

Heights of simple loops and pseudo-Anosov homeomorphisms

(阪大 理 工 学 部 数 学 科)

Tsuyoshi Kobayashi

1. Introduction

Let H be a 3-dimensional, orientable handlebody of genus g (>1) and ℓ ($\subset \partial H$) be a connected essential simple closed curve. A *simple loop* is the ambient isotopy class of ℓ and we often abbreviate it by denoting ℓ . A *surface* is a connected 2-manifold. Let F be a 2-sided surface properly embedded in a 3-manifold M . F is *essential* if it is incompressible and not parallel to a subsurface of ∂M . Then we define the *height* of ℓ , $h(\ell)$, as follows;

$$h(\ell) = \min \{ -\chi(F) \mid (F, \partial F) (\subset (H, \partial H - \ell)) \text{ is an essential surface} \},$$
 where $\chi(F)$ denotes the Euler characteristic of F .

In section 2 we see that $h(\ell)$ can be defined.

Fix a hyperbolic metric on ∂H . Let L ($\subset \partial H$) be a geodesic lamination. We say that L is of *full type* if there is a system of mutually disjoint incompressible disks $\{D_1, \dots, D_{3g-3}\}$ in H such that $\partial D_1 \cup \dots \cup \partial D_{3g-3}$ cuts ∂H into $2g-2$ pants P_1, \dots, P_{2g-2} , which satisfies;

(i) $\cup \partial D_i$ and L intersect transversely and there is no 2-gon B in ∂H such that $\partial B = \alpha \cup \beta$ where α is a subarc of $\partial D_1 \cup \dots \cup \partial D_{3g-3}$, β is a subarc of L and

(ii) for each P_i we have;

for each pair of boundary components of P_i , there is a subarc of L properly embedded in P_i , which joins the components.

Then the first result of this paper is;

Theorem 1. *Let $f: \partial H \rightarrow \partial H$ be a pseudo-Anosov homeomorphism and $\ell \subset \partial H$ a simple loop. Suppose that the stable lamination of f is of full type. Then $\lim_{p \rightarrow \infty} h(f^p(\ell)) = \infty$, i.e. for each $n (\geq -1)$ there exists a constant N such that if $m > N$ then $h(f^m(\ell)) > n$.*

We note that the assumption on the stable lamination of f is essential. In fact Fathi-Laudenbach [4] showed that there exists a pseudo-Anosov homeomorphism $\varphi: \partial H \rightarrow \partial H$ which extends to a homeomorphism of H . Then clearly we have $h(\varphi^n(\ell)) = h(\ell)$, for each ℓ and n .

Let $f: \partial H \rightarrow \partial H$ be a homeomorphism and $D_1 \cup \dots \cup D_g$ a union of mutually disjoint incompressible disks in H such that $D_1 \cup \dots \cup D_g$ cuts H into a 3-cell. Then we get a compact 3-manifold whose boundary is a sphere by attaching 2-handles to H along the simple loops $f(\partial D_1), \dots, f(\partial D_g)$. We note that the obtained manifold does not depend on the choice of D_1, \dots, D_g (Lemma 4.1). Hence we denote the manifold by \bar{M}_f . Then M_f denotes the manifold obtained from \bar{M}_f by capping off the boundary by a 3-cell. Roughly speaking, M_f is obtained from H by attaching a copy of H by f .

As an application of Theorem 1, we have;

Theorem 2. *Let $f: \partial H \rightarrow \partial H$ be a pseudo-Anosov homeomorphism. Suppose that the invariant laminations of f are of full type. Then, for each $n (\geq 0)$, there is a constant N such that if $m > N$, then M_{f^m} does not contain a 2-sided incompressible surface whose genus is less*

than or equal to n .

2. Preliminaries

For the definitions of the standard terms in the 3-dimensional topology we refer to [6,7]. We assume that the reader is familiar with [1].

Let H, ℓ be as in section 1. First we will show that the height of ℓ can be defined.

Lemma 2.1. *There exists a 2-sided non-separating surface S properly embedded in $(H, \partial H - \ell)$.*

Proof. Let M be the 3-manifold obtained from H by attaching a 2-handle $D^2 \times I$ along ℓ . Since the genus of H is greater than 1, the first Betti number of M is greater than 0. Hence by [6, Lemma 6.6], M contains a properly embedded, 2-sided, non-separating incompressible surface S' . Then, by moving S' by an ambient isotopy, we may suppose that S' intersects the 2-handle in horizontal disks. Hence $S = S' - \text{Int}(S' \cap (D^2 \times I))$ is a 2-sided non-separating surface properly embedded in H . Moreover, by moving S by a tiny isotopy, we may suppose that S is properly embedded in $(H, \partial H - \ell)$.

This completes the proof of Lemma 2.1.

Then we have;

Corollary. *There exists an essential surface properly embedded*

in $(H, \partial H - \ell)$.

Proof. Let S be the surface obtained in Lemma 2.1. If necessary, by applying the loop theorem and performing a surgery on S , we may suppose that S is incompressible in H . Since S is non-separating, S is not parallel to a subsurface in ∂H . Hence S is essential.

