数理解析研究所講究録
第 619 巻 1987 年 12-44

Some results restricting the mutual position of the components of a nonsingular real algebraic curve in $\mathbb{R}P^1 \times \mathbb{R}P^1$

Sachiko MATSUOKA

奈良女子大 理 松岡幸子

In this article we study nonsingular real algebraic curves in $\mathbb{R}P^1 \times \mathbb{R}P^1$. We apply some of the techniques used in the recent investigations of algebraic curves in $\mathbb{R}P^2$ (cf. [5], [6]) to our algebraic curves in $\mathbb{R}P^1 \times \mathbb{R}P^1$. The main results of this article are Theorems (1.14), (1.15), (1.19), (1.20), (1.21), and (1.23). These theorems correspond to Rokhlin's congruence, Kharlamov-Gudkov-Krakhnov's congruence, Arnol'd's congruence, Marin-Kharlamov's congruence, Petrovskii's inequality, and Arnol'd's inequalities respectively; which are important to algebraic curves in $\mathbb{R}P^2$ (cf. [5], [6]).

§1. Formulation of our problem and statement of results

Let $F(X_0,X_1;Y_0,Y_1)$ be a homogeneous polynomial of degree d, r with respect to (X_0,X_1), (Y_0,Y_1) respectively, that is

$$F(X_0,X_1;Y_0,Y_1) = \sum_{\substack{i_0+i_1=d \\ j_0+j_1=r}} a_{i_0 i_1 j_0 j_1} X_0^{i_0} X_1^{i_1} Y_0^{j_0} Y_1^{j_1},$$

where d and r are non-negative integers with $dr \neq 0$, $(d,r) = (0,1)$, or $(d,r) = (1,0)$; the $a_{i_0 i_1 j_0 j_1}$ are real numbers, at least one of which is non-zero. We set

$$A = \{([X_0:X_1],[Y_0:Y_1]) \in P^1 \times P^1 \mid F(X_0,X_1;Y_0,Y_1) = 0\} \text{ and}$$
$$\mathbb{R}A = \{([X_0:X_1],[Y_0:Y_1]) \in \mathbb{R}P^1 \times \mathbb{R}P^1 \mid F(X_0,X_1;Y_0,Y_1) = 0\},$$

where P^1 means the one-dimensional complex projective space CP^1. We assume that A is nonsingular, then it is well known (cf. [2]) that A is a compact connected one-dimensional complex manifold of genus

(1.1) $\qquad g(A) = (d-1)(r-1).$

The topology of A depends only on the degree (d,r). But the same is not true of the real curve $\mathbb{R}A$, and the pair $\mathbb{R}A \subset \mathbb{R}P^1 \times \mathbb{R}P^1$. In any case $\mathbb{R}A$ is a closed manifold of real dimension one, hence diffeomorphic to a disjoint union of circles. Our problem can now be **formulated** as follows.

Problem. Describe which arrangements of circles can be realized by an algebraic curve of degree (d,r).

We write ℓ_A for the number of connected components of $\mathbb{R}A$.

Lemma 1.2. ("Harnack-Thom type" inequality) For a curve of degree (d,r), we have

$$\ell_A \leq (d-1)(r-1) + 1.$$

(1.2) is an immediate consequence of (1.1) by the theorem of Harnack (cf. [6]). Note that there exist curves having precisely $(d-1)(r-1) + 1$ components of every degree (d,r) (see [2]).

Definition 1.3. We call curves having $(d-1)(r-1) + 1 - \iota$ components <u>$M - \iota$ curves</u>.

Let E_i $(i=1,\dots,\ell_A)$ denote the components of $\mathbb{R}A$. Then the isotopy type of the embedding $E_i \hookrightarrow \mathbb{R}P^1 \times \mathbb{R}P^1$ is determined by the homology class

$$[E_i] = s_i[\infty \times \mathbb{R}P^1] + t_i\,[\mathbb{R}P^1 \times \infty] \in H_1(\mathbb{R}P^1 \times \mathbb{R}P^1; \mathbb{Z}),$$
$$s_i,\ t_i \in \mathbb{Z}.$$

The following can be shown easily.

(1.4) If $s_i t_i = 0$, then $(s_i, t_i) = (0,0)$, $(\pm 1, 0)$, or $(0, \pm 1)$; and if $s_i t_i \neq 0$, then s_i and t_i are relatively prime.

(1.5) For i, j $(1 \leq i, j \leq \ell_A)$, we have $s_i t_j - t_i s_j = 0$.

From (1.4) and (1.5), we conclude

(1.6) if $E_i \neq 0$ and $E_j \neq 0$ $(1\ i, j\ {}_A)$, then $[E_i] = \pm [E_j]$.

Definition 1.7. We call components E_i with $E_i = 0$ <u>ovals</u>, and otherwise components <u>non-trivial</u>.

We write ℓ' and ℓ'' for the numbers of ovals and non-trivial components respectively. $(\ell_A = \ell' + \ell'')$

For an oval E_i, $\mathbb{R}P^1 \times \mathbb{R}P^1 \setminus E_i$ consists of two connected components, one of which is diffeomophic to an open disk Int D^2.

Definition 1.8. For an oval E_i, we call the connected component of $\mathbb{R}P^1 \times \mathbb{R}P^1 \setminus E_i$ which is diffeomorphic to an open disk the <u>interior</u> of the oval, and the other component the <u>exterior</u> of that. We say an oval E_i <u>surrounds</u> an oval E_j, if the interior of E_i contains E_j (Fig. 1.9).

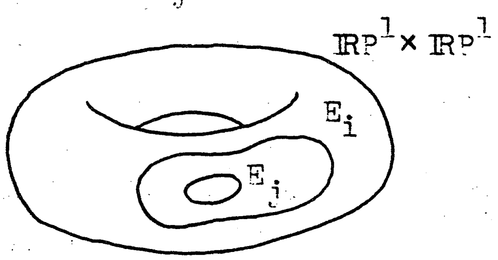

Fig. 1.9.

Definition 1.10. An oval surrounded by even (odd) number of ovals is called <u>even</u> (<u>odd</u>), and an oval surrounded by j ($\in \mathbb{Z}$) ovals is called a <u>j-oval</u>. In particular a 0-oval is called an <u>outermost</u> oval. The numbers of even, odd, even j-, and odd j-ovals are denoted by P, N, ${}_jP$, and ${}_jN$ respectively.

From now on we restrict ourselves to the case of even degree $(d,r) = (2k,2\ell)$. In this case we can say whether the value of the polynomial F at a point of $\mathbb{R}P^1 \times \mathbb{R}P^1$ is positive or negative. The two sides of $\mathbb{R}A$ are given by $F > 0$, $F < 0$; we set

$$B^+ = \{([X_0:X_1],[Y_0:Y_1]) \in \mathbb{R}P^1 \times \mathbb{R}P^1 \mid F(X_0,X_1;Y_0,Y_1) \geqq 0\}$$
$$B^- = \{([X_0:X_1],[Y_0:Y_1]) \in \mathbb{R}P^1 \times \mathbb{R}P^1 \mid F(X_0,X_1;Y_0,Y_1) \leqq 0\}$$

Convention 1.11. (i) In the case $\ell'' = 0$, we make the convention that $F < 0$ in the intersection of the exteriors of all the ovals of $\mathbb{R}A$. (ii) In the case $\ell'' > 0$, we write E_i $(i=1,\ldots,\ell')$ and E_i $(i=\ell'+1,\ldots,\ell'+\ell'')$ for ovals and non-trivial components of $\mathbb{R}A$ respectively. Then $\mathbb{R}P^1 \times \mathbb{R}P^1 \setminus (E_{\ell'+1} \cup \cdots \cup E_{\ell'+\ell''})$ consists of ℓ'' components, each of which is diffeomorphic to $S^1 \times \mathrm{Int}\, I$. We write R_i $(i=1,\ldots,\ell'')$ for the closures of these components; and make the convention that $F < 0$ in the intersection of $\mathrm{Int}\, R_1$ and the exteriors of all the ovals in R_1,

$E_{\ell'+\ell''} \vee E_{\ell'+1} = \partial R_1$, $E_{\ell'+1} \vee E_{\ell'+2} = \partial R_2, \ldots$, $E_{\ell'+i} \vee E_{\ell'+i+1} = \partial R_{i+1}, \ldots$, and $E_{\ell'+\ell''-1} \vee E_{\ell'+\ell''} = \partial R_{\ell'+\ell''}$. Then it turns out that ℓ'' is even. (See Fig. 1.12.)

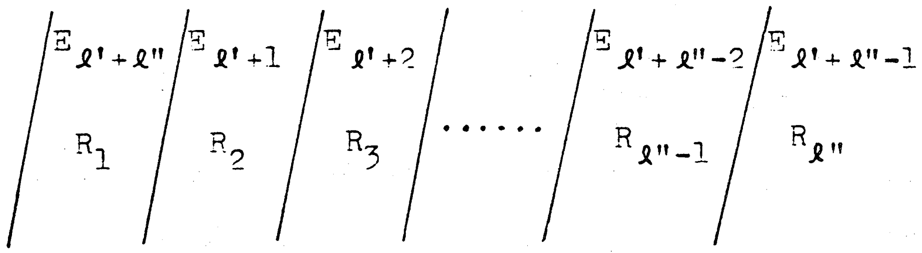

Fig. 1.12.

The numbers of even, odd, even j-, and odd j-ovals in R_i are denoted by P^i, N^i, ${}_jP^i$, and ${}_jN^i$ respectively.

Note 1.13. In the case $\ell'' = 0$, we have

$$\#\{\text{components of } B^+\} = P,$$
$$\#\{\text{components of } B^-\} = 1 + N,$$
$$\chi(B^+) = P - N, \text{ and}$$
$$\chi(B^-) = N - P;$$

and in the case $\ell'' > 0$, we have

$$\#\{\text{components of } B^+\} = \sum_{i:\text{odd}} P^i + \sum_{i:\text{even}} N^i + \frac{\ell''}{2},$$
$$\#\{\text{components of } B^-\} = \sum_{i:\text{even}} P^i + \sum_{i:\text{odd}} N^i + \frac{\ell''}{2},$$
$$\chi(B^+) = \sum_i (-1)^{i+1}(P^i - N^i), \text{ and}$$
$$\chi(B^-) = \sum_i (-1)^i (P^i - N^i).$$

Now we state our main results. First for M curves and M-1 curves we obtain the following congruences.

