

On Primitive Ideal Spaces of C^* -Algebras over
Certain Locally Compact Groupoids

by

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Introduction. Let Γ be a locally compact Hausdorff second countable groupoid with a left Haar system $(\nu^x)_{x \in X}$ in the sense of [9] ($X =$ the unit space of Γ). By analogy with Fell's algebraic bundles over groups, we define the notion of C^* -algebras over Γ and, given a C^* -algebra A over Γ , we can form a C^* -algebra $C^*(\Gamma, A)$ as the completion of the cross sectional algebra of A . In this note, under some stringent assumptions on Γ , we present a concrete realization of the primitive ideal space of $C^*(\Gamma, A)$. This is a C^* -version of [12]. For the proofs, we refer to [13].

§1. Assumptions.

We impose a series of conditions on Γ . First we assume that

(A1) Γ is minimal, i.e., for all $x \in X$, $[x] = \{y \in X; \text{there exists } \gamma \in \Gamma \text{ such that } s(\gamma) = x, t(\gamma) = y\}$ is dense in X .

(A2) Stabilizers of Γ are discrete and uniformly abelian, i.e., there are a discrete abelian group G and a family of topological isomorphisms $(\iota_x : G \rightarrow \Gamma_x^x)_{x \in X}$ such that for $g \in G$, $X \ni x \rightarrow \iota_x(g) \in \Gamma$ is

continuous and $t_{t(\gamma)}(g)\gamma = \gamma t_{s(\gamma)}(g)$ holds for $\gamma \in \Gamma$.

(A3) Γ is amenable, i.e., for any quasi-invariant measure μ in X there is a net $\{f_i\}_{i \in I}$ of functions in $C_c(\Gamma)$ such that

(i) for all $x \in X$ and all $i \in I$, we have $\int \nu^x(d\gamma) |f_i(\gamma)|^2 = 1$

(ii) the function $\gamma \mapsto \int \nu^{s(\gamma)}(d\gamma') f_i(\gamma') \overline{f_i(\gamma\gamma')}$ converges to 1 as $i \rightarrow \infty$, relative to the weak* topology of $L^\infty(\Gamma, \mu \circ \nu)$.

Comment. The amenability in (A3) is due to [9]. (A2) means that Γ is a central extension of a principal groupoid by an abelian group. We write $t_{t(\gamma)}(g)\gamma$ and $\gamma t_{s(\gamma)}(g)$ as $g\gamma$ and γg in the following.

Definition 1. Let $A = \{A_\gamma\}_{\gamma \in \Gamma}$ be a separable continuous field of Banach spaces over Γ . A is called a C^* -algebra over Γ if the following structures are specified:

(i) For $(\gamma_1, \gamma_2) \in \Gamma$, a multiplication $m: A_{\gamma_1} \times A_{\gamma_2} \ni (a_1, a_2) \mapsto a_1 a_2 \in A_{\gamma_1 \gamma_2}$ is defined. It is associative and satisfies $\|a_1 a_2\| \leq \|a_1\| \|a_2\|$.

(ii) For $\gamma \in \Gamma$, a conjugate linear map $*$: $A_\gamma \rightarrow A_{\gamma^{-1}}$ is defined and satisfies $(a^*)^* = a$, $\|a^*\| = \|a\|$, and $(a_1 a_2)^* = a_2^* a_1^*$.

(iii) If φ_1, φ_2 are continuous sections of A , then

$$\Gamma^{(2)} \ni (\gamma_1, \gamma_2) \mapsto \varphi_1(\gamma_1) \varphi_2(\gamma_2) \in A_{\gamma_1 \gamma_2}$$

is a continuous section of $m^*(A)$, where $m^*(A)$ is the pull back of A under the map $m: \Gamma^{(2)} \rightarrow \Gamma$.

(iv) If φ is a continuous section of A , $\varphi^*(\gamma) = \varphi(\gamma^{-1})^*$ is also a continuous section of A .

Let $C_c(\Gamma, A)$ be the set of continuous sections of A with compact supports. We make $C_c(\Gamma, A)$ into a $*$ -algebra in the following way:

$$(\varphi_1 \varphi_2)(\gamma) = \int \nu^{t(\gamma)}(d\gamma') \varphi_1(\gamma') \varphi_2(\gamma'^{-1}\gamma)$$

$$\varphi^*(\gamma) = \varphi(\gamma^{-1})^*$$

$C_c(\Gamma, A)$ is completed to the C^* -algebra $C^*(\Gamma, A)$ according to the maximal continuous C^* -norm (the topology of $C_c(\Gamma, A)$ is that of uniform convergence on compact subsets.). Analogously as in [3], we define the multiplier bundle $\{M(A_\gamma)\}_{\gamma \in \Gamma}$. We assume

(A4) for all $\gamma \in \Gamma$, $U(A_\gamma) = \{u \in M(A_\gamma); b^*b = 1_{s(\gamma)}, bb^* = 1_{t(\gamma)}\}$ is not empty and there is a Borel section of $\{U(A_\gamma)\}_{\gamma \in \Gamma}$ in the sense that we can find a section $u(\gamma) \in U(A_\gamma)$ such that for any continuous section a of A , $\Gamma \ni \gamma \longmapsto u(\gamma)^*a(\gamma) \in A_{s(\gamma)}$ is a Borel section.