Let F be a closed, orientable surface of genus $g (>1)$ with a hyperbolic metric and $PL(F)$ the space of projective measured laminations of F with an appropriate topology. Then $PL(F)$ is homeomorphic to the $6g-7$ dimensional sphere S^{6g-7} . A simple loop is the ambient isotopy class of a closed, connected, 1-submanifold of F which is not contractible in F . Then it is known that the set of all simple loops together with the counting measure consist a dense subset of $PL(F)$. Let ℓ be a simple loop. $\mathcal{G}_\ell (\subset PL(F))$ denotes the set of all simple loops which are disjoint from ℓ . Then let $\mathcal{G}(\ell;1) = \mathcal{G}_\ell$, and $\mathcal{G}(\ell;k) = \bigcup_{\ell' \in \mathcal{G}(\ell;k-1)} \mathcal{G}_{\ell'}$, ($k > 1$).

In this section we will prove;

Proposition 2.2. *Let $f:F \rightarrow F$ be a pseudo-Anosov homeomorphism with an unstable lamination L^- and ℓ a simple loop. Then there exists a sequence of neighborhoods of L^- in $PL(F)$, $\{U_i\}_{i=1}^\infty$, such that $U_1 \supset U_2 \supset \dots$, and $U_k \cap \mathcal{G}(\ell;k) = \emptyset$, for each k .*

Let $\tau (\subset F)$ be a maximal train track, i.e. the closure of each

component of $F - \tau$ is a 3-gon. Then the set of all non-negative weights on τ defines a $6g-7$ dimensional ball B_τ in $PL(F)$. And the set of all positive weights on τ corresponds to the interior of B_τ [1,10].

Lemma 2.3. *If ℓ is carried by τ with all weights positive, then $\mathcal{G}_\ell \subset B_\tau$.*

Proof. Let $N(\tau)$ be a standard neighborhood of τ . Since ℓ

[Figure 1]

is carried by τ , we may suppose that $\ell \subset N(\tau)$ and ℓ is transverse to the fibers. Let $m \in \mathcal{G}_\ell$. Since each component of $F - N(\tau)$ is contractible, we can isotope m into $N(\tau)$ with $m \cap \ell = \emptyset$. If m is transverse to the fibers of $N(\tau)$, then $m \in B_\tau$. Hence we suppose that m is not transverse to the fibers. We can isotope m so that m is transverse to the fibers except neighborhoods of the switches which are unions of fibers. See Figure 2. Then the neighborhood of

[Figure 2]

a switch will look like as in Figure 3. We note that the subarcs of

[Figure 3]

m as a_1 or a_2 in Figure 3 play a bad role in this situation. Then we will show that we can eliminate such arcs by an isotopy.

Let Δ be the closure of a component of $F - N(\tau)$ and e an edge of the triangle Δ . Then we have;

Assertion. There is a rectangle R in F such that $R \subset N(\tau)$, $\text{Int } R \cap \ell = \emptyset$, one edge of R is a subarc of ℓ , one edge of R is

e , and the rest edges of R are subarcs of two fibers of $N(\tau)$.

We note that the universal cover of F is isometric to the hyperbolic plane \mathbb{H}^2 . Let $\tilde{\Delta} (\subset \mathbb{H}^2)$ be a lift of Δ , \tilde{e} the edge of $\tilde{\Delta}$

[Figure 4]

corresponding to e and $\tilde{N}(\tau) (\subset \mathbb{H}^2)$ the lift of $N(\tau)$. Let $I_p (p \in \tilde{e})$ be the fiber of $\tilde{N}(\tau)$ such that $p \in I_p$. Then, by [1, Lemma 5.8], we see that $I_p \cap \tilde{\Delta} = p$. Hence $\bigcup_{p \in \tilde{e}} I_p$ is a disk which is a disjoint union of the intervals I_p . Since ℓ is carried by τ with all weights

[Figure 5]

positive, we have a rectangle $\tilde{R} (\subset \tilde{N}(\tau))$ such that one edge of \tilde{R} is \tilde{e} , one edge of \tilde{R} is a subarc of a lift of ℓ , and the rest edges are subarcs of two fibers of $\tilde{N}(\tau)$. Since \tilde{e} projects homeomorphically onto e , we see that \tilde{R} projects to the rectangle as in Assertion.

This completes the proof of Assertion.

Let F' be the component of $F - \ell$ such that $F' \supset m$, F^* the surface obtained from F' by adding a boundary and $N (\subset F^*)$ the image of $N(\tau)$. Then we can reduce the subarcs of type a_1 in Figure 3 by an isotopy as in Figure 6. By Assertion, we can reduce

[Figure 6]

the subarcs of type a_2 in Figure 3 by an ambient isotopy on F^* as in Figure 7.

[Figure 7]

It is easy to see that these operations terminates in finitely many steps and we see that m is carried by τ .

This completes the proof of Lemma 2.3.

Proof of Proposition 2.2. First we will show that U_1 actually exists. Then we will construct U_2, U_3, \dots inductively.

Assume that U_1 does not exist. Then there is a sequence of elements of \mathcal{G}_ℓ , $\{m_i\}_{i=1}^\infty$, such that $m_i \rightarrow L^-$ ($i \rightarrow \infty$). By [10, Chapitre 2, Proposition], there exists a maximal train track τ and a positive integer n such that the stable lamination L^+ is carried by τ with all weights positive and $f^n(\tau)$ is carried by τ . If necessary, by taking $f^{nm}(\tau)$ for τ , we may suppose that $B_\tau \cap L^- = \emptyset$ [10, Chapitre 2]. Hence there exists a neighborhood of L^- in $PL(F)$ which is disjoint from B_τ . There is an integer N such that $f^N(\ell) \subset \text{Int } B_\tau$. On the other hand, since $f: PL(F) \rightarrow PL(F)$ is a continuous map, we have $f^N(m_i) \rightarrow L^-$ ($i \rightarrow \infty$), contradicting Lemma 2.3. Hence U_1 exists.

