数理解析研究所講究録
第 628 巻 1987 年 89-138



# Rings with only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules, II

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## 1. Introduction.

The purpose of this paper is to determine the structure of one-dimensional local rings that have finite Buchsbaum-representation type and our main results are summarized into the following

Theorem (1.1). Let $R$ be a Noetherian local ring of $\dim R = 1$. Suppose that $R$ is complete and the residue class field of $R$ is infinite. Then the following two conditions are equivalent.

(1) $R$ has finite Buchsbaum-representation type, that is $R$ possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules;

(2) $R \cong P/fI$, where $P$ is a two-dimensional complete regular local ring with maximal ideal $n$, $f \in n$, and $I$ an ideal of $P$ such that $I$ contains some power of $n$, $f \notin n^3$, and $P/fP$ is reduced.

In particular, $R$ is a Cohen-Macaulay ring of finite Buchsbaum-representation type if and only if $R$ is a reduced ring of multiplicity at most 2.

As an immediate consequence of (1.1), we have

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*) Partially supported by Grant-in-Aid for Co-operative Research.

Corollary (1.2). Let $R$ be a Cohen-Macaulay complete local ring of $\dim R = 1$ and assume that $R$ contains an algebraically closed coefficient field $k$. Then $R$ has finite Buchsbaum-representation type if and only if $R$ is a simple curve singularity of type $(A_n)$, that is $R$ is isomorphic to one of the following rings:

$k[[X,Y]]/(X^2 + XY^n)$ $(n \geq 0)$,

$k[[X,Y]]/(X^2 + Y^{2n+1})$ $(n \geq 1)$,

$k[[X,Y]]/(X^2 + XY^{n+i} + Y^{2n+1})$ $(1 \leq i < n$, $\operatorname{ch} k = 2)$.

Accordingly, combining (1.2) and the main result of the previous paper [10] of the author and K. Nishida, one knows all the Cohen-Macaulay complete local rings $R$ of $\dim R \geq 1$ that have finite Buchsbaum-representation type, provided the rings $R$ contain algebraically closed coefficient fields $k$ of $\operatorname{ch} k \neq 2$. But, before going into the detail, let us recall some basic notion.

Let $R$ be a Noetherian local ring and $M$ a finitely generated R-module. Then $M$ is said to be Buchsbaum, if the difference

$$I_R(M) = l_R(M/qM) - e_q(M)$$

is an invariant of $M$ which does not depend on the particular choice of a parameter ideal $q$ for $M$ (here $l_R(M/qM)$ and $e_q(M)$ respectively denote the length of $M/qM$ and the multiplicity of $M$ relative to $q$). Consequently, $M$ is Cohen-Macaulay if and only if $M$ is Buchsbaum and $I_R(M) = 0$. A Buchsbaum R-module $M$ is called maximal, if $\dim_R M = \dim R$. The ring $R$ is said to be a Buchsbaum ring, if $R$ is a Buchsbaum module over itself.

The notion of Buchsbaum ring was introduced by W. Vogel [21], while he studied a problem posed by D. A. Buchsbaum [4]. Nowadays it has been recognized that Buchsbaum rings and modules behave themselves as well as the Cohen-Macaulay ones and the researches on Buchsbaum rings and modules are gathered into the monumental book [20] of J. Stückrad and W. Vogel, which the readers may consult for the general references too.

Let us say that a Noetherian local ring R has finite Buchsbaum-representation type (resp. finite CM-representation type), if R possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum (resp. Cohen-Macaulay) modules. Inspired by the recent drastic progress of the research on Cohen-Macaulay local rings possessing finite CM-representation type (see [1, 2, 3, 5, 8, 11, 14, 15, 18, 22] etc.), the rings of finite Buchsbaum-representation type have begun to be explored. The fundamental theorem is due to D. Eisenbud and the author [7] (see [9] also), which claims that any regular local ring R has finite Buchsbaum-representation type —— more precisely, the syzygy modules of the residue class field of R are the representatives of indecomposable maximal Buchsbaum R-modules and any maximal Buchsbaum R-module is a direct sum of them. Subsequently, by the help of the recent results of [5] and [15] concerning maximal Cohen-Macaulay modules on simple hypersurface singularities, the author and K. Nishida [10] succeeded in showing that the converse of our fundamental theorem is also true, provided R is a Cohen-Macaulay complete local ring of $\dim R \geq 2$ and R contains an algebraically closed coefficient field k of $\operatorname{ch} k \neq 2$ . It would be quite interesting, if one can remove (or replace by a weaker one) the assumption in [10] that R is Cohen-Macaulay. But the one-dimensional case seems to be a more urgent theme, which we have chosen as the target of the present paper.

