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An example of one-dimensional Cohen-Macaulay local rings that possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

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In our paper [GN] with K. Nishida we proved the following

Theorem 1. *Let* $P = k[[X_1, X_2, \dots, X_n]]$ *be a formal power series ring over an algebraically closed field* k *of* $\mathrm{ch}\, k \neq 2$. *Let* $R = P/I$, *where* I *is an ideal of* P *and suppose* $\dim R = d \geq 2$. *Then the following two conditions are equivalent.*

(1) R *is a regular local ring.*

(2) R *is a Cohen-Macaulay ring that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.*

When this is the case, the syzygy modules of the residue class field k *of* R *are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly* d *non-isomorphic indecomposable maximal Buchsbaum modules over* R.

Our contribution in the above theorem is the implication (2) => (1). The last assertion and the implication (1) => (2) are due to [G] (see also [EG, Theorem 3.2]), where some consequences of the result are discussed too.

I would like to note here that the assumption $\dim R \geq 2$ in Theorem 1 is not superfluous. When $\dim R = 1$, maximal Buchsbaum R-modules M are characterized by the condition that

$$\dim_R M = 1 \quad \text{and} \quad \underline{m}.H^0_{\underline{m}}(M) = (0)$$

(here $H^i_{\underline{m}}(.)$ denotes the i th local cohomology functor of R relative to the maximal ideal $\underline{m}$ of R). This condition (is of course not too much weak but) seems not quite strong. Nevertheless in some sense surprisingly, there exist such Cohen-Macaulay local rings R of $\dim R = 1$ that are non-regular but possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. In my lecture I will explore the typical example $R = k[[t^2, t^3]]$.

Now let k be a field and $S = k[[t]]$ a formal power series ring over k . We put $R = k[[t^2, t^3]]$. Then R and S are the only indecomposable maximal Cohen-Macaulay R-modules (cf. [H, Satz 1.6]) and the R-module S has a resolution of the following form

$$\cdots \to R^2 \xrightarrow{\begin{pmatrix} t^3 & t^4 \\ -t^2 & -t^3 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} t^3 & t^4 \\ -t^2 & -t^3 \end{pmatrix}} R^2 \xrightarrow{\varepsilon} S \to 0 ,$$

where $\varepsilon\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = a + bt$. Therefore we have an embedding $\sigma : S \to R^2$ which sends 1 (resp. t) to $\begin{pmatrix} t^3 \\ -t^2 \end{pmatrix}$ (resp. $\begin{pmatrix} t^4 \\ -t^3 \end{pmatrix}$) and makes the diagram

$$\begin{array}{ccccc} S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S \\ \downarrow t & & \downarrow \rho & & \downarrow -t \\ S & \xrightarrow[\sigma]{} & R^2 & \xrightarrow[\varepsilon]{} & S \end{array}$$

commutative, where $\rho = \begin{pmatrix} 0 & -t^2 \\ -1 & 0 \end{pmatrix}$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times m$ matrices with entries in k and let

$\phi : (R^2)^m \to (R^2)^m$ be the homomorphism defined by

$$\phi((x_i)) = (\sum_{j=1}^{m} a_{ij}x_j + \sum_{j=1}^{m} b_{ij}\rho(x_j)) .$$

Then we clearly have

Lemma 2. *The diagram*

$$\begin{array}{ccccc} S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\varepsilon^m} & S^m \\ \downarrow A+tB & & \downarrow \phi & & \downarrow A-tB \\ S^m & \xrightarrow[\sigma^m]{} & (R^2)^m & \xrightarrow[\varepsilon^m]{} & S^m \end{array}$$

is commutative (here σ^m *and* ε^m *respectively denote the direct sum of* m *copies of* σ *and* ε *).*

Let $\underline{m}$ ($= t^2S$) denote the maximal ideal of R and let N be an R-submodule of S such that $\underline{m}$ is contained in N . We put $M = R^2/\sigma(N)$. Then

Proposition 3. *M is an indecomposable maximal Buchsbaum R-module with* $H^0_{\underline{m}}(M) = S/N$.

Proof. Considering the exact sequence

$$0 \to S/N \to M \to S \to 0 ,$$

we get $H^0_{\underline{m}}(M) = S/N$ as $\underline{m}.(S/N) = (0)$ and as S is Cohen-Macaulay ; so M is a maximal Buchsbaum R-module. Assume that $M = M_1 \oplus M_2$ for some non-zero submodules M_1 and M_2 . Then M_i's are cyclic, since M is generated by two elements. If $\dim_R M_i = 1$ for $i = 1, 2$, the isomorphisms $S \cong M/H^0_{\underline{m}}(M) \cong M_1/H^0_{\underline{m}}(M_1) \oplus M_2/H^0_{\underline{m}}(M_2)$ claim that S is decomposable. Hence

$\dim_R M_i = 0$ for some i, say $i = 2$. Then M_2' is contained in $H^0_{\underline{m}}(M)$ and so S is a homomorphic image of M_1 — this is impossible, because M_1 is cyclic while S is not. Thus we see M is indecomposable.

We define

$$M_1 = R^2/\sigma(\underline{m}) \ , \quad M_2 = R^2/\sigma(R) \ , \quad \text{and} \quad M_3 = R^2/\sigma(tS) \ .$$

By Proposition 3 M_i's are indecomposable maximal Buchsbaum R-modules and $M_1 \not\cong M_i$ ($i = 2, 3$), since

$$\dim_k H^0_{\underline{m}}(M_i) = 2 \quad (i = 1),$$

$$= 1 \quad (i = 2, 3).$$

M_2 is of homological dimension 1 but M_3 is not ; so $M_2 \not\cong M_3$. The goal of my lecture is the following

Theorem 4. *M_1, M_2, M_3, S and R are the indecomposable maximal Buchsbaum R-modules.*

To prove this theorem we need one more lemma 5, the proof of which is routine (use the induction on the size of matrices C) and shall be omitted.