Suppose that we have constructed U_1, \dots, U_{k-1} . Let τ, n be as above. Then there exists a positive integer N_1 such that $f^{nN_1}(PL(F) - U_{k-1}) \subset \text{Int } B_\tau$. Let $U_k = f^{-nN_1}(PL(F) - B_\tau)$. Since $\mathcal{G}(\ell; k-1) \cap U_{k-1} = \emptyset$, we have $f^{nN_1}(\mathcal{G}(\ell; k-1)) \cap f^{nN_1}(U_{k-1}) = \mathcal{G}(f^{nN_1}(\ell); k-1) \cap f^{nN_1}(U_{k-1}) = \emptyset$. Hence $\mathcal{G}(f^{nN_1}(\ell); k-1) \subset \text{Int } B_\tau$. Then, by Lemma 2.3, we have $\mathcal{G}(f^{nN_1}(\ell); k) \subset B_\tau$. Then we have $U_k \cap \mathcal{G}(\ell; k) = \emptyset$.

This completes the proof of Proposition 2.2.

3. Proof of Theorem 1

In this section we will give a proof of Theorem 1. Throughout this section, let H, ℓ be as in section 1. The key of the proof is Lemma 3.2 which is an easy estimation of $h(\ell)$ and Lemma 3.4.

Lemma 3.1. *Suppose that $h(\ell) > -1$. Then there exists an element ℓ' of \mathcal{G}_ℓ such that $h(\ell') < h(\ell)$.*

Proof. There exists an essential surface F properly embedded in $(H, \partial H - \ell)$ such that $-\chi(F) = h(\ell)$. Since every incompressible, ∂ -incompressible surface in H is a disk, there is a disk D in H such that $\text{Int } D \cap F = \emptyset$, $D \cap \partial H = \partial D \cap \partial H = \alpha$ an arc and $D \cap F = \partial D \cap F = \beta$ an arc such that $\partial\alpha = \partial\beta$, $\alpha \cup \beta = \partial D$. Let F' be the 2-manifold obtained from F by performing a surgery along D and ℓ' a component of ∂F . Clearly $\ell' \in \mathcal{G}_\ell$. By moving ℓ' by a tiny isotopy, we may suppose that $\ell' \cap \partial F' = \emptyset$. Since F is essential, we see that a component of F' , say F'' , is essential. And $\chi(F'') > \chi(F)$. Hence we have $h(\ell') < h(\ell)$.

This completes the proof of Lemma 3.1.

As an immediate consequence of Lemma 3.1, we have ;

Lemma 3.2. *Assume that $\mathcal{G}(\ell; n)$ does not contain a simple loop of height -1 , i.e. if $m \in \mathcal{G}(\ell; n)$, then $\partial H - m$ is incompressible in H . Then $h(\ell) > n$.*

Lemma 3.3. *If ℓ is of full type, then $h(\ell) > -1$, i.e. $\partial H - \ell$*

is incompressible in H .

For the definition of ℓ being of full type, see section 1.

Proof. Assume that $\partial H - \ell$ is compressible in H . Then there is a compression disk D properly embedded in $(H, \partial H - \ell)$. Let $\{D_1, \dots, D_{3g-3}\}$ be a system of disks with respect to which ℓ is of full type. Then we may suppose that $\partial D_i, \partial D, \ell$ are geodesics for a fixed hyperbolic metric on ∂H . Assume that $\partial D = \partial D_i$ for some i . Then ∂D intersects ℓ , a contradiction. Hence we suppose that $\partial D \neq \partial D_i$ for each i . By cut and paste argument, we may suppose that D intersects $\cup D_i$ in transverse arcs. Let Δ be an innermost disk in D , i.e. $\Delta \cap (\cup D_i) = \partial \Delta \cap (\cup D_i) = \alpha$, an arc and $\Delta \cap \partial H = \beta$, an arc such that $\alpha \cup \beta = \partial \Delta$. Let P be the closure of the component of $H - N(\cup D_i)$ such that $\Delta \cap P \neq \emptyset$. Then $\beta \cap P$ is an arc and separates two boundary components of the pants $P \cap \partial H$. On the other hand, since ℓ is of full type, there is a subarc γ of ℓ properly embedded in $P \cap \partial H$ such that γ joins the two boundary components. Hence $\beta \cap \gamma \neq \emptyset$, a contradiction.

This completes the proof of Lemma 3.3.

We say that a train track τ on ∂H is of *full type* if there is a system of disks D_1, \dots, D_{3g-3} in H and a system of pants P_1, \dots, P_{2g-2} as in section 1 which satisfies;

(i) τ and $\partial D_1 \cup \dots \cup \partial D_{3g-3}$ intersect transversely and there is no 2-gon B in ∂H such that $\partial B = a \cup b$, where a is a differentiable arc on τ , and b is a subarc of $\partial D_1 \cup \dots \cup \partial D_{3g-3}$

and

(ii) for each pair of boundary components of each pants P_i , there is a differentiable arc a on τ such that a is properly embedded in P_i , and a joins the boundary components.