Theorem 1.14. ("Rokhlin type" congruence) For an M curve of degree $(d,r) = (2k, 2\ell)$ with $\ell'' = 0$, we have

$$\chi(B^+) \ (= P - N) \equiv \frac{dr}{2} \pmod 8.$$

Theorem 1.15. ("Kharlamov-Gudkov-Krakhnov type" congruence) For an M-1 curve of degree $(d,r) = (2k, 2\ell)$ with $\ell'' = 0$, we have

$$\chi(B^+)\ (= P - N) \equiv \frac{dr}{2} \pm 1 \pmod 8.$$

Next, we consider the embedding $\mathbb{R}A \hookrightarrow A$.

Definition 1.16. We say $\mathbb{R}A$ is a <u>dividing</u> curve if $A \setminus \mathbb{R}A$ is not connected.

We write $\tau : P^1 \times P^1 \to P^1 \times P^1$ for the complex conjugation. Since F is a real polynomial, we have $\tau(A) = A$. The following lemma is basic.

Lemma 1.17. (i) An M **curve is always a dividing curve.** (ii) For a dividing curve $\mathbb{R}A$, $A \setminus \mathbb{R}A$ consists of two connected components, which are interchanged by τ, and we have

$$\ell_A \equiv (d-1)(r-1) + 1 \pmod 2.$$

For a dividing curve $\mathbb{R}A$, we write A^+ and A^- for the closures of components of $A \setminus \mathbb{R}A$. The natural orientations of $A^{\pm}$ determine on $\mathbb{R}A$, as on their common boundary, two opposite orientations.

Definition 1.18. For a dividing curve $\mathbb{R}A$, its two opposite orientations as stated above are called <u>complex</u>.

In the case $\ell'' > 0$, we set

$\hat{\ell} = \#\{i : \text{even} \mid$ Any orientation of ∂R_i induced by an orientation of R_i does not coincide with any complex orientation of ∂R_i.}

Theorem 1.19. ("Arnol'd type" congruences) For a dividing curve of degree $(d,r) = (2k,2\ell)$, we have

$$\chi(B^+) \equiv \frac{dr}{2} \begin{cases} (\text{mod } 4) & (\text{if } \ell'' = 0, \text{ or } \ell'' > 0 \text{ and } \hat{\ell} \text{ is even}) \\ (\text{mod } 2) & (\text{if } \ell'' > 0 \text{ and } \hat{\ell} \text{ is odd}) \end{cases}$$

Theorem 1.20. ("Marin-Kharlamov type" congruences) If $\mathbb{R}A$ is a not dividing M-2 curve of degree $(d,r) = (2k,2\ell)$ with $\ell'' = 0$, or an M curve of degree $(d,r) = (2k,2\ell)$ with $\ell'' > 0$ and $\hat{\ell}$ is odd; then we have

$$\chi(B^+) \equiv \frac{dr}{2} + 0, \pm 2 \pmod 8.$$

Theorem 1.21. ("Petrovskii type" inequality) For a curve of degree $(d,r) = (2k,2\ell)$, we have

$$|\chi(B^+)| \leqq (k-1)(\ell-1) + 2k\ell.$$

To formulate our last theorem we divide the ovals into three classes.

Definition 1.22. An oval is called <u>positive</u> (<u>zero</u>, <u>negative</u>) if the Euler characteristic of the intersection of its interior and the exteriors of all the ovals surrounded by it is 1 (0, negative). The numbers of positive, zero, and negative even (odd) ovals are denoted by P_+ (N_+), P_0 (N_0), and P_- (N_-) respectively, and we define the notations P^i_+, P^i_0, P^i_-, N^i_+, N^i_0, and N^i_- in the same way.

In the case $\ell'' > 0$ we may assume that

$$[E_i] = \pm(s[\infty \times \mathbb{R}P^1] + t[\mathbb{R}P^1 \times \infty])$$

for every non-trivial component E_i. (Recall (1.6).) We set

$$\ell'''_{even} = \#\{i:\text{even} \mid R_i \text{ contains some ovals.}\}$$
$$\ell'''_{odd} = \#\{i:\text{odd} \mid R_i \text{ contains some ovals.}\}$$

Theorem 1.23. ("Arnol'd type" inequalities) Let $\mathbb{R}A$ be a curve of degree $(d,r) = (2k, 2\ell)$. We consider the following inequalities.

(1) $P_- + P_0 \leqq (k-1)(\ell-1)$

(2) $P_+ + P_0 \leqq (k-1)(\ell-1) + 2k\ell + (P - N)$

(3) $N_- + 1 + N_0 \leqq (k-1)(\ell-1)$

(4) $N_+ + N_0 \leqq (k-1)(\ell-1) + 2k\ell - (P - N)$

(1') $\sum_{i:odd} P^i_- + \sum_{i:even} N^i_- + \sum_{i:odd} P^i_0 + \sum_{i:even} N^i_0 + \frac{\ell''}{2} \leqq (k-1)(\ell-1)$

(2') $\sum_{i:odd} P^i_+ + \sum_{i:even} N^i_+ + \sum_{i:odd} P^i_0 + \sum_{i:even} N^i_0 + \frac{\ell''}{2} - \ell'''_{even} \leqq (k-1)(\ell-1) + 2k\ell + \sum_i (-1)^{i+1}(P^i - N^i)$

(3') $\sum_{i:odd} N^i_- + \sum_{i:even} P^i_- + \sum_{i:odd} N^i_0 + \sum_{i:even} P^i_0 + \frac{\ell''}{2} \leqq (k-1)(\ell-1)$

(4') $\sum_{i:odd} N^i_+ + \sum_{i:even} P^i_+ + \sum_{i:odd} N^i_0 + \sum_{i:even} P^i_0 + \frac{\ell''}{2} - \ell'''_{odd} \leqq (k-1)(\ell-1) + 2k\ell - \sum_i (-1)^{i+1}(P^i - N^i)$

(i) If $\mathbb{R}A$ is not dividing and $\ell'' = 0$ ($\ell'' > 0$), then (1), (2), (3), and (4) ((1'), (2'), (3'), and (4')) are correct.

(ii) If $\mathbb{R}A$ is dividing, k and ℓ are even, $\ell'' = 0$, and B^+ (B^-)

has a component whose Euler characteristic is not zero; then (1) and (2) ((3) and (4)) are correct.

(ii') (a) If $\mathbb{R}A$ is dividing, k and ℓ are even, $\ell'' > 0$, $\hat{\ell}$ is even, and B^+ (B^-) has a component whose Euler characteristic is not zero; then (1') and (2') ((3') and (4')) are correct.

(b) If $\mathbb{R}A$ is dividing, $\ell'' > 0$, $s \equiv k \pmod 2$, $t \equiv \ell$ (mod 2), $\hat{\ell}$ is odd, and B^+ (B^-) has a component whose Euler characteristic is not zero; then (1') and (2') ((3') and (4')) are correct.

(iii) In the case $\ell'' = 0$ ($\ell'' > 0$) (1), (2), (3), and (4) ((1'), (2'), (3'), and (4')) are correct if we add one to the right-hand side of each of them.

This completes the statement of our main results. Owing to limited space we give only the proofs of our congruences in the following sections. In §2 we prove (1.14), (1.15), and (1.19) by using a double covering Y of $P^1 \times P^1$ branching along A. In §3 we prove all our congruences simultaneously by using another method (Marin's method). In §4 we try classifying curves of degree (4,4) by applying our results to them.

§2. Proofs of congruences I

1°. A double covering Y of $P^1 \times P^1$ branching along A. Let $p_i : P^1 \times P^1 \to P^1$ $(i=1,2)$ be the projection to the i-th component. In the case $(d,r) = (2k,2\ell)$, we have

$$[A] = p_1{}^*\mathcal{O}_{P^1}(d) \otimes p_2{}^*\mathcal{O}_{P^1}(r) = (p_1{}^*\mathcal{O}_{P^1}(k) \otimes p_2{}^*\mathcal{O}_{P^1}(\ell))^{\otimes 2},$$

where $[A]$ is the line bundle over $P^1 \times P^1$ associated to A. Hence A can be the branch locus of a double covering of $P^1 \times P^1$. In fact such a covering is obtained as follows. We set $E = \{[X_0:X_1:X_2] \in CP^2 \mid X_0 \neq 0, \text{ or } X_1 \neq 0\} \times \{[Y_0:Y_1:Y_2] \in CP^2 \mid Y_0 \neq 0, \text{ or } Y_1 \neq 0\}$. Let $E_{k,\ell}$ be the set of equivalence classes of E with respect to the equivalence relation $([X_0:X_1:X_2],[Y_0:Y_1:Y_2]) \underset{k,\ell}{\sim} ([X_0':X_1':X_2'],[Y_0':Y_1':Y_2'])$ if $([X_0:X_1],[Y_0:Y_1]) = ([X_0':X_1'], [Y_0':Y_1'])$ in $P^1 \times P^1$, and

$$\left(\frac{X_2}{X_i}\right)^k \left(\frac{Y_2}{Y_j}\right)^\ell = \left(\frac{X_2'}{X_i'}\right)^k \left(\frac{Y_2'}{Y_j'}\right)^\ell \quad (i,j = 0,1).$$

We write $\pi : E_{k,\ell} \longrightarrow P^1 \times P^1$ for the natural projection. This is nothing but the line bundle $p_1^*\mathcal{O}_{P^1}(k) \otimes p_2^*\mathcal{O}_{P^1}(\ell)$. Now we set

$$Y = \left\{F(X_0,X_1;Y_0,Y_1) + X_2^d Y_2^r = 0\right\} \quad (\subset E_{k,\ell}).$$

Then Y is a compact connected 2-dimensional complex manifold, and the restriction $\pi : Y \longrightarrow P^1 \times P^1$ is a required double covering of $P^1 \times P^1$ branching along A. We write $\theta : Y \rightarrow Y$ for the covering transformation, which is a holomorphic involution. The complex conjugation on E induces an anti-holomorphic involution on Y, which is denoted by T^-. (Note that F is a real polynomial.) We set $T^+ = \theta \circ T^-$ $(= T^- \circ \theta)$, then $\pi \circ T^\pm = \tau \circ \pi$. Let $Y_R^\pm$ denote the fixed point sets of $T^\pm$, which are closed manifolds of real dimension 2. We obtain double coverings branching along $\mathbb{R}A$:

$$\pi : Y_R^\pm \longrightarrow B^\pm$$

Since $B^\pm$ **are orientable,** $Y_R^\pm$ are orientable and regarded as the doubles of $B^\pm$. $\theta : Y_R^\pm \longrightarrow Y_R^\pm$ are orientation reversing.