In the one-dimensional case one must be slightly careful, because non-regular Cohen-Macaulay rings may have finite Buchsbaum-representation type. The first example was given by [10, Theorem (5.3)] and the ring

$$k[[X,Y]]/(X^2 + Y^3)$$

is, where $X$, $Y$ are indeterminates over a field $k$. However as is stated in our corollary (1.2), such rings as $k[[X,Y]]/(X^2 + Y^3)$ are in some sense the only one-dimensional Cohen-Macaulay local rings possessing finite Buchsbaum-representation type. A general structure theorem is now supplied by our theorem (1.1) for any one-dimensional local rings to have finite Buchsbaum-representation type.

Let us now explain how to organize this paper. The last assertion of Theorem (1.1) will be proved in Sections 3, 4 and 5. We will prove the implication (1) => (2) (resp. (2) => (1)) of Theorem (1.1) in Section 6 (resp. Section 7). Section 2 is devoted to some preliminary steps, which we need in the proof of Theorem (1.1).

Throughout this paper let $R$ be a Noetherian local ring with maximal ideal $m$ and $\dim R = 1$. Let $H^i_m(.)$ denote the $i$-th local cohomology functor of $R$ relative to $m$. For each finitely generated R-module $M$, let $\mu_R(M)$ denote the number of elements in a minimal system of generators for $M$.

## 2. Preliminaries.

To begin with we note

Lemma (2.1). Let $M$ be a finitely generated R-module. Then

(1) $M$ is a maximal Buchsbaum R-module if and only if

$$\dim_R M = 1 \quad \text{and} \quad m.H_m^0(M) = (0) .$$

(2) Suppose that $M$ is an indecomposable maximal Buchsbaum R-module. Then $H_m^0(M)$ is contained in $mM$ .

Proof. (1) See [19, Proposition 15].

(2) See [10, Proof of Claim in Theorem (5.3)].

The next result is due to [2], when $R$ is complete.

Proposition (2.2). Let $R$ be a Cohen-Macaulay ring. Then $R$ is reduced, if $R$ has finite CM-representation type.

Proof. Let $p$ be a minimal prime ideal of $R$ and let $p^{(2)} = p^2R_p \cap R$ . Then as $R/p^{(2)}$ has finite CM-representation type, passing to the ring $R/p^{(2)}$ , we may assume that $p$ is a <u>unique</u> minimal prime ideal of $R$ with $p^2 = (0)$ . Choose a regular element $x$ of $R$ and put

$$I_i = x^iR + p$$

for each $i \geq 1$ . Then since $I_i$ is an indecomposable maximal Cohen-Macaulay R-module, there must be an isomorphism $\phi : I_i \to I_j$ for some $1 \leq i < j$ .

Claim. $\phi(p) = p$ and $\phi(x^i) \equiv \varepsilon x^j \mod p$ with a unit $\varepsilon$ of $R$.

For, let $f \in p$ . Then as $f.\phi(f) = \phi(f^2) = 0$ , the element $\phi(f)$ is a zerodivisor of $R$ . Hence $\phi(f) \in p$ and so we get $\phi(p) = p$ . Because $I_j = \phi(x^i)R + p$ , the second assertion is clear.

Now let $f \in p$ . Then as

$$x^i.\phi(f) = f.\phi(x^i) = \varepsilon f.x^j$$

by the claim, we get $x^i p = x^i \phi(p) = x^j p$ . Hence $p = x^{j-i} p$ , which implies (by Nakayama's lemma) that $p = (0)$ .

Lemma (2.3) (cf. [10, Proposition (5.2)]). Let $L$ be a maximal Cohen-Macaulay R-module and let

$$0 \to M \to F \to L \to 0$$

denote the initial part of a minimal free resolution of $L$ . Then

(1) For any R-submodule $N$ of $M$ containing $mM$ , the R-module $F/N$ is a maximal Buchsbaum module and $H_m^0(F/N) = M/N$ . If $L$ is indecomposable, then so is $F/N$ .

(2) Let $N$ and $N'$ be R-submodules of $M$ containing $mM$ . Then $F/N \cong F/N'$ if and only if $\phi(N) = N'$ for some automorphism $\phi$ of $F$ . When this is the case, one always has $\phi(M) = M$ too.