Then, by Lemma 3.3, we have;

Lemma 3.4. *Suppose that the train track τ ($\subset \partial H$) is of full type. If ℓ is a simple loop which is carried by τ with all weights positive, then $h(\ell) > -1$.*

Proof of Theorem 1. Let L^+ , L^- be the stable, and unstable laminations of f . By Proposition 2.2, there is a neighborhood U of L^- in $PL(F)$ such that $U \cap \mathcal{G}(\ell; n) = \emptyset$. Since L^+ is of full type, there is a train track τ ($\subset \partial H$) which is of full type such that L^+ is carried by τ with all weights positive ([1, Lemma 5.2]). Let N be a positive integer such that $f^N(PL(F) - U) \subset \text{Int } B_\tau$. Then we have $f^N(\mathcal{G}(\ell; n)) = \mathcal{G}(f^N(\ell); n) \subset \text{Int } B_\tau$. Hence, by Lemmas 3.2 and 3.4, we see that $h(f^N(\ell)) > n$. Moreover, by [10, Chapitre 2], we may assume that $f(B_\tau) \subset \text{Int } B_\tau$. Hence we see that if $m \geq N$, then $h(f^m(\ell)) > n$.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section we will prove Theorem 2. First we will show that M_f is well defined.

Let H be a genus g (>1) handlebody and $f: \partial H \rightarrow \partial H$ a homeomorphism. Let $\{D_1, \dots, D_g\}, \{D_1', \dots, D_g'\}$ be systems of mutually disjoint incompressible disks in H such that $\cup D_i$ ($\cup D_i'$ resp.) cuts H into a 3-cell. Let M (M' resp.) be the 3-manifold obtained from H by attaching 2-handles along the union of simple loops $\cup f(\partial D_i)$ ($\cup f(\partial D_i')$ resp.).

Then we have;

Lemma 4.1. M and M' are homeomorphic.

Proof. First we prepare a terminology. Let B be a rectangle in ∂H such that $\text{Int } B \cap (\cup \partial D_i) = \emptyset$, one edge of B is a subarc α of ∂D_j , another edge of ∂B is a subarc β of ∂D_k , where $j \neq k$, with $B \cap (\cup D_i) = \alpha \cup \beta$. Then $D_j \cup B \cup D_k$ is a disk D in H . By moving D by a tiny isotopy, we may suppose that $D \cap \partial H = \partial D$ and $D \cap (\cup D_i) = \emptyset$. It is easy to see that the disks $\{D_1, \dots, \hat{D}_j, \dots, D_g, D\}$ cuts H into a 3-cell, where $\hat{}$ means removing the element. Then we say that $\{D_1, \dots, \hat{D}_j, \dots, D_g, D\}$ is obtained from $\{D_1, \dots, D_g\}$ by a *band move*. It is known that $\bigcup_{i=1}^g D_i'$ is ambient isotopic to the union of disks which is obtained from $\{D_1, \dots, D_g\}$ by a sequence of band moves ([9, Korrolar 2]). Hence we may suppose that $\{D_1', \dots, D_g'\}$ is obtained from $\{D_1, \dots, D_g\}$ by a band move. We may suppose that $D_i = D_i'$ ($2 \leq i \leq g$), and $D_1 \cap D_1' = \emptyset$.

Then \dot{M} and \dot{M}' are homeomorphic to the manifold obtained from H by attaching 2-handles along the union of simple loops

$\partial D_1 \cup \partial D_1' \cup \partial D_2 \cup \dots \cup \partial D_g$, where \dot{M} denotes the manifold obtained from M by removing a 3-cell from its interior. This shows that M and M' are homeomorphic.

This completes the proof of Lemma 4.1.

By Lemma 4.1, we may denote the manifold obtained as above \bar{M}_f . Then M_f denotes the closed 3-manifold obtained from \bar{M}_f by capping off the boundary by a 3-cell. It is easy to see that $H' = \text{cl}(M_f - H)$ is a genus g handlebody. Hence $(H, H'; F)$, where $F = \partial H = \partial H'$ ($\subset M_f$) is a Heegaard splitting of M_f . $(H, H'; F)$ is called a *canonical Heegaard splitting* of M_f .

Then we will introduce the concept of rectangle condition of a Heegaard splitting by Casson-Gordon [2].

Let S be the genus g orientable surface and P_i ($i=1,2$) be a pants (:disk with two holes) embedded in S , with $\partial P_i = l_1^i \cup l_2^i \cup l_3^i$. We suppose that ∂P_1 and ∂P_2 intersect transversely. Let Q_1, \dots, Q_m be the compact surfaces obtained from $S - (\partial P_1 \cup \partial P_2)$ by adding a boundary. We say that P_1 and P_2 are *tight* if;

(i) each Q_i is not a 2-gon,

(ii) for each pair of, pair of boundary components,

$((l_s^1, l_t^1), (l_p^2, l_q^2))$ of P_1 and P_2 with $s \neq t, p \neq q$, there exists a rectangle Q_u embedded in P_1 and P_2 such that

$\text{Int } Q_u \cap (P_1 \cup P_2) = \emptyset$, and the edges of Q_u are subarcs of l_s^1, l_t^1, l_p^2 , and l_q^2 .

Let l_1, \dots, l_{3g-3} (l_1', \dots, l_{3g-3}' resp.) be a system

of mutually disjoint simple loops such that the closure of the complement of a regular neighborhood of $\cup \ell_i$ ($\cup \ell_i'$ resp.) is a union of $2g-2$ pants P_1, \dots, P_{2g-2} (P_1', \dots, P_{2g-2}' resp.). We say that $\{\ell_1, \dots, \ell_{3g-3}\}$ and $\{\ell_1', \dots, \ell_{3g-3}'\}$ are *tight* if, for each pair (i, j) , P_i and P_j are tight.

