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4. Remarks on Smoothings of Four-Spaces

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One of the striking consequences of Michael Freedman's topological theory of 4-manifolds and Simon Donaldson's non-existence results of certain smooth 4-manifolds is the existence of an exotic smoothing of the euclidean 4-space $\mathbb{R}^4$. Then an example of a manifold with finitely generated homology groups admitting infinitely many smooth structures was found along the same lines [G] [K]. More recently Taubes showed the existence of uncountably many smooth structures on $\mathbb{R}^4$ [T]. Actually Taubes' argument applies to a fairly large class of open 4-manifolds, and it may now be possible to expect uncountably many smooth structures on every non-compact 4-manifold.

In this informal note, we give some remarks and observations concerning smooth structures on non-compact 4-manifolds which seem to indicate the complicated nature of the problem: In §1 we provide a natural construction which possibly produces uncountably many smoothings and discuss some problems on the construction; In §2 we give some observations which shows some difficulties to reasonably topologize the set of smooth structures on a non-compact 4-manifold.

§1 Given a non-compact 4-manifold V^4, consider the following construction which is an immediate generalization of Taubes' construction to arbitrary non-compact manifold and actually produces uncountably many smoothings in many cases (e.g. when an end of V is diffeomorphic to $S^3 \times \mathbb{R}$).

<u>1.1</u> Construction: Fix an exotic smoothing of $\mathbb{R}^4$, denoted R, which is standard on $(-\infty,0) \times \mathbb{R}^3$, and a smooth properly embedded half-open ray A in V^4. Identifying an open tubular neighborhood of A with $(-\infty,0) \times \mathbb{R}^3$ R so that the open end of A goes to $0 \times \mathbb{R}^3$, we can form an end connected sum of V^4 with R, denoted $V^4 \natural R$, which is homeomorphic to V^4. Then we can define a continuously parametrized smooth submanifolds V(r) of $V^4 \natural R$ for $0 < r \leq \infty$ by setting V(r) = $V^4 \cup$ (open ball of radius r centered at the origin in R), which are homeomorphic to V^4.

<u>1.2</u> Remark: When the end of V is diffeomorphic to an end of a punctured definite 1-connected 4-manifold with non-standard intersection form, uncountably many distinct smoothings of V^4 can be obtained by taking continuously parametrized parallel ends as in [T]. It is not clear, however, that the above construction (where R is connected to V along the standard structure) yields uncountably many smoothings in these cases.

<u>1.3</u> Remark: If the smooth structure on the end of V^4 is sufficiently complicated, the above construction fails. For example, set V^4 = a universal smoothing of $\mathbb{R}^4$ in [FT]. Then, for any choice of R, V(r)'s are all diffeomorphic to V^4, i.e., the above construction cannot produce any new smoothing.

<u>1.4</u> $P = \#_{n=1}^{\infty} S^2 \times S^2$: Also, if the end of V^4 is topologically complicaded, the above construction might fail. A candidate to this is an infinite connected sum of $S^2 \times S^2$. More specifically, consider a countable sequence of small disjoint 4-balls D^4_n in the standard $\mathbb{R}^4$

centered at points $(n,0,0,0)$, $n = 1,2, \cdots$, and take connected sums with countably many copies of $S^2 \times S^2$, denoted $(S^2 \times S^2)_n$, at D^4_n's. The resulting manifold $P^4 = \#_{n=1}^{\infty}(S^2 \times S^2)_n$ is an open smooth 4-manifold with one end whose homology groups are infinitely generated. The following observation is an easy consequence of techniques in [FT] which shows the complicatedness of the smooth manifold P.

1.5 Proposition: If Q^4 is a smooth 4-manifold topologically homeomorphic to P (i.e. a possibly different smoothing of P). Then Q can be smoothly embedded into P in such a way that int(P − Image(Q)) is topologically an open 4-ball.

Proof First observe that any smoothing of $\mathbb{R}^4$, say U, can be smoothly embedded into P. In fact, one can construct a proper h-cobordism consisting of (small) 2- and 3-handles between P and $U \# (\#_{n=1}^{\infty}(S^2 \times S^2)_n)$ which is topologically a product and smoothly a product near $\bigcup_{n=1}^{\infty}((S^2 \times S^2)_n - D^4_n)$. The smooth Whitney tricks may be performed after removing self-intersections of Whitney disks by Norman tricks in $(S^2 \times S^2)_n - D^4_n$ in the middle level and we get a diffeomorphism $U \#(\#_{n=1}(S^2 \times S^2)_n) \cong P$. Then we get an embedding of $U \cong U$ − (tubular neighborhood of a smooth proper half-open arc which joins D^4_n's in U) into P.