Proof. (1) Consider the exact sequence

$$0 \to M/N \to F/N \to L \to 0$$

and we get $H_m^0(F/N) = M/N$ , since $m.(M/N) = (0)$ (and since $L$ is Cohen-Macaulay). So $F/N$ is, by (2.1), a maximal Buchsbaum R-module. Assume that $F/N = A_1 \oplus A_2$ for some non-zero R-submodules $A_1$ and $A_2$ . If $\dim_R A_i = 1$ for any $i = 1, 2$ , then the isomorphisms

$$L \cong (F/N)/H_m^0(F/N) \cong A_1/H_m^0(A_1) \oplus A_2/H_m^0(A_2)$$

claim that $L$ is decomposable. If $\dim_R A_i = 0$ for some $i$ , say $i = 2$ , then $A_2$ is contained in $H_m^0(F/N)$ and so $L$ is a homomorphic image of $A_1$ . Hence $\mu_R(L) \leq \mu_R(A_1)$ —— this is impossible, because $\mu_R(L) = \mu_R(A_1) + \mu_R(A_2)$ and $\mu_R(A_2) \geq 1$ . Thus $F/N$ is indecomposable, if so is $L$ .

(2) The first assertion is standard. Let us check the second one and let $\overline{\phi} : F/N \to F/N'$ denote the isomorphism induced by $\phi$ . Then as $H_m^0(.)$ is a functor, we get

$$\overline{\phi}(H_m^0(F/N)) = H_m^0(F/N') ,$$

whence $\phi(M) = M$ because $H_m^0(F/N) = M/N$ and $H_m^0(F/N') = M/N'$ .

Corollary (2.4). Suppose that $R$ has finite Buchsbaum-representation type and that the field $R/m$ is infinite. Let $I$ be an ideal of $R$ such that $R/I$ is a Cohen-Macaulay ring of $\dim R/I = 1$ . Then $\mu_R(I) \leq 1$ and $I = \sqrt{I}$ .

Proof. Let $\underline{F}$ denote the set of ideals $J$ in $R$ satisfying $mI \subset J \subset I$ . Then for each $J$ in $\underline{F}$ , applying (2.3) to the exact sequence $0 \to I \to R \to R/I \to 0$ , we get that $R/J$ is an indecomposable maximal Buchsbaum R-module. Hence the set $\underline{F}$ must be finite, which forces $\mu_R(I) \leq 1$ as $R/m$ is infinite. See (2.2) for the second assertion.

Let us recall the following

Definition (2.5). Let $K$ be an R-module. Then $K$ is called a canonical module of $R$ and denoted by $K_R$ , if

$$\hat{R} \otimes_R K \cong \mathrm{Hom}_R(H_m^1(R),E)$$

as $\hat{R}$-modules (here $\hat{R}$ (resp. $E$ ) denotes the completion of $R$ (resp. the injective envelope of $R/m$ )).

The canonical module of $R$ is uniquely (up to isomorphisms) determined by $R$ , if it exists. When $R$ is a homomorphic image of a Gorenstein local ring $S$ , $R$ has a canonical module $K_R = \mathrm{Ext}_S^t(R,S)$ ( $t = \dim S - 1$ ) (cf. [13, Satz 5.12]). Various properties of canonical modules are discussed in [13]. Let us summarize below some of them, which we shall use in the proof of (2.7).

Proposition (2.6) ([13]). Let $R$ be a Cohen-Macaulay ring possessing a canonical module $K_R$ . Then

(1) $K_R$ is an indecomposable maximal Cohen-Macaulay R-module with

$$\mathrm{Ext}_R^i(K_R,K_R) = R \quad ( i = 0 ), \qquad = (0) \quad ( i > 0 ).$$

(2) For any prime ideal $p$ of $R$ the local ring $R_p$ has the canonical module $K_{(R_p)} = (K_R)_p$ .

(3) The following conditions are equivalent: (a) $R$ is a Gorenstein ring, (b) $\mu_R(K_R) = 1$ , and (c) $K_R$ is free.

Let $v(R)$ denote the embedding dimension of $R$ .

Theorem (2.7). Let $R$ be a Cohen-Macaulay ring of finite Buchsbaum-representation type. Suppose that $R$ possesses a canonical module $K_R$ and that the field $R/m$ is infinite. Then $v(R) \leq 2$ .

Proof. As $R$ has finite CM-representation type, we have only to show that $R$ is a Gorenstein ring (cf. [12, Satz 1.2]) — so it suffices to check that $\mu_R(K_R) = 1$ (cf. (2.6) (3)).