Let $(H, H'; F)$ be a Heegaard splitting of a 3-manifold M . We say that $(H, H'; F)$ satisfies a *rectangle condition* if there are two systems of mutually disjoint simple loops $\{\ell_1, \dots, \ell_{3g-3}\}$ and $\{\ell_1', \dots, \ell_{3g-3}'\}$ which are tight and each ℓ_i (ℓ_i' resp.) is a boundary of a disk properly embedded in H (H' resp.).

We say that a Heegaard splitting $(H, H'; F)$ is *strongly irreducible* if there does not exist incompressible disks $D \subset H$, $D' \subset H'$ such that $\partial D \cap \partial D' = \emptyset$.

Then Casson-Gordon proved;

Theorem [2]. *If a Heegaard splitting $(H, H'; F)$ satisfies a rectangle condition, then $(H, H'; F)$ is strongly irreducible.*

In this paper we will show;

Theorem 4.2. *Let H be as in section 1 and $f: \partial H \rightarrow \partial H$ be a pseudo-Anosov homeomorphism. Suppose that the invariant laminations of f are of full type. Then there is an integer N_1 such that if $m > N_1$, then the canonical Heegaard splitting of M_{f^m} satisfies a rectangle condition.*

Proof. Let L^+ (L^- resp.) be the stable (unstable resp.) lamination of f . We suppose that L^+ (L^- resp.) is of full type with respect to a system of disks $\{D_1, \dots, D_{3g-3}\}$ ($\{D_1', \dots, D_{3g-3}'\}$ resp.) in H . Then we may suppose that L^+ , L^- , $\cup \partial D_i$, $\cup \partial D_i'$ are unions of geodesics on ∂H . The universal cover of ∂H is isometric to the hyperbolic plane \mathbb{H}^2 . Let $\tilde{f}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ be a lift of f , and S_∞^1 be the circle at ∞ of \mathbb{H}^2 . Then it is known that \tilde{f} has a unique continuous extension to $\mathbb{H}^2 \cup S_\infty^1 = D^2$ ([1, Lemma 3.7]). Moreover, there is a positive integer n such that if $g: D^2 \rightarrow D^2$ is the extension of a lift of f^n then $g|_{S_\infty^1}$ has finitely many fixed points A_1, \dots, A_{2n} ($n \geq 1$) on S_∞^1 , alternately contracting and expanding. We suppose that A_1, \dots, A_{2n} are on S_∞^1 in this order, A_1, \dots, A_{2n-1} are contracting and A_2, \dots, A_{2n} are expanding. Let I_j ($j=1, \dots, 2n-1$) be a closed interval in S_∞^1 bounded by A_j and A_{j+1} such that $\text{Int } I_j \cap (\cup A_i) = \emptyset$, and $I_{2n} = \text{cl}(S_\infty^1 - \bigcup_{j=1}^{2n-1} I_j)$. Let \tilde{L}^+ (\tilde{L}^- resp.) ($\subset \mathbb{H}^2$) be the lift of L^+ (L^- resp.), and γ_1 (δ_1 resp.) be the geodesics of \tilde{L}^+ (\tilde{L}^- resp.) which joins A_1 and A_3 (A_2 and A_4 resp.).

Then we will show;

Assertion. For each ∂D_i ($\partial D_i'$ resp.), there exists a lift e ($\subset \mathbb{H}^2$) of it such that the endpoints of e are contained in I_1 and I_2 (I_1 and I_{2n} resp.).

Proof. Since each leaf of L^+ is dense in L^+ ([1, Theorem 4.8]), there is a sequence of lifts of the points of

$L^+ \cap \partial D_i$, $\{a_i\}_{i=1}^\infty$ such that $a_i \in \delta_1$, and $a_i \rightarrow A_2$ ($i \rightarrow \infty$). Let $\{\varepsilon_i\}_{i=1}^\infty$ be the sequence of the lifts of ∂D_i such that $\varepsilon_i \cap \delta_1 = a_i$. Since each ε_i is a geodesic, we see that at least one endpoint of ε_i is contained in I_1 or I_2 for sufficiently large i . Assume that the other endpoints of ε_i are not contained in I_2 or I_1 . Then we have $\lim_{i \rightarrow \infty} \arg(\varepsilon_i, \delta_1) = 0$, where $\arg(\varepsilon_i, \delta_1)$ denotes the angle between the two geodesics. Since L^+ is a closed set, we have $\partial D_i \subset L^+$, a contradiction.

This completes the proof of Assertion.

