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Casson's λ-invariant and its application

Shinji Fukuhara and Noriko Maruyama

Tsuda College

A. Casson [C] defined an integer valued invariant $\lambda(M)$ for an oriented homology 3-sphere M.

We give a sum formula to calculate Casson's λ-invariant for an oriented homology 3-sphere which is constructed by gluing two knot exteriors in homology 3-spheres with some diffeomorphism between their boundaries. Our result is just the λ-invariant version of C. Gordon's theorem [G1. Theorem 2] for μ-invariant.

§1. Casson's λ-invariant.

Casson proved the following theorem.

Theorem 1 (A. Casson).

Let M be an oriented homology 3-sphere. There exist an integer valued invariant $\lambda(M)$ with the following properties.

(1) If $\pi_1(M) = 1$, then $\lambda(M) = 0$.

(2) $\lambda(M) = -\lambda(M)$, where $-M$ denotes M with the opposite orientation.

(3) Let K be a knot in M and $(K_n;M)$ be the oriented homology 3-sphere obtained by performing $1/n$-Dehn surgery on M along K, $n \in \mathbb{Z}$. $\lambda(K_{n+1};M) - \lambda(K_n;M)$ is determined independently to n.

(4) $\lambda(M)$ reduces, mod 2, to the Rohlin invariant $\mu(M)$.

By the property (3), $\lambda'(K;M) = \lambda(K_{n+1};M) - \lambda(K_n;M)$ is well defined. By the fact that $(K_0;M) = M$ and the induction on n,

Corollary 2. $\lambda(K_n;M) = \lambda(M) + n\,\lambda'(K;M)$.

Let $\Delta_{K;M}(t)$ be the normalized Alexander polynomial of a knot K in M. "normalized" means that the followings hold,

(1) $\Delta_{K;M}(1) = 1$,

(2) $\Delta_{K;M}(t) = \Delta_{K;M}(t^{-1})$.

Moreover, let V be a $2h \times 2h$ Seifert matrix of K, then the normalized Alexander polynomial is given as follows.

$$\Delta_{K;M}(t) = t^{-h} \det(V - tV^T).$$

Casson related $\lambda'(K;M)$ to the above classical invariant of K as follows.

Theorem 3 (A. Casson). $\lambda'(K;M) = \frac{1}{2}\Delta''_{K;M}(1)$, where $\Delta''_{K;M}(t)$ is the second derivative of normalized Alexander polynomial of K.

Corollary 2 and Theorem 3 are useful to calculate Casson's λ-invariant of an oriented homology 3-sphere. For example, we have the following.

Example. $(\Sigma(p,q,pqr \pm 1)) = -r(p^2-1)(q^2-1)/24$, where $\Sigma(p,q,pqr \pm 1)$ is the Brieskorn homology sphere and p, q, r are pairwise coprime integers $\geqq 2$.

§2. A sum formula for Casson's λ-invariant.

To give the formula precisely, we need some notations.

We study oriented homology 3-spheres which are constructed by C. Gordon [G1]. Let M_i be an oriented homology 3-sphere and K_i be an oriented knot in M_i with the exterior X_i, for $i = 1, 2$. We always identify ∂X_i with $S^1 \times \partial D^2$ and parametrize it by an angular coordinate (θ, φ). If A be a 2×2 integral matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with det A = -1, then A defines an orientation reversing diffeomorphism $h; \partial X_1 \to \partial X_2$ by $h(\theta, \varphi) = (\alpha\theta + \beta\varphi, \gamma\theta + \delta\varphi)$. We denote the naturally oriented closed 3-manifold obtained by gluing two knot exteriors with h, $X_1 \cup_h X_2$ as $M(K_1, K_2; A)$ or $M(K_1, K_2; \alpha, \beta, \gamma, \delta)$. By the explicit computation of the first homology group of $M(K_1, K_2; A)$, it is known that $M(K_1, K_2; \alpha, \beta, \gamma, \delta)$ becomes a homology 3-sphere if and only if $|\gamma| = 1$. Hereafter, we assume always this. Since $\det A = \alpha\delta - \beta\gamma = -1$ and $|\gamma| = 1$, $\beta = \pm(\alpha\delta + 1)$, so it is detemined by α, β and the sign ε. Hence we denote the homology 3-sphere obtained this construction by $M^{\varepsilon}(K_1, K_2; \alpha, \delta)$.

Our result is the following.