Next we consider the topology of Y. From the branched covering $Y \longrightarrow P^1 \times P^1$ we get

$$(2.1)\qquad \begin{cases} \pi_1(Y) = 0, \quad \chi(Y) = 6 + 2(d-1)(r-1), \text{ and} \\ \text{the signature } \sigma(Y) = -dr. \end{cases}$$

Hence Y is torsion free, and we have

$$(2.2)\qquad \operatorname{rank} H_2(Y\,;\,\mathbb{Z}) = 4 + 2(d-1)(r-1).$$

2°. Proofs of Theorems (1.14) and (1.15).

Definition 2.3. Let X be an almost complex manifold, and T be an anti-holomorphic involution on X. We say (X,T) is an <u>M-ι manifold</u> if

$$\dim H_*(X_R\,;\,\mathbb{Z}_2) = \dim H_*(X\,;\,\mathbb{Z}_2) - 2\iota,$$

where X_R denotes the fixed point set of T.

Theorem 2.4 (Rokhlin). (See [6].) Let (X,T) be an M manifold of real dimension 4n. Then

$$\chi(X_R) \equiv \sigma(X) \pmod{16}.$$

Theorem 2.5 (Kharlamov, Gudkov, Krakhnov). (See [6].) Let (X,T) be an M-1 manifold of real dimension 4n. Then

$$\chi(X_R) \equiv \sigma(X) \pm 2 \pmod{16}.$$

The relations **between** $\mathbb{R}A$ and $(Y,T^{\pm})$ are as follows.

Lemma 2.6. In the case $\ell'' = 0$, the following three conditions are equivalent. (i) $\mathbb{R}A$ is an $M-\iota$ curve. (ii) (Y,T^-) is an $M-\iota$ manifold. (iii) (Y,T^+) is an $M-(\iota+2)$ manifold. And in the case $\ell'' > 0$, the following three conditions are equivalent. (i') $\mathbb{R}A$ is an $M-\iota$ curve. (ii') (Y,T^-) is an $M-(\iota+2)$ manifold. (iii') (Y,T^+) is an $M-(\iota+2)$ manifold.

(2.6) is shown by the argument of 1°.

Now we give the proofs of (1.14) and (1.15).

If $\mathbb{R}A$ is an M curve of degree (d,r) with $\ell'' = 0$, then by (2.6), (Y,T^-) is an M manifold. By (2.4), we have $\chi(Y_R^-) \equiv \sigma(Y)$ (mod 16), where $\chi(Y_R^-) = 2\,\chi(B^-) = 2(N - P)$ and $\sigma(Y) = -dr$ (see (2.1)). Hence we have $P - N \equiv \frac{dr}{2}$ (mod 8). This completes the proof of (1.14). (1.15) is shown by (2.5) in the same way.

3°. Proof of Theorem (1.19). Recall that for a dividing curve $\mathbb{R}A$, $A \setminus \mathbb{R}A$ consists of two connected components, which are interchanged by τ, and the closures of these components are denoted by A^+ and A^-. A^+, A^-, B^+, and B^- have the common boundary $\mathbb{R}A$. We set

$$W = A^+ \cup B^+ \quad (\subset P^1 \times P^1). \tag{2.7}$$

W is a closed PL submanifold of $P^1 \times P^1$, and orientable if and only if an orientation of B^+ determine a complex orientation.

The proof of (1.19) rests on the next lemma.

Lemma 2.8. We have

$$[W] = \begin{cases} k[\infty \times P^1] + \ell[P^1 \times \infty] & \text{if } \ell'' = 0 \\ (k+\hat{\ell}s)[\infty \times P^1] + (\ell+\hat{\ell}t)[P^1 \times \infty] & \text{if } \ell'' > 0 \end{cases}$$

in $H_2(P^1 \times P^1 ; \mathbb{Z}_2)$.

(2.8) is shown at the chain level. We fix a triangulation of $P^1 \times P^1$ such that (i) the various subspaces arising are all subcomplexes (ii) τ is a simplicial map. We shall allow the following abuse of notation : A (for example) may denote either the space A or the corresponding ($\mathbb{Z}$-, or $\mathbb{Z}_2$-) chain (sum of all the 2-simplexes contained in A).

We now lift our triangulation of $P^1 \times P^1$ to a triangulation of the double covering Y. We define the transfer

$$\mathrm{tr} : (\text{chains of } P^1 \times P^1) \longrightarrow (\text{chains of } Y)$$

as follows : if σ is a simplex of A (the branching locus), $\mathrm{tr}\,\sigma$ is twice the corresponding simplex in Y; if σ is not in A, then $\mathrm{tr}\,\sigma$ is the sum of the two simplexes lying over it in Y. Then tr is a chain map. We set $(\infty \times P^1)_Y = \mathrm{tr}(\infty \times P^1)$, $(P^1 \times \infty)_Y = \mathrm{tr}(P^1 \times \infty)$.

Lemma 2.9. We have

$$[A] = k\left[(\infty \times P^1)_Y\right] + \ell\left[(P^1 \times \infty)_Y\right] \quad \text{in } H_2(Y ; \mathbb{Z}).$$

From (2.8) and (2.9) we get the next lemma.

Lemma 2.10. In $H_2(Y ; \mathbb{Z}_2)$ we have

$$[Y_R^{\pm}] = \begin{cases} [A] & \text{if } \ell'' = 0 \\ [A] + \hat{\ell} s\left[(\infty \times P^1)_Y\right] + \hat{\ell} t\left[(P^1 \times \infty)_Y\right] & \text{if } \ell'' > 0. \end{cases}$$

Now we prove (1.19). We define the unimodular integral symmetric bilenear form

$$\langle\ ,\ \rangle : H_2(Y ; \mathbb{Z}) \times H_2(Y ; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

by

$$\langle a,b\rangle = a \circ \theta_* b,$$

where $\circ$ is the intersection form of Y. Then for any element a,

(2.11) $$\langle a,a\rangle \equiv \langle a,[A]\rangle \pmod 2.$$

If $\ell'' = 0$, or $\ell'' > 0$ and $\hat{\ell}$ is even, then by (2.10) there exists an element L of $H_2(Y\,;\mathbb{Z})$ such that

$$[Y_R^+] = [A] + 2L \quad \text{in } H_2(Y\,;\mathbb{Z}).$$

Hence we have $\langle [Y_R^+],[Y_R^+]\rangle = \langle [A],[A]\rangle + 4\langle L,[A]\rangle + 4\langle L,L\rangle$, and by (2.11) we obtain

$$\langle [Y_R^+],[Y_R^+]\rangle \equiv \langle [A],[A]\rangle \pmod 8,$$

where $\langle [Y_R^+],[Y_R^+]\rangle = -[Y_R^+]\circ[Y_R^+] = \chi(Y_R^+)$ (Poincaré-Hopf Theorem.), and $\langle [A],[A]\rangle = [A]\circ[A] = \frac{1}{2}(2dr) = dr$. Thus it follows that

$$\chi(B^+) \equiv \frac{dr}{2} \pmod 8.$$

In the case $\ell'' > 0$ and $\hat{\ell}$ is odd, the required result follows from Lemma 1.17 (ii). This completes the proof of (1.19).

3. Proofs of congruences II

In this section we prove all our cogruences at once. First we give an outline of the proof. We consider the quotient space $P^1 \times P^1/\tau$ and its subspace $W = A/\tau \cup B^+$ (cf.(2.7)). We shall define the element $\alpha(P^1\times P^1/\tau, W)$ of $\mathbb{Z}_8$ for the pair $(P^1\times P^1/\tau, W)$ and show the following lemmas.

Lemma 3.1. For a curve $\mathbb{R}A$ of degree $(d,r) = (2k,2\ell)$, we have $$\chi(B^+) - \frac{dr}{2} \equiv (P^1\times P^1/\tau, W) \pmod 8.$$

Lemma 3.2. In the case $\ell'' = 0$

1) If $\mathbb{R}A$ is an M curve, then $\alpha(P^1\times P^1/\tau, W) = 0$

2) If $\mathbb{R}A$ is an $M-1$ curve, then $\alpha(P^1 \times P^1/\tau, W) = \pm 1$

3) If $\mathbb{R}A$ is an $M-2$, not dividing curve; then

$$\alpha(P^1 \times P^1/\tau, W) = 0, \pm 2$$

4) If $\mathbb{R}A$ is a dividing curve, then

$$\alpha(P^1 \times P^1/\tau, W) = 0, 4$$

In the case $\ell'' > 0$

1') If $\mathbb{R}A$ is an M curve, and

i) $\hat{\ell}$ is even, then $\alpha(P^1 \times P^1/\tau, W) = 0, 4$

ii) $\hat{\ell}$ is odd, then $\alpha(P^1 \times P^1/\tau, W) = 0, \pm 2$

2') If $\mathbb{R}A$ is a dividing curve, and

i) $\hat{\ell}$ is even, then $\alpha(P^1 \times P^1/\tau, W) = 0, 4$

ii) $\hat{\ell}$ is odd, then $\alpha(P^1 \times P^1/\tau, W)$ is even

Then (1.14), (1.15), (1.19), and (1.20) follow from 1), 2), 4) and 2'), and 3) and 1') - (ii) respectively.