Let P_1, \dots, P_{2g-2} (P_1', \dots, P_{2g-2}' resp.) be a system of pants obtained from ∂H by cutting along $\cup \partial D_i$ ($\cup \partial D_i'$ resp.). Let ∂D_α , ∂D_β and ∂D_γ , ∂D_δ be pairs of boundary components of a pair (P_i, P_j') . By Assertion, there are lifts $\tilde{\alpha}$, $\tilde{\beta}$ ($\tilde{\gamma}$, $\tilde{\delta}$ resp.) of ∂D_α , ∂D_β (∂D_γ , ∂D_δ resp.) in H^2 such that $\tilde{\alpha} \cap \gamma_1 \neq \emptyset$, $\tilde{\beta} \cap \gamma_1 \neq \emptyset$, $\tilde{\gamma} \cap \delta_1 \neq \emptyset$, $\tilde{\delta} \cap \delta_1 \neq \emptyset$, the endpoints of $\tilde{\alpha}$, $\tilde{\beta}$ ($\tilde{\gamma}$, $\tilde{\delta}$ resp.) are contained in I_1 and I_2 (I_1 and I_{2n-1} resp.), and the subarc of γ_1 (δ_1 resp.) bounded by $\tilde{\alpha} \cap \gamma_1$, $\tilde{\beta} \cap \gamma_1$ ($\tilde{\gamma} \cap \delta_1$, $\tilde{\delta} \cap \delta_1$ resp.) projects to an arc properly embedded in P_i (P_j' resp.). Then there is a constant $N_{i,j,\alpha,\beta,\gamma,\delta}$ such that if $m > N_{i,j,\alpha,\beta,\gamma,\delta}$, then the endpoints of $\tilde{f}^m(\tilde{\alpha})$ and $\tilde{f}^m(\tilde{\beta})$ are separated by the endpoints of $\tilde{\gamma}$ and $\tilde{\delta}$. Let $N_1 = \max \{N_{i,j,\alpha,\beta,\gamma,\delta} \mid 1 \leq i, j \leq 2g-2, \alpha, \beta, \gamma, \delta\}$. Then N_1 satisfies the conclusion of Theorem 4.2.

This completes the proof of Theorem 4.2.

Lemma 4.3. *Suppose that a Heegaard splitting $(H, H'; F)$ of M is strongly irreducible and that M contains a 2-sided incompressible*

surface S . Then there is a surface S' which is ambient isotopic to S and satisfies:

- (i) S' intersects F in transverse loops,
- (ii) every component of $S' \cap H$ is not a disk, and
- (iii) just one component of $S' \cap H'$ is a disk.

Proof. This lemma is proved by using the argument of [6, Chapter III], which was used to prove the Haken's theorem. We assume that the reader is familiar with the proof. We note that Jaco considered the case of S being the 2-sphere, but the argument works in this situation. We also note that if $(H, H'; F)$ is strongly irreducible, then M is irreducible ([3, Theorem 2.1]).

We may suppose that each component of $S \cap H$ is a disk and the number of the component is minimal among all surfaces which are ambient isotopic to S . By the minimality, we see that $S \cap H'$ is an incompressible surface in H' . Hence we have a hierarchy for $S \cap H'$, $(S_2^{(0)}, \alpha_0), (S_2^{(1)}, \alpha_1), \dots, (S_2^{(p)}, \alpha_p)$, and a sequence of isotopies of type A in M which realizes the hierarchy. Let $S^{(q)}$ ($0 \leq q \leq p+1$) be the image of S after the q -th isotopy of type A, i.e. $S^{(q)} \cap H' = S_2^{(q)}$. Let k be the integer such that $S_2^{(k)}$ does not contain a disk and $S_2^{(k+1)}$ contains a disk. We note that $S_2^{(k+1)}$ contains just one disk. Then, by Theorem 4.2 and a theorem of Casson-Gordon, we see that each component of $S^{(k+1)} \cap H$ is not a disk. Hence $S' = S^{(k+1)}$ satisfies the the conclusions of Lemma 4.3.

This completes the proof of Lemma 4.3.

Lemma 4.4. Let $f: \partial H \rightarrow \partial H$ be a pseudo-Anosov homeomorphism.

Suppose that the stable lamination L^+ of f is of full type. Then, for each $n (\geq -1)$, there exists a neighborhood U of L^+ in $PL(\partial H)$ such that the height of every simple loop contained in U is greater than n .

Proof. Assume that the conclusion of Lemma 4.3 does not hold. Then there exists an integer $n (\geq -1)$, and a sequence of simple loops $\{\ell_i\}_{i=1}^{\infty}$ on ∂H such that $\ell_i \rightarrow L^+$ ($i \rightarrow \infty$), and $h(\ell_i) \leq n$. By [10], there is a maximal train track τ on ∂H and a positive integer m such that L^+ is carried by τ with all weights positive and $f^m(\tau)$ is carried by τ . Set $B_i = B_{f^{mi}(\tau)}$ ($i \geq 1$). Then, by [10], we see that $B_{p+1} \subset \text{Int } B_p$, and B_p converges to L^+ . If necessary, by taking a subsequence of $\{\ell_i\}$, we may suppose that $\ell_i \subset \text{Int } B_i$ ($i \geq 1$). By Lemmas 3.1 and 2.2, we have a sequence of simple loops $\{\ell_i'\}_{i=1}^{\infty}$ such that $h(\ell_i') \leq n-1$, $\ell_i' \in B_i$, hence, $\ell_i' \rightarrow L^+$ ($i \rightarrow \infty$). By applying this argument finitely many times, we get a sequence of simple loops $\{\ell_i^*\}_{i=1}^{\infty}$ such that $h(\ell_i^*) = -1$ and $\ell_i^* \rightarrow L^+$ ($i \rightarrow \infty$). But this contradicts Lemma 3.4.

This completes the proof of Lemma 4.4.

Proof of Theorem 2. By Lemma 4.4, there is a neighborhood U^+ (U^- resp.) of L^+ (L^- resp.) such that the height of every simple loop in U^+ (U^- resp.) is greater than $2n+1$ (1 resp.). Let N' be an integer such that $f^m(PL(\partial H) - U^-) \subset U^+$ for each $m (> N')$. And let $N = \max\{N', N_1\}$, where N_1 is the constant obtained in Theorem 4.2. Then we will see that this N satisfies the conclusion of Theorem 2.