Let

$$0 \to M \xrightarrow{i} F \to K_R \to 0$$

denote the initial part of a minimal free resolution of $K_R$. First of all we will show that

Claim 1. $\mu_R(M) \leq 1$.

For, assume $\mu_R(M) \geq 2$ and choose elements $f$, $g$ of $M$ so that the classes $\overline{f}$, $\overline{g}$ of $f$, $g$ in $M/mM$ are linearly independent over $R/m$. For each $\lambda \in R/m$, let $c_\lambda \in R$ with $\lambda = c_\lambda$ mod $m$ and put

$$N_\lambda = mM + Rh_\lambda ,$$

where $h_\lambda = f + c_\lambda g$. Then $F/N_\lambda$'s are indecomposable maximal Buchsbaum R-modules by (2.3)(1) (cf. (2.6)(1) too). Hence there must be an isomorphism

$$F/N_\lambda \cong F/N_\mu$$

for some $\lambda$, $\mu \in R/m$ with $\lambda \neq \mu$. Take an automorphism $\phi$ of $F$ so that

$$\phi(N_\lambda) = N_\mu \quad \text{and} \quad \phi(M) = M$$

(cf. (2.3)(2)). Let $\overline{\phi}$ denote the automorphism of $K_R$ induced from $\phi$ and we write

$$\overline{\phi} = \varepsilon 1_{K_R}$$

with a unit $\varepsilon$ of $R$ (cf. (2.6)(1)). Then as both the automorphisms $\phi$ and $\varepsilon 1_F$ lift $\overline{\phi}$, we have

$$\phi = \varepsilon 1_F + i \circ \delta$$

with a homomorphism $\delta : F \to M$ (here $i : M \to F$ denotes the inclusion map). Notice that $\delta(M) \subset mM$, as $M \subset mF$. Then we get

$$\phi(h_\lambda) \equiv \varepsilon h_\lambda \mod mM ,$$

whence $h_\lambda \in N_\mu$ because $\phi(h_\lambda) \in N_\mu$ and $\varepsilon \notin m$. Consequently we see $\overline{f} + \lambda\overline{g} \in R/m.(\overline{f} + \mu\overline{g})$, which forces $\lambda = \mu$ as (by our choice) $\overline{f}$ and $\overline{g}$ are linearly independent over $R/m$ —— this is a contradiction.

We put $r = \mu_R(K_R)$ . Let us assume that $r \geq 2$ . Then $M \neq (0)$ by (2.6)(3), whence $\mu_R(M) = 1$ by Claim 1. Write

$$M \cong R/I$$

with an ideal $I$ of $R$ —— so $R/I$ is a Cohen-Macaulay ring of $\dim R/I = 1$ . Let $p \in \mathrm{Ass}_R R/I$ . Then since $IR_p = (0)$ and since $(K_R)_p = R_p$ by (2.6)(2) and (3) (recall that $R$ is a reduced ring, cf. (2.2)), we readily get by the exact sequence

$$(*) \quad 0 \to R/I \to F \to K_R \to 0$$

that

Claim 2. $r = 2$ .

Now take the $K_R$-dual of the sequence (*). Then because $\mathrm{Hom}_R(R/I, K_R)$ is a canonical module $K_{R/I}$ of $R/I$ (cf. [13, Satz 5.12]), we have an exact sequence of the following form:

$$(**) \quad 0 \to R \to K_R \oplus K_R \to K_{R/I} \to 0$$

(cf. (2.6)(1)). Notice that $R/I$ is also a Cohen-Macaulay ring of finite Buchsbaum-representation type. Then we get by Claim 2 that $\mu_R(K_{R/I}) \leq 2$ too, whence $2r \leq 3$ by the exact sequence (**) — this is of course impossible, since $r = 2$ by Claim 2. Thus $\mu_R(K_R) = 1$ , as desired.

For the rest of this section let $P$ denote a regular local ring of $\dim P = 2$ and assume that

$$R = P/fP$$

with an element $f$ of $P$ .

We note

Proposition (2.8) ([6] and [12, Lemma 1.3]). Let $M$ be an indecomposable maximal Cohen-Macaulay R-module such that $M \not\cong R$ . Then the minimal free resolution of $M$ is periodic of period 2 and the first syzygy module of $M$ is indecomposable too.

The next corollary is fairly obvious. However its use is so crucial that let us give a proof for completeness.