1°. Definition of $\alpha(P^1 \times P^1/\tau, W)$ and Proof of (3.1). First the quotient $P^1 \times P^1/\tau$ is, as $P^1 \times P^1$, a naturally oriented smooth manifold without boundary. The following fact is known ([3]).

(3.3) $P^1 \times P^1/\tau$ is diffeomorphic to the 4-sphere S^4.

Next A/τ is also, as A, a smooth manifold, whose boundary is regarded as $\mathbb{R}A$. The following is obtained.

(3.4) A/τ is orientable if and only if $\mathbb{R}A$ is dividing.

Now we set $W = A/\tau \cup B^+$. Then W is a connected PL closed submanifold of $P^1 \times P^1/\tau$, which is orientable if and only if A/τ is orientable and an orientation of B^+ determine a complex orientation.

Remark 3.5. Although W is possibly non-orientable, $\mathbb{R}A$ is embedded in W two-sidedly.

By (3.3) we can define the "Rokhlin form" (see 1)

$$q : H_1(W ; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$$

for the pair $(P^1 \times P^1/\tau, W)$. Let $\alpha(P^1 \times P^1/\tau, W)$ denote the "Brown invariant" (see [1]) of q, which is an element of $\mathbb{Z}_8$. Then by the formula of Rokhlin-Guillou-Marin ([1]) we obtain

$$(3.6) \qquad \sigma(P^1 \times P^1/\tau) - (W \circ W)_{P^1 \times P^1/\tau} \equiv 2\,\alpha(P^1 \times P^1/\tau, W) \pmod{16},$$

where $\sigma(P^1 \times P^1/\tau)$ denotes the signature of $P^1 \times P^1/\tau$, that is zero by (3.3), and $(W \circ W)_{P^1 \times P^1/\tau}$ denotes the self-intersection number of W in $P^1 \times P^1/\tau$. And "2" means the homomorphism $\mathbb{Z}_8 \longrightarrow \mathbb{Z}_{16}$ such that $2(1) = 2$. Since $(W \circ W)_{P^1 \times P^1/\tau} = (A/\tau \circ A/\tau)_{P^1 \times P^1/\tau} + (B^+ \circ B^+)_{P^1 \times P^1/\tau} = \frac{1}{2}(A \circ A)_{P^1 \times P^1} + 2(-\chi(B^+)) = dr - 2\chi(B^+)$, (3.1) follows.

2°. The subspace L. To calculate $\alpha(P^1 \times P^1/\tau, W)$ we consider the following subspace L of $H_1(W ; \mathbb{Z}_2)$.

$$(3.7) \quad L = \begin{cases} \langle [E] \in H_1(W ; \mathbb{Z}_2) \mid E \text{ is a component (oval) of } \mathbb{R}A. \rangle_{\mathbb{Z}_2} & \text{if } \hat{\ell} = 0 \\ \langle [\partial R_i] \in H_1(W ; \mathbb{Z}_2) \rangle_{\mathbb{Z}_2} + \langle [E] \in H_1(W ; \mathbb{Z}_2) \mid E \text{ is an oval of } \mathbb{R}A. \rangle_{\mathbb{Z}_2} & \text{if } \hat{\ell} > 0 \end{cases}$$

Lemma 3.8. The Rokhlin form q is zero on L.

Proof. The following equality is easy to verify.

$$L = \langle [\partial B_i^-] \in H\,(W ; \mathbb{Z}_2) \mid B_i^- \text{ is a component of } B^-. \rangle_{\mathbb{Z}_2}$$

B_i^- is a membrane ([1]) for W. Hence by the definition of q (see [1]), we have

$$q([\partial B_i^-]) = (\vartheta + 2\#((\mathrm{Int}\ B_i^-)\cap W))_{\mathrm{mod}\ 4},$$

where ϑ denotes the obstruction number to extend the projective normal bundle of ∂B_i^- in W to a subbundle of the projective normal bundle of B_i^- in $P^1 \times P^1/\tau$. Whereas B_i^- is embedded in W two-sidedly (recall (3.5)), hence we have $\vartheta = 2\vartheta_V$, where ϑ_V denotes the obstruction number to extend a nowhere zero section of the normal bundle of ∂B_i^- in W to a section of the normal bundle of B_i^- in $P^1 \times P^1/\tau$. Since $\vartheta_V = 2(-\chi(B_i^-))$, and $\#((\mathrm{Int}\ B_i^-)\cap W) = 0$; we have

$$q([\partial B_i^-]) = (4(-\chi(B_i^-)))_{\mathrm{mod}\ 4} = 0. \qquad \text{Q.E.D.}$$

The Brown invariant has the following properties. (cf.[1])

(3.9) Let V be a finite dimensional vector space over $\mathbb{Z}_2$, $\circ : V \otimes V \longrightarrow \mathbb{Z}_2$ be a inner product, and $\varphi : V \longrightarrow \mathbb{Z}_4$ be a quadratic function (cf.[1]).

(1) If $V = V_1 \oplus V_2$ is a orthogonal decomposition, then for the Brown invariant $\sigma(\varphi)$ we have $\sigma(\varphi) = \sigma(\varphi|_{V_1}) + \sigma(\varphi|_{V_2})$.

(2) If $\dim_{\mathbb{Z}_2} V = 1$, then $\sigma(\varphi) = \pm 1$.

(3) If the matricial representation of $\circ$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\sigma(\varphi) = 0, 4$.

Remark 3.10. Let $\circ : H_1(W ; \mathbb{Z}_2) \times H_1(W ; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ be the intersection form. Then our Rokhlin form q has the following property. $q(u+v) = q(u) + q(v) + 2u\circ v$, where $2 : \mathbb{Z}_2 \to \mathbb{Z}_4$, $2(1)=2$.

After the arguments in the following subsections 3°—8°, we shall obtain the following lemma.

Lemma 3.11. There exist elements u_i, v_i (i=1,..,dimL) such that $\{u_i, v_i \mid i=1,..,\dim L\}$ is a basis of L, the matricial representation of $\circ$ with respect to $\{u_i, v_i\}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ ($\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), $\tilde{L} = \bigoplus_i \langle u_i, v_i \rangle_{\mathbb{Z}_2}$ is a orthogonal decomposition, and $\tilde{L}$ is a orthogonal summand.

By Lemma 3.8, Remark 3.10, and the definition of the Brown invariant ([1]), we have $\sigma(q|_{\langle u_i, v_i \rangle}) = 0$. Hence from (3.9) and Lemma 3.11 we conclude

$$\alpha(P^1 \times P^1/\tau, W) = \sigma(q|_{\tilde{L}}) + \sigma(q|_{\tilde{L}^{\perp}}) = \sigma(q|_{\tilde{L}^{\perp}}). \tag{3.12}$$

3°. To investigate the aspect of W we decompose it into **some handles.** First we consider A/τ. We assume that $\mathbb{R}A$ is an M-l curve. From the double covering $A \to A/\tau$, which is branching along $\mathbb{R}A$, we get $\chi(A/\tau) = 2-l-\ell_A$. Hence by classical arguments we get a handlebody representation of the triad $(A/\tau; \phi, \mathbb{R}A)$ as follows.

$$h^0 \cup (h_1 \cup \dots \cup h_{g(A)+1}) \cup h_1^2, \tag{3.13}$$

where h^0, h_j, and h_1^2 mean a 0-handle D^2, a 1-handle $(D_j^1 \times D_j^1, \alpha_j)$, and a 2-handle D_1^2 respectively.

We say a 1-handle h_j is attached <u>orientably</u> (<u>non-orientably</u>) if $h^0 \cup h_j$ is orientable (non-orientable).

Then (1) if A/τ is orientable, our way of attaching the handles is as follows.

First we attach h_1 to ∂D^2 orientably, h_2 to the boundary component which contains $D_1^1 \times \{1\}$ orientably so

(3.14) $\left\{\begin{array}{l}\text{that } D_1^1\times\{1\} \text{ and } D_2^1\times\{-1\} \text{ will be in the same boundary} \\ \text{component, and } h_3,\ldots,h_{\frac{l}{2}} \text{ in the same way.}\end{array}\right.$

Next we attach $h_{\frac{l}{2}+j}$ $(j=1,\ldots,\frac{l}{2})$ to $D_j^1\times\{\pm 1\}$ orientably so that $\{\pm 1\}\times D^1_{\frac{l}{2}+j}$ will be in $D_j^1\times\{\pm 1\}$ respectively.

And we attach h_{l+1} to the boundary component which contains $D^1_{\frac{l}{2}}\times\{1\}$ orientably so that $D^1_{\frac{l}{2}}\times\{1\}$ and $D^1_{l+1}\times\{-1\}$ will be in the same boundary component, and h_{l+2}, $h_{l+3},\ldots,h_{g(A)+1}$ in the same way. (cf. Fig. 3.15)

Last we attach h_1^2 to the boundary component which contains $D^1_{\frac{l}{2}+1}\times\{-1\}$.

(2) If A/τ is non-orientable, our way of attaching the handles is as follows. First we attach $h_1,\ldots,h_l$ $(h_{l+1},\ldots$ $\ldots,h_{g(A)+1})$ non-orientably (orientably) to the same place as (3.14). (cf. Fig. 3.16) Last we attach h_1^2 to the boundary component which contais $D_1^1\times\{-1\}$.

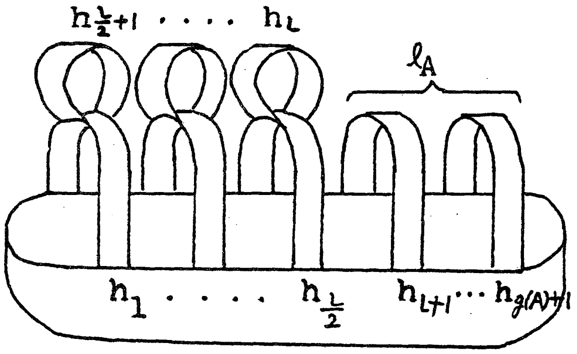

Fig. 3.15.