Assume that there is an integer $m (>N)$ such that M_{f^m} contains an incompressible surface S whose genus is less than or equal to n , i.e. $\chi(S) \geq 2-2n$. We may suppose that S satisfies the conclusions of Lemma 4.2. Let $S_1 = S \cap H$ and $S_2 = S \cap H'$. Then $\chi(S) = \chi(S_1) + \chi(S_2) \geq 2-2n$. Let D be the disk component of S_2 . Then D is the only component which makes a positive contribution in $\chi(S_1) + \chi(S_2)$. Hence we have $\chi(S_1) + \chi(S_2) \leq \chi(S_2) + 1$. Then $2-2n \leq \chi(S_2) + 1$ and $\chi(S_2) \geq 1-2n$. Let S^* be a component of S_2 . Then $\chi(S^*) \geq 1-2n$. Hence $f^m(\partial D) \not\subseteq U^+$, contradicting the fact that ∂D does not contained in U^- .

This completes the proof of Theorem 2.

5. Examples

In this section we will give ways of constructing pseudo-Anosov homeomorphisms which satisfies the assumptions of Theorems 1 and 2. Throughout this section let H be as in section 1 and T_ℓ the Dehn twist along the simple loop ℓ in ∂H .

Example 1. Let ℓ ($\subset \partial H$) be a simple loop which is of full type and m ($\subset \partial H$) a simple loop such that $\ell \cup m$ fills up ∂H , i.e. the number of intersection of ℓ and m is minimal among all simple loops which are isotopic to m and each component of $\partial H - (\ell \cup m)$ is an open disk. Then, by [4, Exposé 13], we see that $T_m \circ T_\ell^{-n}$ is isotopic to a pseudo-Anosov homeomorphism for each positive integer n and the stable lamination tends to ℓ if n grows larger. Hence $T_m \circ T_\ell^{-n}$ satisfies the assumption of Theorem 1 if n is sufficiently large.

Example 2. Let φ be a pseudo-Anosov homeomorphism such that φ^{-1} satisfies the assumption of Theorem 1, i.e. the unstable lamination L^- of φ is of full type. Let D be a component of the system of disks with respect to which L^- is of full type. Then, as observed in the proof of [9, Lemma 2.5], $\varphi^{-k} \circ T_{\partial D} \circ \varphi^k \circ T_{\partial D}^{-1}$ is isotopic to a pseudo-Anosov homeomorphism for a sufficiently large k , and the invariant laminations tend to L^- and $T_{\partial D}(L^-)$ if k grows larger. Clearly $T_{\partial D}(L^-)$ is of full type. Hence $\varphi^{-k} \circ T_{\partial D} \circ \varphi^k \circ T_{\partial D}^{-1}$ satisfies the assumption of Theorem 2 if k is sufficiently large.

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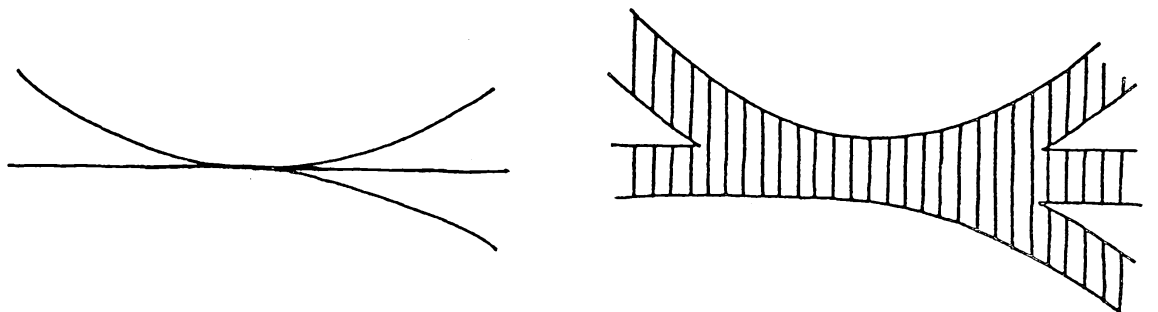