Lemma 5. *Let C be an $m \times n$ matrix with entries in S/t^2S. Then there exist an invertible $m \times m$ matrix P with entries in S/t^2S and an invertible $n \times n$ matrix Q with entries in k such that PCQ has the following form*

$$\left(\begin{array}{c|c|c|c}
\begin{matrix}1 & & \\ & \cdot & \\ & & \cdot\end{matrix} & 0 & \begin{matrix}t & & \\ & \ddots & \\ & & t\end{matrix} & 0 \\ \hline
\begin{matrix}\cdot & & \\ & \cdot & \\ & & 1\end{matrix} & 0 & 0 & 0 \\ \hline
0 & \begin{matrix}t & & \\ & \ddots & \\ & & t\end{matrix} & 0 & 0 \\ \hline
0 & 0 & 0 & 0
\end{array}\right) \quad mod\ t^2S\ .$$

Proof of Theorem 4.

Let M be an indecomposable maximal Buchsbaum R-module such that $M \ncong R$. Let $V = H^0_{\underline{m}}(M)$. Then $\underline{m}V = (0)$.

Claim. V is contained in $\underline{m}M$ *and* $M/V \cong S^m$ *for some* $m \geq 1$.

For let W be the intersection of V and $\underline{m}M$ and write $V = W \oplus W'$. Then $W' \cap \underline{m}M = (0)$ and we have an embedding $W' \to M \to M/\underline{m}M$, which naturally splits. Hence $W' = (0)$ as M is indecomposable and thus V is contained in $\underline{m}M$. Since M/V is Cohen-Macaulay, the second assertion is clear.

By the above claim we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & V & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & (R^2)^m & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \| & & \downarrow & & \\
0 & \longrightarrow & S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\varepsilon^m} & S^m & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. Here we consider N to be an R-submodule of S^m and the homomorphism $i : N \to S^m$ to be the inclusion map. Hence $\underline{m}.S^m$ is contained in N, as $V \cong S^m/N$. Let $\tau : S^m \to S^m/\underline{m}S^m = (S/t^2S)^m$ denote the canonical epimorphism. We put $U = \tau(N)$ and $n = \dim_k U$. If $n = 0$, then $N = \underline{m}S^m$ and so $M = (R^2/\sigma(\underline{m}))^m$. Consequently, we get $m = 1$ and $M = M_1$.

Now suppose that $n \geq 1$ and let $v_1, v_2, \ldots, v_n$ be a k-basis of U. Let us apply Lemma 5 to the $m \times n$ matrix $C = (v_1, v_2, \ldots, v_n)$. Then Lemma 5 asserts that by some automorphism P of $(S/t^2S)^m$, U is mapped onto the k-subspace U' which is spanned by the columns of an $m \times n$ matrix of the following form:

$$(\#) \qquad \left(\begin{array}{c|c|c}
 & 0 & \begin{matrix} t & & \\ & \ddots & \\ & & t\end{matrix} \\ \cline{2-3}
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 & 0 \\ \hline
0 & \begin{matrix} t & & \\ & \ddots & \\ & & t\end{matrix} & 0 \\ \hline
0 & 0 & 0
\end{array}\right) \mod t^2S .$$

Let L be the R-submodule of S^m generated by the columns of the above matrix (#) and put $N' = \underline{m}S^m + L$. Then clearly $U' = \tau(N')$.

We write $P = A + tB \mod t^2S$ with $m \times m$ matrices A

and B with entries in k . Then since the following diagram

$$\begin{array}{ccc} S^m & \xrightarrow{\tau} & (S/t^2S)^m \\ \downarrow A+tB & & \downarrow P \\ S^m & \xrightarrow[\tau]{} & (S/t^2S)^m \end{array}$$

is commutative and since $U' = \tau(N')$, we get that $N' = (A+tB)N$.

Let us now recall the diagram in Lemma 2:

$$\begin{array}{ccccccccc} 0 & \to & S^m & \xrightarrow{\sigma^m} & (R^2)^m & \xrightarrow{\varepsilon^m} & S^m & \to & 0 \\ & & \downarrow A+tB & & \downarrow \phi & & \downarrow A-tB & & \\ 0 & \to & S^m & \xrightarrow[\sigma^m]{} & (R^2)^m & \xrightarrow[\varepsilon^m]{} & S^m & \to & 0 \end{array} .$$

Then as the rows of this diagram are exact and as both the matrices $A + tB$ and $A - tB$ are invertible, the middle ϕ has to be an isomorphism whence, via ϕ , we find that

$$M = (R^2)^m/\sigma^m(N)$$
$$\cong (R^2)^m/\sigma^m(N') .$$

Consequently we may assume that $N = N'$. The condition that M is indecomposable now causes a very tight restriction on the form of the matrix (#) above. We readily see that $m = 1$ and the matrix (#) must be one of

$$(1\ t),\ (1) \text{ and } (t) .$$

Thus $M = R^2/\sigma(S)$ ($= S$) , $M = R^2/\sigma(R)$ ($= M_2$) , or $M = R^2/\sigma(tS)$ ($= M_3$) as claimed. This completes the proof of Theorem 4.

References

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