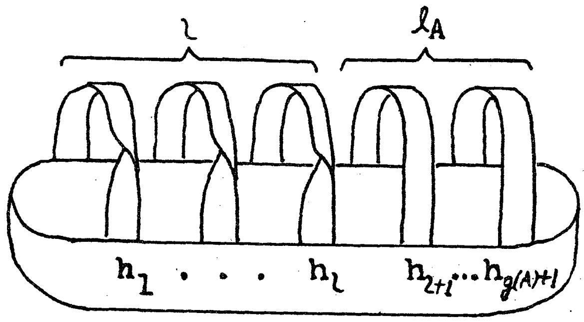

Fig. 3.16.

4°. Now we introduce <u>new notations for the ovals</u> $E_1,\ldots$ $\ldots,E_{\ell'}$. We assume that $\ell'' > 0$. (The case $\ell'' = 0$ can be regarded as a special case of that.) First let

$$(3.17)\qquad {}_jE^i_1,\dots,{}_jE^i_{s_{ij}}$$

denote the j-ovals in R_i, where $i - j \equiv 1 \pmod 2$. For such a oval, the closure of the intersection of the interior of it and the exteriors of all the ovals surrounded by it is a component of B^+. Next let

$$(3.18)\qquad {}_{jk}E^i_1,\dots,{}_{jk}E^i_{r_{ijk}}$$

denote the j+1-ovals surrounded by ${}_jE^i_k$. Last let

$$(3.19)\qquad E^{2j}_1,\dots,E^{2j}_{{}_0p^{2j}}$$

denote the 0-ovals in R_{2j}.

Now we order all the components of $\mathbb{R}A$ in the following order. (cf. Fig. 3.21)

$$(3.20)\begin{cases} {}_{jk}E^i_1,\dots,{}_{jk}E^i_{r_{ijk}} \ ;\ {}_jE^i_k \quad (i - j \equiv 1 \pmod 2),\ k = 1,\dots,s_{ij}) \\ \Big|\ E_{\ell'+2j-1} \ ;\ E^{2j}_1,\dots,E^{2j}_{{}_0p^{2j}} \ ;\ E_{\ell'+2j} \quad (j = 1,\dots,\frac{\ell''}{2}) \end{cases}$$

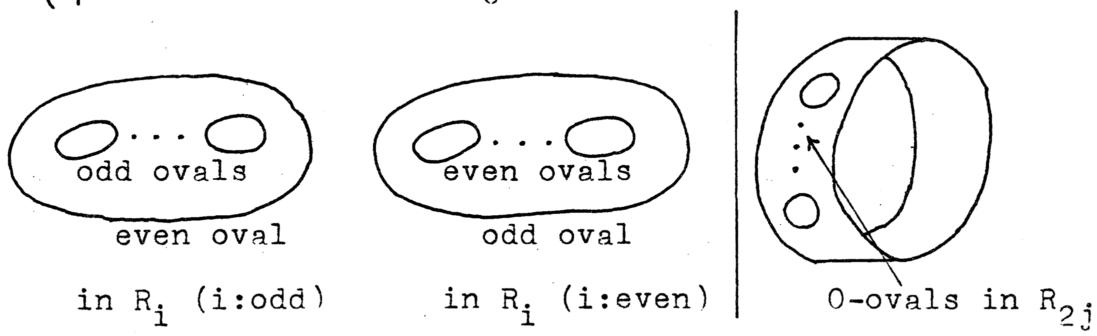

Fig. 3.21. The components of B^+.

In the case where A/τ is orientable (i.e. $\mathbb{R}A$ is dividing) we make the following convention for the order (3.20).

Convention 3.22. We fix an orientation of B^+ and a complex orientation of $\mathbb{R}A$, and divide the components of $\mathbb{R}A$ into two classes: (1) the components on which the orientation of B^+ determines the complex orientation (2) the otherwise components. Let C_{ijk} (C_j) denote the class which contains ${}_jE^i_k$ ($E_{\ell'+2j}$). We gather up the ${}_{jk}E^i_a$'s which belong to C_{ijk}, and push them backward. We write ${}_{jk}E^i_1, \dots, {}_{jk}E^i_{a_{ijk}}$ for the forward components, where $0 \leqq a_{ijk} \leqq r_{ijk}$. Similarly we gather up the E^{2j}_b's which belong to C_j, and push them backward. We write $E^{2j}_1, \dots, E^{2j}_{b_{2j}}$ for the forward components, where $0 \leqq b_{2j} \leqq {}_0P^{2j}$.

We may accept the following assumption.

(3.23) For the handlebody (3.13); the boundary component which contains $D^1_{l+1} \times \{1\}$, that which contains $D^1_{l+2} \times \{1\}, \dots$, and that which contains $D^1_{g(A)+1} \times \{1\}$ correspond to the components of $\mathbb{R}A$ precisely in the order (3.20).

5°. By attaching some handles to the handlebody (3.13) in the following way, we shall obtain a handlebody decomposition of W. The 1-handles attached anew are denoted by $\tilde{h}_t = (\tilde{D}^1_t \times \tilde{D}^1_t, \tilde{\alpha}_t)$, where the t's satisfy the following condition.

(3.24) $l+2 \leqq t \leqq g(A)+1$ and $D^1_t \times \{-1\}$ is in neither ${}_jE^i_k$ nor $E_{\ell'+2j}$.

We attach $\tilde{h}_t$ to $D^1_t \times \{\pm 1\}$ so that $\{\pm 1\} \times \tilde{D}^1_t$ will be in $D^1_t \times \{\pm 1\}$ respectively. In the case where A/τ is orientable (i.e. $\mathbb{R}A$ is dividing) we attach the handles in the following way. If t satisfies the condition that

(3.25) $D^1_t \times \{-1\}$ is in ${}_{jk}E^i_{a_{ijk}}$, $E^{2j}_{b_{2j}}$, or $E_{\ell'+2j-1}$ (with the proviso that $E_{\ell'+2j-1}$ belongs to C_j and $b_{2j} \geqq 1$); (cf. 3.22) then we attach $\tilde{h}_t$ non-orientably. For an otherwise t, we attach $\tilde{h}_t$ orientably. (cf. Fig. 3.26)

Last we attach some 2-handles in the trivial way.

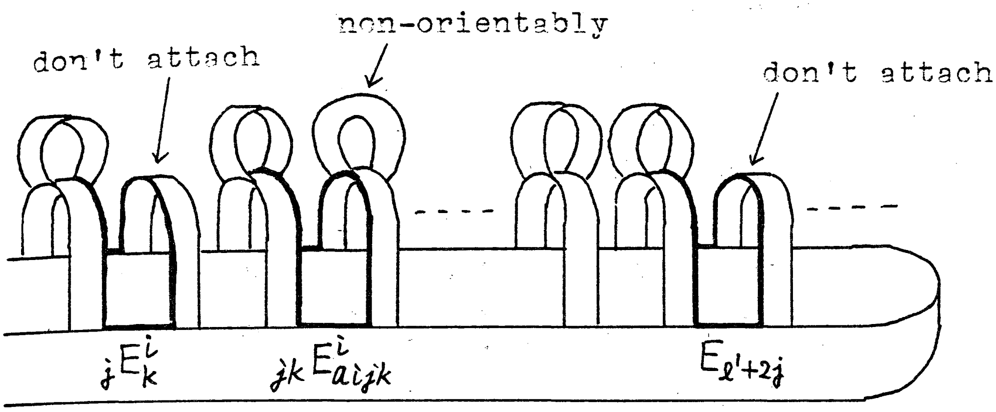

Fig. 3.26.

6^o. Next to study $H_1(W\ ;\mathbb{Z}_2)$, we consider some embedded circles in W. First in the case where A/τ is orientable, for each t ($1\leqq t\leqq \frac{l}{2}$, $l+1\leqq t\leqq g(A)+1$), we choose a embedding f_t : $I=[-1,1]\longrightarrow D^2$(the 0-handle) such that $f_t(\pm 1) = (\pm 1,0)$ (in $D^1_t\times D^1_t$) and $f_t(I)$ are mutually disjoint, and we set

(3.27) $$S_t = f_t(I)\cup D^1_t\times\{0\}.$$

For each t ($\frac{l}{2}+1\leqq t\leqq l$), we choose a embedding $f_t : I\longrightarrow D^1_{t-\frac{l}{2}}\times D^1_{t-\frac{l}{2}}$ such that $f_t(\pm 1) = (\pm 1,0)$ (in $D^1_t\times D^1_t$) and $f_t(I)$ intersects $S_{t-\frac{l}{2}}$ at only one point transversely, and we set

(3.28) $$S_t = f_t(I)\cup D^1_t\times\{0\}.$$

For each t which satisfies the condition (3.24), we choose a embedding $\tilde{f}_t : I\longrightarrow D^1_t\times D^1_t$ such that $\tilde{f}_t(\pm 1) = (\pm 1,0)$ (in $\tilde{D}^1_t\times\tilde{D}^1_t$) and $\tilde{f}_t(I)$ intersects S_t at only one point transversely, and we set

(3.29) $$\tilde{S}_t = \tilde{f}_t(I)\cup\tilde{D}^1_t\times\{0\}.$$

Next in the case where A/τ is non-orientable, for each t ($1\leqq t\leqq g(A)+1$), we choose a embedding f_t in the same way as (3.27), and we set

(3.30) $\quad S_t = f_t(I) \quad D_t^1 \quad 0.$

For each t which satiafies the condition (3.24), we choose a embedding $\tilde{f}_t$ in the same way as (3.29), and we set

(3.31) $\quad \tilde{S}_t = \tilde{f}_t(I) \cup \tilde{D}_t^1 \quad 0.$

Then we have the following lemmas.