Next consider the universal smoothing H of $[0,\infty) \times \mathbb{R}^3$ constructed in [FT]. As above we can smoothly embed H into P, call the image H_+, so that the end of H_+ goes to the end of P.(i.e. a proper embedding). Consider a smooth proper h-cobordism W^5 between P and Q consisting of (small) 2- and 3-handles which is topologically a product (Note that we can extend the smooth structure on P and Q to W, since the obstruction $H^4(\text{relative}; \pi_3(TOP/PL)) = 0$). Then as in [FT], resmoothing a neighborhood N of properly embedded half-open ray A in H_+ in P crossed with [0,1] in W^5 by plugging $H\times[0,1]$ into $N\times[0,1]$ in W^5, smooth Whitney tricks may be performed after smoothing the cores of sutably chosen Casson Handles in the middle level using the resmoothing $H\times(1/2)$. Then we get a diffeomorphism $(Q - (N\times(1)))\underset{\partial}{\cup}(H\times(1)) \cong (P - (N\times(0)))\underset{\partial}{\cup}(H\times(0))$, and hence a smooth embedding of $Q \cong (Q - N\times(1))$ into the latter manifold (P with $N\times0$ replaced by $H\times0$) so that the compliment of the closure of the image is topologically homeomorphic to the open 4-ball. Finally this manifold, P with $N\times0$ replaced by $H\times0$, is actually diffeomorphic to P, since $H\times0$ is absorbed into the universal smoothing H_+ originally embedded in P |

§ 2 Since there are uncountably many smoothings of a non-compact 4-manifold (at least for many cases), the following seems to be a natural question: Can we find a reasonable (Hausdorff, etc.) topology on the set of smooth structures on a non-compact 4-manifold ($\mathbb{R}^4$, for example)?

Let S_4 be the set of smooth structures on $\mathbb{R}^4$. One can define a natural distance (admitting ∞), call Lipschitz-Shikata distance, on S_4 as follows: For U, V $\in S_4$, define $d(U,V) \in [0,\infty]$ by $d(U,V) = \inf_{d,\bar{d}} (\inf_h (\log \max(|h|,|h^{-1}|)))$, where d and $\bar{d}$ run through all complete riemannian metrics compatible with the smooth structures U and V respectively, and h runs through all onto homeomorphism $U \to V$, and $|h| = \sup_{x\neq y, \in U} (\bar{d}(h(x),h(y)/d(x,y)) \in [0,\infty]$. (It is non-trivial that $d(U,V) = 0$ implies U = V, which is a consequence of the following proposition.) One can readily generalize a result of Shikata [S] to non-compact manifolds and get:

2.1 Proposition: S_4 is a discrete space with respect to the Lipschitz-Shikata distance. In other words, if two exotic $\mathbb{R}^4$'s admit a sufficiently small lipeomorphism with respect to some riemannian metrics, then they are diffeomorphic.

Proof Fix an isometric imbedding of V into the euclidean space $\mathbb{R}^m$ for sufficiently large m (Nash imbedding theorem is valid for non-compact manifolds), and take a tubular neighborhood N of V in $\mathbb{R}^m$ so that the fibers of $\pi : N \to V$ are orhtogonal to V in $\mathbb{R}^m$. Then, as usual, the continuous $\mathbb{R}^m$-valued function $f = i \circ h: U \to V \hookrightarrow \mathbb{R}^m$ (where $h: U \to V$ is a small lipeomorphism), is approximated by a C^∞ $\mathbb{R}^m$-valued function $f_t(x) = \int_U f(y)g_t(x,y)dy$, $x \in U$, for $t > 0$, a coordinatewise integration over the riemannian manifold (U,d), where $g_t(x,y)$ is given by $g_t(x,y) = g(d(x,y)/t)/(\int_u g(d(x,y)/t)dx)$ and g is a non-negative C^∞function: $\mathbb{R} \to [0,\infty)$ with compact support and g = constant near $0 \in \mathbb{R}$.

Consider the composition $F(x,t) = \pi \circ f_t(x)$ for $x \in U$, $t > 0$, and $F(x,0) = h(x)$, for $x \in U$, where π is the orthogonal projection of N onto V. Although this composition is not defined everywhere, there is a positive number t(L) for each compact subset L of U such that F(x,t) is well-defined on $L\times[0,t(L)]$, i.e., $f_t(x) \in N$ for $x \in L$, $0 \leq t \leq t(L)$. Furthermore, Shikata's proof in [S] shows that there is a positive constant c (independent of the manifolds U, V or the choice of isometric imbedding of V) such that if $|h|$ and $|h^{-1}|$ are both $< c$, and if $t(L) > 0$ is sufficiently small (depending on L), then F defines a C^∞embedding of (neighborhood of L)x(t) for $0 < t \leq t(L)$.

