5] __________; Gaussian sample functions: uniform dimension and Hölder conditions nowhere.Nagoya Math.J.46(1972),63-86.

[6] __________; Nonincrease almost everywhere of certain measurable functions with applications to stochastic processes.Proc.Amer.Math.Soc.88(1983),141-144.

[7] D.Geman and J.Horowitz; Local times for real and random functions. Duke Math. J.43(1976),809-828.

[8] D.Geman; A note on the continuity of local times. Proc.Amer.Math.Soc. 57(1976),321-326.

[9] D.Geman and J.Horowitz; Occupation densities. Special invited paper. Annal.Prob.8(1980),1-67.

[10]K.Itô and H.P.McKean; Diffusion processes and their sample paths.1965 Die Grund.Math.Wissenschaften Band 125,Springer.

[11]T.Kamae; A characterization of self-affine functions. Japan J.Appl.Math. 3(1986).(in print)

[12]N.Kôno; On self-affine functions.Japan J.Appl.Math.3(1986).(in print)

[13]G.Peano;Sur une courbe,qui remplit toute une aire plane.Math.Annalen 36(1890),157-160.