Lemma 3.32. We can adopt the following elements as a basis of $H_1(W ; \mathbb{Z}_2)$. $[S_t]$ $(1 \leqq t \leqq l)$; $[S_t]$, $[\tilde{S}_t]$ (for all t which satisfy the condition (3.24))

Corollary 3.33. We have

$$\dim H_1(W ; \mathbb{Z}_2) = \begin{cases} l + 2N & \text{if } \ell'' = 0 \\ l + 2\left(\sum_{i:\text{even}} P^i + \sum_{i:\text{odd}} N^i + \frac{\ell''}{2}\right) & \text{if } \ell'' > 0 \end{cases}$$

Lemma 3.34. (1) If A/τ is orientable, then

$$H_1(W ; \mathbb{Z}_2) = \bigoplus_{1 \leqq t \leqq \frac{l}{2}} \left\langle [S_t],\ [S_{t+\frac{l}{2}}] \right\rangle_{\mathbb{Z}_2} \quad \bigoplus_{t_{*)}} \left\langle [S_t],\ [\tilde{S}_t] \right\rangle_{\mathbb{Z}_2}$$

is a orthogonal decomposition, and we have

*): "which satisfy (3.24)"

$[S_t] \circ [S_t] = 0 \qquad (1 \leqq t \leqq l),$

$[S_t] \circ [S_{t+\frac{l}{2}}] = 1 \qquad (1 \leqq t \leqq \frac{l}{2}),$

$[S_t] \circ [S_t] = 0 \quad [S_t] \circ [\tilde{S}_t] = 1$ (t which satisfy (3.24)),

$$[\tilde{S}_t] \circ [\tilde{S}_t] = \begin{cases} 0 & (t \text{ which satisfy (3.24) and do not satisfy (3.25)}) \\ 1 & (t \text{ which satisfy (3.24) and (3.25)}). \end{cases}$$

(2) If A/τ is non-orientable, then

$$H_1(W ; \mathbb{Z}_2) = \bigoplus_{1 \leqq t \leqq l} \left\langle [S_t] \right\rangle_{\mathbb{Z}_2} \quad \bigoplus_{t_{*)}} \left\langle [S_t],\ [\tilde{S}_t] \right\rangle_{\mathbb{Z}_2}$$

is a orthogonal decomposition, and we have

$[S_t]\circ[S_t] = 1\ (1 \leqq t \leqq l)$,

$[S_t]\circ[S_t] = 0$, $[S_t]\circ[\tilde{S}_t] = 1$, and $[\tilde{S}_t]\circ[\tilde{S}_t]$ cannot be determined (t which satisfy (3.24)).

7°. In this subsection we consider the subspace L. We see

(3.35) $L = \left\langle [{}_{jk}E^i_a] \right\rangle_{\mathbb{Z}_2} \oplus \left(\left\langle [E^{2j}_b] \right\rangle_{\mathbb{Z}_2} + \left\langle [\partial R_i] \,\middle|\, i : \text{odd} \right\rangle_{\mathbb{Z}_2} \right)$, and

(3.36)
$$\begin{cases} [{}_{01}E^1_1] = [S_{l+2}] \\ [{}_{01}E^1_2] = [S_{l+2}] + [S_{l+3}] \\ [{}_{01}E^1_3] = [S_{l+3}] + [S_{l+4}] \\ \cdots\cdots\cdots\cdots \\ [{}_{01}E^1_{r_{101}}] = [S_{l+r_{101}}] + [S_{l+r_{101}+1}] \\ \hline [{}_{02}E^1_1] = [S_{l+r_{101}+3}] \\ [{}_{02}E^1_2] = [S_{l+r_{101}+3}] + [S_{l+r_{101}+4}] \\ [{}_{02}E^1_3] = [S_{l+r_{101}+4}] + [S_{l+r_{101}+5}] \\ \cdots\cdots\cdots\cdots \\ [{}_{02}E^1_{r_{102}}] = [S_{l+r_{101}+r_{102}+1}] + [S_{l+r_{101}+r_{102}+2}] \\ \hline \cdots\cdots\cdots\cdots . \end{cases}$$

Hence $\dim_{\mathbb{Z}_2} \left\langle [{}_{jk}E^i_1], \ldots, [{}_{jk}E^i_{r_{ijk}}] \right\rangle_{\mathbb{Z}_2} = r_{ijk}$, therefore,

$$\dim_{\mathbb{Z}_2} \left\langle [{}_{jk}E^i_a] \right\rangle_{\mathbb{Z}_2} = \sum_{ijk} r_{ijk} = \sum_{i;\,\text{even}} (P^i - {}_0P^i) + \sum_{i:\text{odd}} N^i .$$

In fact, the following is a basis of $\left\langle [{}_{jk}E^i_a] \right\rangle_{\mathbb{Z}_2}$.

(3.37) $\left\{ [S_t] \,\middle|\, l+2 \leqq t \leqq l + l' - \sum_{i:\text{even}} {}_0P^i + 1, \text{ and satisfies (3.24).} \right\}$

And we see

$$
(3.38)\left\{
\begin{array}{l}
[E^2_1] = \left[S_{l+\ell'-\sum_{i:\text{even}} {}_0p^i+2}\right] + \left[S_{l+\ell'-\sum_{i:\text{even}} {}_0p^i+3}\right] \\
[E^2_2] = \left[S_{l+\ell'-\sum_{i:\text{even}} {}_0p^i+3}\right] + \left[S_{l+\ell'-\sum_{i:\text{even}} {}_0p^i+4}\right] \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
[E^2_{{}_0p^2}] = \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 4}} {}_0p^i+1}\right] + \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 4}} {}_0p^i+2}\right] \\
\hline
[E_{\ell'+2}] + [E_{\ell'+3}] = \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 4}} {}_0p^i+2}\right] \\
\qquad\qquad + \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 4}} {}_0p^i+4}\right] \\
\hline
[E^4_1] = \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 4}} {}_0p^i+4}\right] + \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 4}} {}_0p^i+5}\right] \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
[E^4_{{}_0p^4}] = \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 6}} {}_0p^i+3}\right] + \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 6}} {}_0p^i+4}\right] \\
\hline
[E_{\ell'+4}] + [E_{\ell'+5}] = \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 6}} {}_0p^i+4}\right] \\
\qquad\qquad + \left[S_{l+\ell'-\sum_{\substack{i:\text{even}\\ i\geq 6}} {}_0p^i+6}\right] \\
\hline
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
[E^{\ell''}_{{}_0p^{\ell''}}] = \left[S_{l+\ell_A-1}\right] + \left[S_{l+\ell_A}\right] \\
\hline
[E_{\ell'+\ell''}] + [E_{\ell'+1}] = \left[S_{l+\ell_A}\right] + \left[S_{l+\ell'-\sum_{i:\text{even}} {}_0p^i+2}\right].
\end{array}
\right.
$$

Hence if we remove $[E_{\ell'+\ell''}] + [E_{\ell'+1}]$ from (3.38), then the remainder is a basis of $\left\langle [E_b^{2j}] \right\rangle_{\mathbb{Z}_2} + \left\langle [\partial R_i] \,\middle|\, i : \text{odd} \right\rangle_{\mathbb{Z}_2}$. Hence,

$$\dim_{\mathbb{Z}_2}\left(\left\langle [E_b^{2j}] \right\rangle_{\mathbb{Z}_2} + \left\langle [\partial R_i] \,\middle|\, i:\text{odd} \right\rangle_{\mathbb{Z}_2} \right) = \sum_{i:\text{even}} 0^{P^i} + \frac{\ell''}{2} - 1.$$

Thus we have

$$(3.39)\quad \dim L = \begin{cases} N & \text{if } \ell'' = 0 \\ \displaystyle\sum_{i:\text{even}} P^i + \sum_{i:\text{odd}} N^i + \frac{\ell''}{2} - 1 & \text{if } \ell'' > 0. \end{cases}$$

8°. Now we prove Lemma 3.11. To do this, we let

$$[\mathcal{S}_1] + [\mathcal{S}_2],\ [\mathcal{S}_2] + [\mathcal{S}_3], \ldots\ldots, [\mathcal{S}_{m-1}] + [\mathcal{S}_m]$$

denote the right-hand sides of (3.38) in order respectively, where we set $m = \sum_{i:\text{even}} 0^{P^i} + \frac{\ell''}{2}$. And if $[\mathcal{S}_i]$ denotes $[S_t]$, let $[\widetilde{\mathcal{S}}_i]$ denote $[\widetilde{S}_t]$.

We can adopt (3.37) and (3.38) from which we remove $[E_{\ell'+\ell''}] + [E_{\ell'+1}]$ as $\{u_i\}$; and adopt

$$(3.40)\quad \left\{ [\widetilde{S}_t] \,\middle|\, \iota+2 \leq t \leq \iota+\ell' - \sum_{i:\text{even}} 0^{P^i} + 1, \text{ and satisfies (3.24).} \right\}$$

and

$$(3.41)\quad \left\{ \left(\sum_{\substack{j=1,\ldots,p-1 \\ [\widetilde{\mathcal{S}}_j]\circ[\widetilde{\mathcal{S}}_j]=1}} [\mathcal{S}_j] \right) + \left(\sum_{j=1,\ldots,p} [\widetilde{\mathcal{S}}_j] \right) \,\middle|\, 1 \leq p \leq m-1 \right\}$$

as $\{v_i\}$.