Theorem 4.

$$\lambda(M^{\varepsilon}(K_1, K_2; \alpha, \delta)) = \lambda(M_1) + \lambda(M_2) - \varepsilon\delta\lambda'(K_1; M_1) + \varepsilon\alpha\lambda'(K_2; M_2).$$

Remarks.

(1) It is known that $\lambda'(K;M)$ reduces, mod 2, to the Arf invariant $c(K;M)$. The theorem above is λ-invariant version of Gordon's formula [G1, Theorem 2] for μ-invariant of the oriented homology 3-sphere $M^{\varepsilon}(K_1, K_2; \alpha, \delta)$.

(2) Using Corollary 2, the formula in the theorem is also denoted as follows,

$$\lambda(M^{\varepsilon}(K_1, K_2; \alpha, \delta)) = \lambda((K_1)_{-\varepsilon\delta}; M_1) + \lambda((K_2)_{\varepsilon\alpha}; M_2).$$

This is the reason why we call it " a sum formula for Casson's λ-invariant".

(3) S. Akbult and J. McCarthy obtained the same formula independently.

§3. Outline of the proof.

Changing the gluing map, we have the following topological lemma.

Lemma 5. $M = M^{\varepsilon}(K_1, K_2; \alpha, \delta) = ((K_1^* \# K_2^*)_{\mp 1}; N_1 \# N_2)$, where K_i^* is a 0-parallel knot of K_i in M_i and $N_i = ((K_i^*)_{n_i}; M_i)$, for $i = 1, 2$, and $n_1 \mp 1 = -\varepsilon\delta$, $n_2 \mp 1 = \varepsilon\alpha$.

We omit the proof and refer to [FM] and [G2].

Next we need a couple of properties of λ and λ'.

Since the Seifert matrix of a connected sum of two knots is a block sum of Seifert matrices,

$$\Delta_{K_1 \# K_2; M_1 \# M_2}(t) = \Delta_{K_1;M_1}(t) \cdot \Delta_{K_2;M_2}(t).$$

By the fact that $\Delta'_{K;M}(1) = 0$ and computing second derivatives of the above equqtion, we have,

Lemma 6. $\lambda'(K_1 \# K_2; M_1 \# M_2) = \lambda'(K_1;M_1) + \lambda'(K_2;M_2)$.

Considering a framed link description of M_1 such that the corresponding framed link satisfy the condition that linking numbers of any two components are zero and using Lemma 6,

Lemma 7. $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.

Let M be an oriented homology 3-sphere and K be a knot in M. Consider two disjoint two simple closed curves a, b in $M - K$. Let N be $(K_n;M)$. Then the following fact is known L .

$$lk_N(a,b) = lk_M(a,b) - n\cdot lk_M(a,K)\cdot lk_M(b,K),$$

where a and b can be seen naturally curves in N.

Using this fact, the next follows.

Lemma 8. Let K be a knot in an oriented homology 3-sphere M and K^* be a zero parallel knot of K in M. Let $N = (K_n;M)$, then

$$\lambda'(K^*;N) = \lambda'(K;M).$$

We prove Theorem 4 as follows:

By Lemma 5 and Corollary 2,

$$\lambda(M^{\varepsilon}(K_1,K_2;\alpha,\delta)) = \lambda(N_1 \# N_2) \mp \lambda'(K_1^* \# K_2^*;N_1 \# N_2),$$

by Lemma 6 and Lemma 7,

$$= \lambda(N_1) + \lambda(N_2) \mp \lambda'(K_1^*;N_1) \mp \lambda'(K_2^*;N_2),$$

by Corollary 2 and Lemma 8,

$$= \lambda(M_1) + (n_1 \mp 1)\lambda'(K_1;M_1)$$
$$+ \lambda(M_2) + (n_2 \mp 1)\lambda'(K_2;M_2),$$

using the definition of n_i for $i = 1, 2$, we have the desired formula.

4. Concluding remark.

By analogous consideration of L. Siebenmann [S], we can prove the following.

Theorem 9. Let M be an oriented homology 3-sphere and T be a 2-torus family which determine a graph Γ. Then

$$\lambda(M) = \sum_{v \in \Gamma^{(0)}} \lambda(M(v)_+).$$

From the view point of this theorem, it seems worthful to compile many data of Casson's λ-invariant of oriented homology 3-spheres. Hence we have the following problem.

Problem. Let M be a graph homology 3-sphere in the sence of Waldhausen [W]. Give a formula to calculate $\lambda(M)$ automatically from the corresponding weighted graph of M.

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