Figure 1

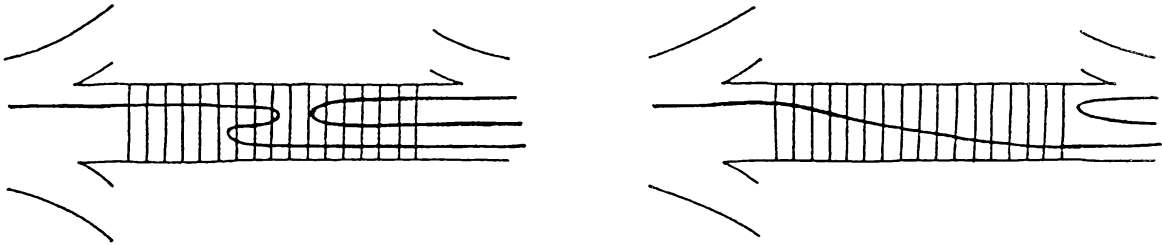


Figure 2

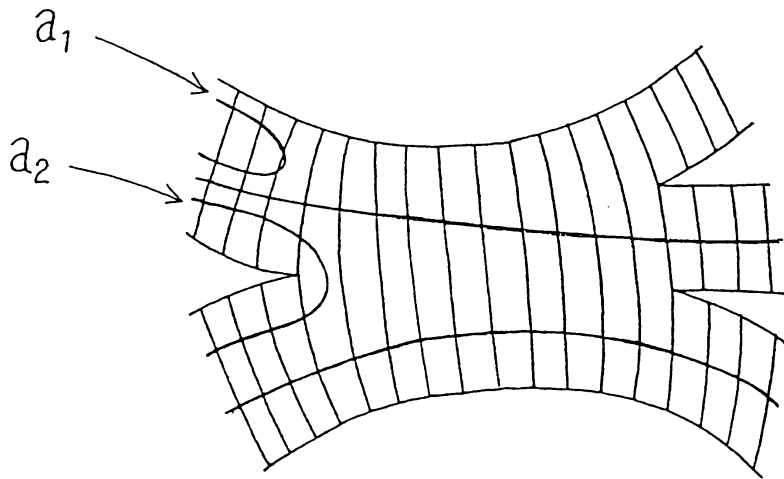


Figure 3

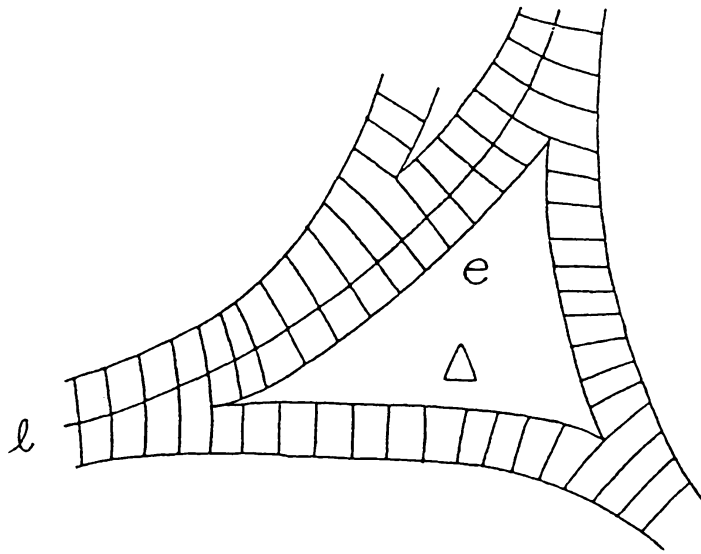


Figure 4

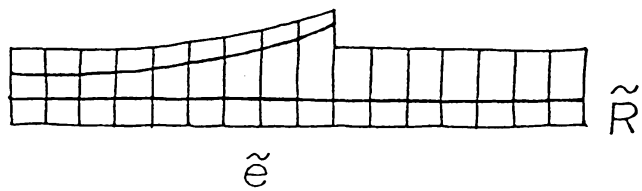


Figure 5

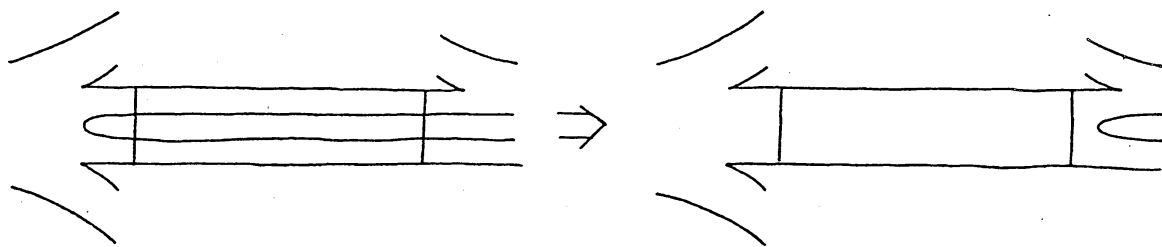


Figure 6

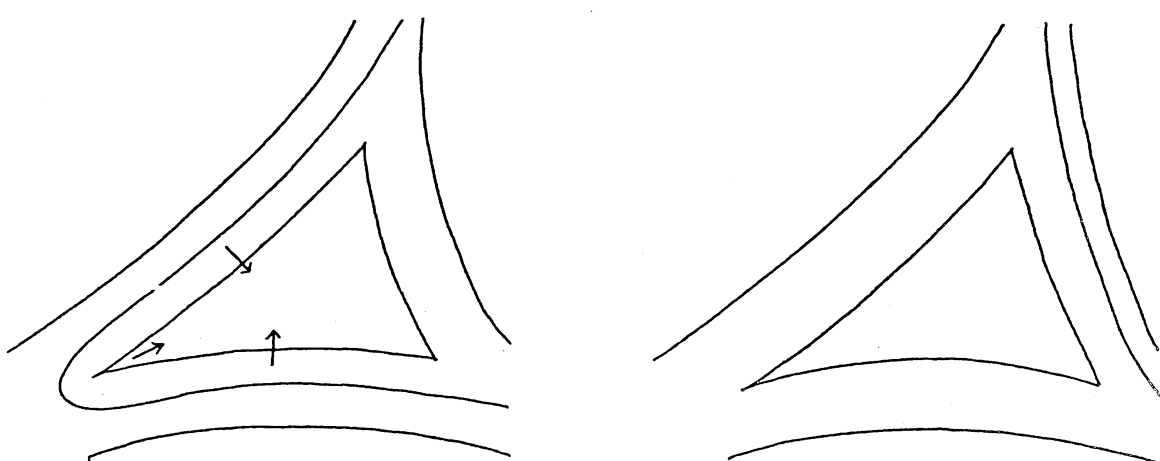


Figure 7