From Lemma 3.34, we get the following.

$$[\mathcal{S}_i]\circ[\mathcal{S}_i] = 0,\ [\mathcal{S}_i]\circ[\widetilde{\mathcal{S}}_i] = 1 \quad \text{for all } i \ (1 \leq i \leq m); \text{ and}$$

$$[\mathcal{S}_i]\circ[\mathcal{S}_k] = [\mathcal{S}_i]\circ[\widetilde{\mathcal{S}}_k] = [\widetilde{\mathcal{S}}_i]\circ[\widetilde{\mathcal{S}}_k] = 0 \quad \text{for all } i \neq k \ (1 \leq i, k \leq m).$$

Hence,

$$(3.42)\quad \left([\mathcal{S}_{i-1}] + [\mathcal{S}_i] \right)\circ\left(\sum_{j=1,\ldots,p} [\widetilde{\mathcal{S}}_j] \right) = [\mathcal{S}_{i-1}]\circ[\widetilde{\mathcal{S}}_{i-1}] + [\mathcal{S}_i]\circ[\widetilde{\mathcal{S}}_i]$$

$$= 1 + 1 = 0 \quad \text{if} \quad i \leqq p,$$

$$(3.43)\quad ([\delta_p] + [\delta_{p+1}]) \circ \Big(\sum_{j=1,\dots,p} [\tilde{\delta}_j]\Big) = [\delta_p] \circ [\tilde{\delta}_p] = 1,$$

$$(3.44)\quad ([\delta_{i-1}] + [\delta_i]) \circ \Big(\sum_{j=1,\dots,p} [\tilde{\delta}_j]\Big) = 0 \qquad \text{if} \quad i \geqq p+2,$$

$$(3.45)\quad \Big(\sum_{j=1,\dots,q} [\tilde{\delta}_j]\Big) \circ \Big(\sum_{j=1,\dots,p} [\tilde{\delta}_j]\Big) = \sum_{j=1,\dots,q} [\tilde{\delta}_j] \circ [\tilde{\delta}_j] \qquad \text{if} \quad q \leqq p;$$

where if we substitute $\left(\sum_{\substack{j=1,\dots,p-1 \\ [\tilde{\delta}_j]\circ[\tilde{\delta}_j]=1}} [\delta_j]\right) + \left(\sum_{j=1,\dots,p} [\tilde{\delta}_j]\right)$ for $\left(\sum_{j=1,\dots,p} [\tilde{\delta}_j]\right)$, then (3.42)—(3.44) are not changed, but (3.45) is changed into "<u>= 0 (if $q \leqq p-1$)</u>".

Thus we obtain $\tilde{L}$ and

$$(3.46)\quad \tilde{L}^{\perp} = \langle [S_t] \mid 1 \leqq t \leqq \iota \rangle_{\mathbb{Z}_2} \oplus \left\langle [\delta_m],\ \left(\sum_{\substack{j=1,\dots,m-1 \\ [\tilde{\delta}_j]\circ[\tilde{\delta}_j]=1}} [\delta_j]\right) + \left(\sum_{j=1,\dots,m} [\tilde{\delta}_j]\right) \right\rangle_{\mathbb{Z}_2}.$$

Hence $H_1(W\,;\mathbb{Z}_2) = \tilde{L} \oplus \tilde{L}^{\perp}$, and

$$(3.47)\qquad \dim \tilde{L}^{\perp} = \begin{cases} \iota & (\text{if } \ell'' = 0) \\ \iota + 2 & (\text{if } \ell'' > 0) \end{cases}$$

This completes the proof of Lemma 3.11. (cf. Fig. 3.48)

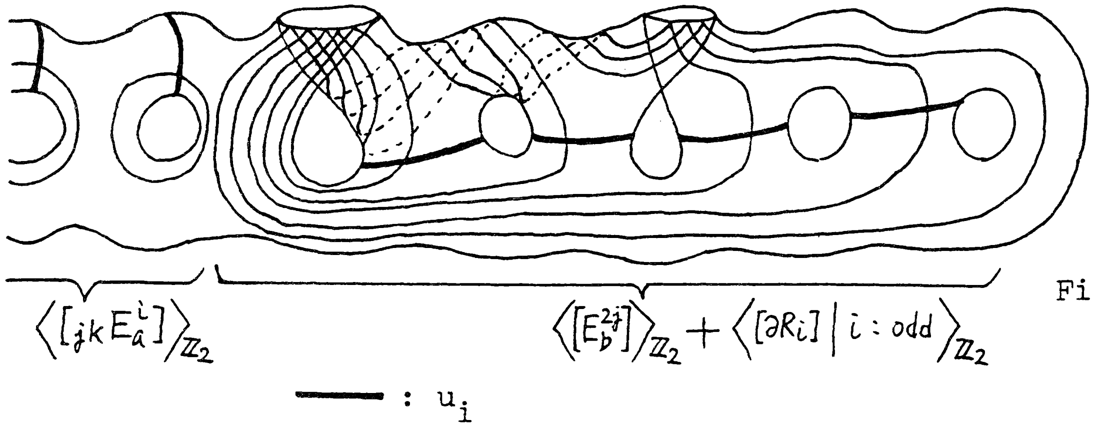

Fig. 3.48.

Moreover we have the following lemma.

Lemma 3.49. In the case where A/τ is orientable and $\ell'' > 0$, the matricial representation of $\circ$ with respect to $\left\{[\delta_m], \left(\sum_{\substack{j=1,\dots,m-1 \\ [\tilde{\delta}_j]\circ[\tilde{\delta}_j]=1}} [\delta_j]\right) + \left(\sum_{j=1,\dots,m} [\tilde{\delta}_j]\right)\right\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (if } \hat{\ell} \text{ is even)} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ (if } \hat{\ell} \text{ is odd)}$$

Proof. The self-intersection number of the second element is as follows.

$$\sum_{j=1,\dots,m} [\tilde{\delta}_j]\circ[\tilde{\delta}_j] = \left(\#\{j \mid 1 \leq j \leq m,\ [\tilde{\delta}_j]\circ[\tilde{\delta}_j] = 1\}\right)_{\mathrm{mod}\ 2}$$
$$= \left(\#\{t \mid \iota + \ell' - \sum_{i:\mathrm{even}} {}_0P^i + 2 \leq t \leq \iota + \ell_A, \text{ and satisfies (3.25).}\}\right)_{\mathrm{mod}\ 2}$$
$$= (\hat{\ell})_{\mathrm{mod}\ 2} \quad \text{(cf. (3.22))}$$
Q.E.D.

9°. Proof of Lemma 3.2. Recall (3.12). In the case where $\ell'' = 0$, since $\dim \tilde{L}^{\perp} = \iota$ by (3.47),

1) if $\mathbb{R}A$ is an M curve (i.e. $\iota = 0$), then $\alpha(P^1 \times P^1/\tau, W) = 0$.

2) if $\mathbb{R}A$ is an M-1 curve (i.e. $\iota = 1$), then by (3.9)-(2) $\alpha(P^1 \times P^1/\tau, W) = \pm 1$.

3) if $\mathbb{R}A$ is an M-2, not dividing curve (i.e. $\iota = 2$ and A/τ is non-orientable), then by (3.34) and (3.46) a matricial representation of $\circ$ on $\tilde{L}^{\perp}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence by (3.9)-(1), (2) $\alpha(P^1 \times P^1/\tau, W) = 0, \pm 2$.

4) if $\mathbb{R}A$ is a dividing curve (i.e. A/τ is orientable), then by (3.34) and (3.46) a matricial representation of $\circ$ on $\tilde{L}^{\perp}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence by (3.9)-(1), (3) $\alpha(P^1 \times P^1/\tau, W) = 0, 4$.

In the case where $\ell'' > 0$, since $\dim \tilde{L}^{\perp} = \iota + 2$ by (3.47),

1') if $\mathbb{R}A$ is an M curve (i.e. $\iota = 0$), then by (3.49) and (3.9) we have the required result.

2') if $\mathbb{R}A$ is a dividing curve (i.e. A/τ is orientable); then by (3.34), (3.46), (3.49), and (3.9) we have the required result.

This completes the proof of Lemma 3.2.

§4. Curves of degree (4,4)

In this section we try classifying curves of degree (4.4) by applying our results stated in §1.

Definition 4.1. We call a set of ovals of $\mathbb{R}A$ totally ordered by the relation of inclusion (Def. 1.8) a <u>nest</u>. A nest which contains m ovals is called a nest of <u>depth</u> m.

By considering the intersection form of $P^1 \times P^1$, we get the following lemma.

Lemma 4.2. **Let $\mathbb{R}A$ be a curve of degree (4,4).**

(i) If $\ell'' = 0$, then the depth of a nest of $\mathbb{R}A$ is at most 2, hence $N_- = N_0 = 0$.

(ii) If $\ell'' > 0$, then $\ell'' = 2$, or 4. And if $\ell'' = 2$, then $(s,t) = (\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 2,\pm 1)$, or $(\pm 1,\pm 2)$. If $\ell'' = 4$, then $(s,t) = (\pm 1,0)$, $(0,\pm 1)$, or $(\pm 1,\pm 1)$. If $\ell'' = 2$, then the depth of a nest of $\mathbb{R}A$ is at most 1, hence $N = P_- = P_0 = 0$. If $\ell'' = 4$ or $(\pm 2,\pm 1)$, $(\pm 1,\pm 2)$; then $\mathbb{R}A$ has no oval, i.e. $\ell' = 0$.

Now we write down the restrictions which are obtained anew from (1.2), (1.14), (1.15), (1.19), (1.20), (1.21), and (1.23).

(4.3) ("Harnack-Thom type" inequality) $\ell_A \leq 10$ (Hence an M curve has 10 components.)

(4.4) ("Rokhlin type" congruence) For an M curve with $\ell'' = 0$, we have $P - N \equiv 0 \pmod 8$.

(4.5) ("Kharlamov-Gudkov-Krakhnov type" congruence) For an M-1 curve with $\ell'' = 0$, we have $P - N \equiv \pm 1 \pmod 8$.

(4.6) ("Petrovskii type" inequality) If $\ell'' = 0$, then $P \leqq 9$.

(4.7) ("Arnol'd type" inequalities) (i) For a not dividing curve, we have $P_- + P_0 \leqq 1$ (if $\ell'' = 0$) and $\ell'' = 2$ (if $\ell'' > 0$). (ii) For a dividing curve with $\ell'' = 0$ and B^+ has a component whose Euler characteristic is not zero, we have $P_- + P_0 \leqq 1$. (iii) For a curve with $\ell'' = 0$, we have $P_- + P_0 \leqq 2$.