Set $\Gamma(X) = \bigcup_{x \in X} \Gamma_x^X$. $\Gamma(X)$ is a closed subgroupoid of Γ and the restriction of A to $\Gamma(X)$ is a C^* -algebra over $\Gamma(X)$, so we can form the cross sectional C^* -algebra as in the above definition, which is denoted by $C^*(\Gamma(X), A)$. Similarly, from the restriction of A to Γ_x^X , we have a C^* -algebra $C^*(\Gamma_x^X, A)$. Let $G^*(x) = \text{Prim } C^*(\Gamma_x^X, A)$ and set $G^* = \bigcup_{x \in X} G^*(x)$ (disjoint union).

Lemma 1 (cf. [4]).

$$\text{Prim } C^*(\Gamma(X), A) = G^* .$$

The topology of G^* is defined by this identification. Let $u \in U(A_\gamma)$ and define an isomorphism from $C^*(\Gamma_x^X, A)$ onto $C^*(\Gamma_y^Y, A)$ ($x = s(\gamma)$, $y = t(\gamma)$) by $C_c(\Gamma_x^X, A) \ni \varphi \longmapsto u\varphi \in C_c(\Gamma_y^Y, A)$ where $(u\varphi)(\gamma') = u\varphi(\gamma'^{-1}\gamma)\gamma u^*$.

Now we define an equivalence relation \simeq in G^* by $\omega_1 \simeq \omega_2$ \iff there exist $\gamma \in \Gamma$ and $u \in U(A_\gamma)$ such that $\omega_1 \in G^*(s(\gamma))$, $\omega_2 \in G^*(t(\gamma))$, and $\omega_2 = u\omega_1$. Here we assume that

(A5) \simeq is an open equivalence relation in G^* .

Comment. This is automatically satisfied if Γ (and A) is arising from a group action.

Definition 2 (cf. [1]). We define an equivalence relation \sim in G^* by

$$\omega_1 \sim \omega_2 \iff \overline{[\omega_1]} = \overline{[\omega_2]} .$$

Here $[\omega]$ denotes the equivalence class of \simeq containing ω and $\overline{[\omega]}$ is the closure in G^* .

Given $\omega \in G^*$, we can construct the induced primitive ideal $\text{ind } \omega$ of $C^*(\Gamma, A)$ (see , for example , [2]). Since $\text{ind } \omega$ depends only on the equivalence class of \sim , we have obtained a map from G^*/\sim into $\text{Prim}(C^*(\Gamma, A))$, which is also denoted by ind .

Now the following is proved by the method of [5], [6], [10] (cf. [2], [4]).

Lemma 2.

$$\text{ind} : G^*/\sim \longrightarrow \text{Prim}(C^*(\Gamma, A))$$

is a homeomorphism.

Our next problem is the concrete description of G^*/\sim^* . For

that purpose, we need one more assumption. Through the isomorphism $\iota_x: G \rightarrow \Gamma_x^X$, $A|_{\Gamma_x^X}$ will be regarded as a C^* -algebraic bundle over G and then, by Theorem 9.1 in [3], we have an action of G on $\text{Prim}(A_x)$. Now suppose that

(A6) for each $x \in X$, all G -orbits in $\text{Prim}(A_x)$ is dense in $\text{Prim}(A_x)$.

For example, if A_x is simple, this condition is trivially satisfied.

§2. Topological decomposition of $C^*(\Gamma, A)$.

For $g \in G$, we denote by A^g the pull-back bundle of A under the map $X \ni x \mapsto \iota_x(g) \in \Gamma$. Let $M(A^g) = \{M(A_{\iota_x(g)})\}_{x \in X}$ be the multiplier bundle of A^g . A section φ of $M(A^g)$ is called strictly continuous if for each continuous section f of A^g , $X \ni x \mapsto \varphi(x)f(x) \in A_{\iota_x(g)}$ is a continuous section of A^g . Let C_g be the set of bounded strictly continuous sections, say φ , of $M(A^g)$ satisfying

$$\varphi(t(\gamma))a = a\varphi(s(\gamma)) \quad \text{for all } \gamma \in \Gamma \text{ and all } a \in A_\gamma.$$

Lemma 3.