Hence we can find codimension 0 compact submanifolds M_n, L_n of U for $n = 1, 2, \cdots$, and a deceasing sequence $t_1 > t_2 > \cdots$ of positive numbers ($0 < t_n < t(L_n)$) such that: (i) $\bigcup_{n=1}^{\infty} M_n = \bigcup_{n=1}^{\infty} L_n = U$,

$M_n \Subset L_n \Subset M_{n+1}$; (ii) $F|_{L_n x(t)}$ is a C^∞ embedding of $L_n \cong L_n x(t)$ into V for $0 < t \leq t_n$; (iii) $F(M_n x[0,t_n]) \subset F(L_n x(t_n)) \subset F(M_{n+1} x(t_{n+1}))$.

Consider the function $\bar{F}(x,t) = (F(x,t),t) \in Vx[0,\infty)$ for (x,t) in a neighborhood of $X \equiv \bigcup_{n=1}^{\infty} (L_n x[0,t_n])$ in $Ux[0,\infty)$. Then $\bar{F}$ C^∞ embeds $X - Ux(0)$ into $Vx(0,\infty)$ (differentiability of $\bar{F}$ is assured by the compactness of support of g in the definition of $f_t(x)$). One can construct a C^∞ vector field $\mathcal{X}$ on $Vx(o,\infty)$ by partition of unity argument with the following properties: (i) $dp_2(\mathcal{X}) = d/dt$ (where p_2 is the projection $Vx(0,\infty) \to (0,\infty)$); (ii) $\mathcal{X} = d\bar{F}(\partial/\partial t)$ on $\bar{F}(\bigcup_{n=1}^{\infty} M_n x(0,t_n])$; (iii) $\mathcal{X} = \partial/\partial t'$ outside a neighborhood of $\bigcup_{n=1}^{\infty} (F(L_n x(t_n)) \times (0,t_n])$.

Fixing a number $t_0 > t_1$, let $\pi : V \times (0,\infty) \to V$ be the projection along the C^∞ flow $\mathcal{X}$ onto $V = Vx(t_0)$, i.e. $\pi(y,t)$ is the unique intersection of $Vx(t_0)$ with the trajectory of $\mathcal{X}$ through (y,t). Then the desired diffeomorphism $\bar{h}$: $U \to V$ may be written as the union $\bar{h}(x) = \bigcup_{n=1}^{\infty} \pi \circ \bar{F}|_{M_n x(t_n)} (x,t_n)$, which is well-defined from the construction. ◻

<u>2.2</u> Remark: The above argument is valid for any non-compact manifolds of any dimension (the constant c depends only on the dimension).

<u>2.3</u> Remark: The Lipschitz-Shikata distance is, hence, too strong. There are some candidates for defining weaker topologies on S_4. Consider the following spaces of embeddings: (i) E_1 = topological embeddings of the unit open 4-ball into a universal smoothing U of $\mathbb{R}^4$ in [FT]; (ii) E_2 = topological embeddings of the unit open 4-ball into $P = \#_{n=1}(S^2 x S^2)_n$ described in 1.4; (iii) E_3 = C^∞ proper embeddings of the universal smoothing H of the half-space $[0,\infty) x \mathbb{R}^3$ in [FT] into H, f: H $\to$ H. Then we have projections $p_i : E_i \to S_4$, defined by $p_i(f)$ = Image of f with the induced smooth structure, for

$i = 1, 2$, and $p_3(f) = \mathrm{Int}(H - \mathrm{Image}(f))$ (p_1 and p_3 are surjective by [FT], and p_2 is surjective from the argument in Proof of 1.5). Hence any topology on E_i induces a quotient topology on S_4. It seems, however, not easy to make this topology Hausdorff. For example, if we put compact-open topology on E_1, the only open sets of S_4 will be the whole set and the empty set.

2.4 Remark: It would be nice if one could define a reasonable topology on S_4 with possibly accessible homotopy groups. Related to this is the following naive question: Is there a reasonable topology on S_4 such that the singular complex $S(S_4)$ is identifiable with the Kan complex $\mathrm{DIFF}(\mathbb{R}^4)$ of sliced families of smooth structures on $\mathbb{R}^4$ (S_4 is the set of vertices $\mathrm{DIFF}(\mathbb{R}^4)^0$)? Again, this topology cannot be Hausdorff, since a universal smoothing U is contained in any neighborhood of any element and the only neighborhood of the standard structure is the whole set.

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