(4.8) ("Arnol'd type" congruences) (1) For a dividing curve with $\ell'' = 0$, we have $P - N \equiv 0 \pmod 4$. (2) For a dividing curve with $\ell'' = 2$ and $\hat{\ell} = 0$, we have $P^1 - P^2 \equiv 0 \pmod 4$. (3) For a dividing curve with $\ell'' = 2$ and $\hat{\ell} = 1$, we have $P^1 - P^2 \equiv 0 \pmod 2$.

(4.9) ("Marin-Kharlamov type" congruences) (1) For an M-2, not dividing curve with $\ell'' = 0$, we have $P - N \equiv 0, \pm 2 \pmod 8$. (2) For an M curve with $\ell'' = 2$ and $\hat{\ell} = 1$, we have $P^1 - P^2 \equiv 0, \pm 2$ (mod 8).

Remark 4.10. (1) If $\ell'' = 0$ and $P_- \geqq 1$, then $P_- = 1$ and $P_0 = 0$. (**Fig.** 4.11) (2) If $\ell'' = 0$ and $P_0 \geqq 1$, then $P_- = 0$. (3) If $\ell'' = 0$ and $P_0 = 2$, then $\mathbb{R}A$ is dividing and $P_+ = 0$. (Fig. 4.12)

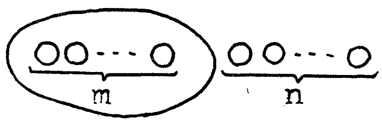

Fig. 4.11. $P_- = 1$, $P_0 = 0$

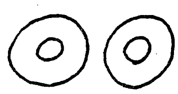

Fig. 4.12. $P_- = 0$, $P_0 = 2$

Owing to limited space we give the table of possible isotopy types of $\mathbb{R}A \hookrightarrow \mathbb{R}P^1 \times \mathbb{R}P^1$ only in the case $\ell'' = 0$. (Table 4.13) In the table, $\frac{m}{1}n$ and $\frac{1}{1}\frac{1}{1}$ mean the isotopy types of Fig. 4.11 and

M	$\frac{9}{I}$ (-8) d.	$\frac{5}{I}4$ (0) d.	$\frac{1}{I}8^*$ (8) d.					
M-1	$\frac{8}{I}$ (-7) n.d.	$\frac{5}{I}3$ (-1) n.d.	$\frac{4}{I}4$ (1) n.d.	$\frac{1}{I}7^*$ (7) n.d.	9 (9) n.d.			
M-2	$\frac{7}{I}$ (-6) n.d.	$\frac{6}{I}1$ (-4) d.	$\frac{5}{I}2$ (-2) n.d.	$\frac{4}{I}3$ (0)	$\frac{3}{I}4$ (2) n.d.	$\frac{2}{I}5$ (4) d.	$\frac{1}{I}6^*$ (6) n.d.	8^* (8)
M-3	$\frac{6}{I}$ (-5) n.d.	$\frac{5}{I}1$ (-3) n.d.	$\frac{4}{I}2$ (-1) n.d.	$\frac{3}{I}3$ (1) n.d.	$\frac{2}{I}4$ (3) n.d.	$\frac{1}{I}5$ (5) n.d.	7^* (7) n.d.	
M-4	$\frac{5}{I}$ (-4)	$\frac{4}{I}1$ (-2) n.d.	$\frac{3}{I}2$ (0)	$\frac{2}{I}3$ (2) n.d.	$\frac{1}{I}4^*$ (4)	6^* (6) n.d.		
M-5	$\frac{4}{I}\,^*$ (-3) n.d.	$\frac{3}{I}1^*$ (-1) n.d.	$\frac{2}{I}2^*$ (1) n.d.	$\frac{1}{I}3^*$ (3) n.d.	5^* (5) n.d.			
M-6	$\frac{3}{I}\,^*$ (-2) n.d.	$\frac{2}{I}1^*$ (0)	$\frac{1}{I}2^*$ (2) n.d.	4^* (4)	$\frac{1}{I}\frac{1}{I}\,^*$ (0) d.			
M-7	$\frac{2}{I}\,^*$ (-1) n.d.	$\frac{1}{I}1^*$ (1) n.d.	3^* (3) n.d					
M-8	$\frac{1}{I}\,^*$ (0)	2^* (2) n.d.						
M-9	1^* (1) n.d.							
M-10	ϕ^* n.d.							

Table 4.13.

$\ell'' = 0$

Fig. 4.12 respectively. The number in each parenthesis means the Euler characteristic of B^+. The notations d. and n.d. mean "dividing" and "not dividing" respectively. The asterisked isotopy types are realized by some curves of degree (4,4). For example the isotopy type $\frac{3}{1}$ is realized in the following way.

First we choose real numbers a_i (i=1,2,3,4), b_i (i=1,2,3,4), α_i (i=1,2,3,4), and β_i (i=1,2,3,4) such that $\alpha_1 < a_1 < a_2 < a_3 < \alpha_2 < a_4 < \alpha_3 < \alpha_4$ and $\beta_1 < b_1 < b_2 < b_3 < \beta_2 < b_4 < \beta_3 < \beta_4$. We set $f_1(X_0,X_1;Y_0,Y_1) = (X_1 - a_2X_0)(Y_1 - b_2Y_0)$, $f_2(X_0,X_1;Y_0;Y_1) = (X_1 - \alpha_1X_0)(Y_1 - \beta_1Y_0)$, and $f_{3,\varepsilon'} = f_1 + \varepsilon' f_2$. For a sufficiently small $\varepsilon' \neq 0$, $\{f_{3,\varepsilon'} = 0\}$ is a nonsingular curve of degree (1,1) with $\ell'' = 1$ and (s,t) = (1,1). Next we set $f(X_0,X_1;Y_0,Y_1) = f_{3,\varepsilon'}(X_0,X_1;Y_0,Y_1) \times \prod_{i=2}^{4}(X_1 - \alpha_i X_0) \times \prod_{i=2}^{4}(Y_1 - \beta_i Y_0)$. $\{f = 0\}$ is a singular curve of degree (4,4). (Fig. 4.14) Now we set $F(X_0,X_1;Y_0,Y_1) = \prod_{i=1}^{4}(X_1 - a_i X_0) \times \prod_{i=1}^{4}(Y_1 - b_i Y_0)$ and $F_\varepsilon = F + \varepsilon f$. Then for a sufficiently small $\varepsilon \neq 0$, $\{F = 0\}$ is a nonsingular curve of degree (4,4) whose isotopy type is $\frac{3}{1}$. (Fig. 4.15)

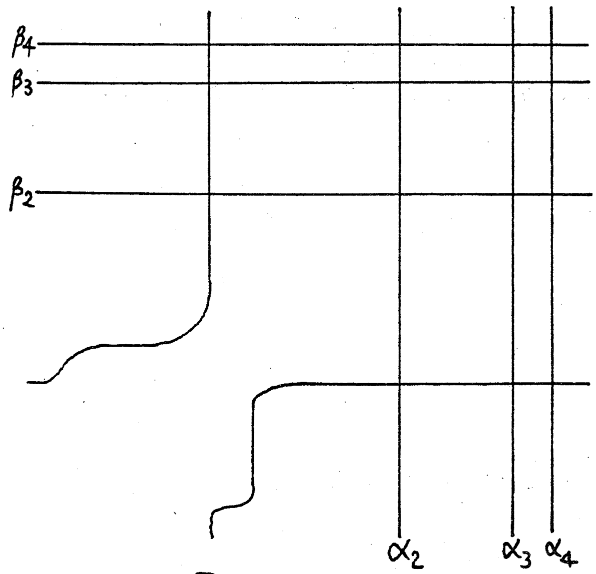

Fig. 4.14.

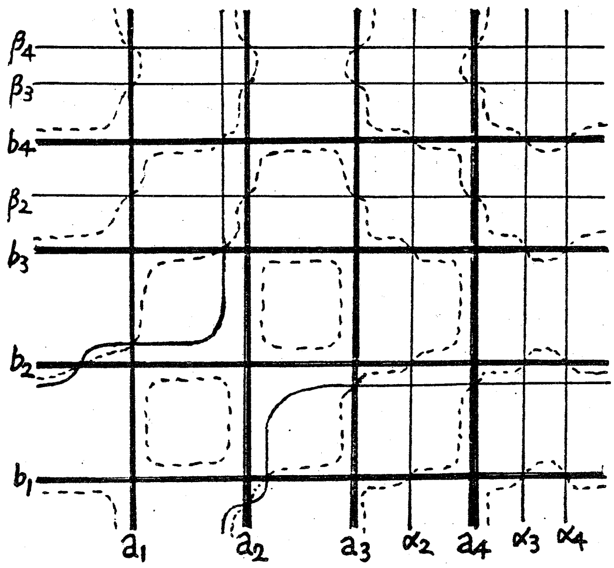

Fig. 4.15.

The author would like to thank G. Ishikawa and S. Izumiya for their great help and constant encouragement.

References

[1] L. Guillou et A. Marin, Une extention d'un théorème de Rohlin sur la signature, C. R. Acad. Sc. Paris, t. 285 (18 juillet 1977), 95-98; or in [5], 69-80.

[2] G. Ishikawa, Construction of extremal real correspondences of $\mathbb{P}^1$, preprint.

[3] M. Letizia, Quotients by complex conjugation of nonsingular quadrics and cubics in $\mathbb{P}^3_{\mathbb{C}}$ defined over $\mathbb{R}$, Pacific J. Math., 110 (1984), 307-314.

[4] A. Marin, Quelques remarques sur les courbes algébriques planes réelles, in [5], 51-68.

[5] J.J. Risler, Séminaire sur la géométrie algébrique réelle, Publications mathématiques de l'Université Paris VII (1980).

[6] G. Wilson, Hilbert's sixteenth problem, Topology, 17 (1978), 53-73.

Sachiko Matsuoka
Department of Mathematics
Nara Women's University
NARA 630, JAPAN

(February 8, 1987)