- (i) $\dim C_g \leq 1$.
- (ii) If $C_g \neq 0$, we can find $\varphi \in C_g$ such that $\varphi(x)$ is a unitary element in $M(A_{\iota_x(g)})$ for all $x \in X$.

Set $S = \{g \in G; C_g \neq 0\}$.

Lemma 4. S is a subgroup of G .

Set $\Omega = \{\omega: S \ni g \mapsto \omega_g \in C_g; \text{ for all } x \in X, \omega_g(x) \in U(A_{t_x(g)})\}$
 and $\omega_{g_1}(x)\omega_{g_2}(x) = \omega_{g_1g_2}(x)$ ($g, g_1, g_2 \in G$). Then S^\wedge (= dual group of S) acts on Ω as a transformation group:

$$(\sigma\omega)_g = \langle \sigma, g \rangle \omega_g, \quad \sigma \in S^\wedge, \quad \omega \in \Omega, \quad g \in S.$$

Lemma 5. Ω is a principal homogeneous space under the action of S^\wedge (in particular, $\Omega \neq \emptyset$).

By this lemma, we can transform the topology of S^\wedge into Ω and Ω becomes a compact Hausdorff space.

Let $\omega \in \Omega$ and define an action of S on A by

$$ga = \omega_g(t(\gamma))a, \quad g \in S, \quad a \in A_\gamma.$$

Taking quotient, we obtain a C^* -algebra A^ω over Γ/S . Let $\varphi \in C_c(\Gamma, A)$ and set

$$\varphi_\omega(\gamma) = \sum_{g \in S} \omega_g(t(\gamma))\varphi(g^{-1}\gamma), \quad \gamma \in \Gamma.$$

Then φ_ω gives rise to an element in $C_c(\Gamma/S, A^\omega)$ and we have

Lemma 6. $\{(\varphi_\omega)_{\omega \in \Omega}; \varphi \in C_c(\Gamma, A)\}$ defines a continuous field structure for the family of C^* -algebras $\{C^*(\Gamma/S, A^\omega)\}_{\omega \in \Omega}$.

Theorem. Let Γ be a locally compact Hausdorff second countable groupoid satisfying (A1)~(A3) and A be a C^* -algebra over Γ satisfying (A4)~(A6). Then

(i) for all $\omega \in \Omega$, $C^*(\Gamma/S, A^\omega)$ is simple,

(ii) $C_c(\Gamma, A) \ni \varphi \mapsto \{\varphi_\omega\}_{\omega \in \Omega} \in \{C^*(\Gamma/S, A^\omega)\}_{\omega \in \Omega}$ is extended to the isomorphism between $C^*(\Gamma, A)$ and the cross section C^* -algebra of $\{C^*(\Gamma/S, A^\omega)\}_{\omega \in \Omega}$.

Sketch of the proof. We define an action of G^\wedge on $C^*(\Gamma(X), A)$ by

$$(\sigma\varphi)(g, x) = \langle \sigma, g \rangle \varphi(g, x).$$

Then the associated action of G^\wedge on G^* preserves the equivalence relation \sim and we can show that this action is transitive. So G^*/\sim is identified with G^\wedge/S'^\perp , where S' is a subgroup of G and $S'^\perp = \{\sigma \in G^\wedge; \sigma|_{S'} = 1\}$. Now imbed $C(G^*/\sim)$ into $C_b(G^*)$ and regard it as a subalgebra of the center Z of the multiplier algebra $M(C^*(\Gamma(X), A))$ (by Dauns-Hofmann Theorem). Then we can explicitly write down the condition when an element in Z belongs to $C(G^*/\sim)$, and this shows that $S' = S$ and $S^\wedge = G^\wedge/S'^\perp = \Omega$. The topological decomposition follows from [11].

Corollary 1. $\Omega = \text{Prim}(C^*(\Gamma, A))$.

Corollary 2. $C^*(\Gamma, A)$ is simple if and only if $S = \{e\}$.

Let G be a locally compact group and N be an open normal subgroup of G with G/N abelian. Let A be a separable C^* -algebra and $\alpha: G \rightarrow \text{Aut}(A)$, $\rho: N \rightarrow U(A)$ be homomorphisms, which form a twisted covariance system in the sense of [6]. Set $S = \{g \in G; \text{there exists } u \in U(A) \text{ such that } \alpha_{h^{-1}}(u)a = \rho(h^{-1}g^{-1}hg)\alpha_{g^{-1}}(a)u \text{ for all } h \in G \text{ and all } a \in A\}$. Due to §9 of [3], we can construct a C^* -algebraic bundle over G/N . Since algebraic bundles are special cases of C^* -algebras over

groupoids, we have

Corollary 3. S is a subgroup of G containing N and, if A is G -simple the primitive ideal space of the twisted covariance algebra $A \times_N G$ is homeomorphic to $(S/N)^\wedge$.

This is a supplementary result to [7],[8].

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