Corollary (2.9). Let $L$ be a maximal Cohen-Macaulay R-module with no free direct summand. Let

$$0 \to M \to F \to L \to 0$$

denote the initial part of a minimal free resolution of $L$ . Then $\mu_R(M) = \mu_R(L)$ , and any automorphism of $M$ can be extended to that of $F$ .

Proof ([6]). Let

$$0 \to F_1 \overset{\phi}{\to} F_0 \overset{\varepsilon}{\to} L \to 0$$

be a minimal free resolution of the P-module $L$ . Then as $fF_0$ is contained in $\phi(F_1)$ , we have a (unique) homomorphism $\psi : F_0 \to F_1$ with $\phi \circ \psi = f1_{F_0}$ . Notice that $\psi \circ \phi = f1_{F_1}$ too. Let $\overline{F}_i = F_i/fF_i$ ( $i = 0, 1$ ) and let $\overline{\phi} : \overline{F}_1 \to \overline{F}_0$ denote the homomorphism induced by $\phi$ . Then a simple use of the snake lemma yields an exact sequence

$$(*) \quad 0 \to L \overset{\partial}{\to} \overline{F}_1 \overset{\overline{\phi}}{\to} \overline{F}_0 \to L \to 0$$

of R-modules, where

$$\partial(\varepsilon(x)) = \psi(x) \mod fF_1$$

for each element $x$ of $F_0$ . Notice that $\partial(L) \subset m.\overline{F}_1$ (otherwise, $L$ contains $R$ as a direct summand) and we have the sequence (*) to be part of a minimal free resolution of $L$ . Hence

$\mu_R(M) = \mu_R(L)$ . Let us identify $F = \overline{F}_0$ and $M = \text{Image } \overline{\phi}$ .

Let $\xi$ be any automorphism of $M$ . Then because the P-module $M$ admits a minimal free resolution

$$0 \to F_0 \overset{\psi}{\to} F_1 \to M \to 0 ,$$

we may choose automorphisms $F_0 \overset{\beta}{\to} F_0$ and $F_1 \overset{\alpha}{\to} F_1$ so that $\alpha$ lifts $\xi$ and $\alpha \circ \psi = \psi \circ \beta$ .

Claim. $\phi \circ \alpha = \beta \circ \phi$ .

For, first recall that $\psi \circ \phi = f1_{F_1}$ . Then

$$\begin{aligned} \psi \circ (\phi \circ \alpha) &= f\alpha \\ &= \alpha \circ (\psi \circ \phi) \\ &= \psi \circ (\beta \circ \phi) , \end{aligned}$$

whence we get $\phi \circ \alpha = \beta \circ \phi$ .

Let $\overline{\alpha}$ (resp. $\overline{\beta}$ ) denote the automorphism of $\overline{F}_1$ (resp. $\overline{F}_0$) induced from $\alpha$ (resp. $\beta$ ). Then by the above claim, the square

$$\begin{array}{ccc} \overline{F}_1 & \xrightarrow{\overline{\phi}} & \overline{F}_0 \\ \downarrow \overline{\alpha} & & \downarrow \overline{\beta} \\ \overline{F}_1 & \xrightarrow{\overline{\phi}} & \overline{F}_0 \end{array}$$

is commutative and therefore, since $M = \text{Image } \overline{\phi}$ and the automorphism $\overline{\alpha}$ of $\overline{F}_1$ lifts $\xi$ , we get that the automorphism $\overline{\beta}$ of $F = \overline{F}_0$ is a required extension of $\xi$ .

3. Cohen-Macaulay rings of finite Buchsbaum-representation type, I.

The purpose of this section is to prove the following

Theorem (3.1). Let $R$ be a Cohen-Macaulay complete local ring with infinite residue class field. Suppose that R has finite Buchsbaum-representation type. Then R is a reduced ring and the multiplicity e(R) of R is at most 2 .

In this theorem the assertion that R is a reduced ring is already known (cf. (2.2)). Because, by virtue of (2.7), our ring R is a homomorphic image of a two-dimensional regular local ring, the assertion that $e(R) \leqq 2$ immediately follows from the next

Theorem (3.2). Let P be a regular local ring of dim P = 2 and assume that R = P/fP with an element f of P . Let $\overline{R}$ denote the integral closure of R in its total quotient ring. If the R-module $\overline{R}$ is finitely generated and if $e(R) \geqq 3$ , there exists a family $\{M_\lambda\}_{\lambda \in R/m}$ of indecomposable maximal Buchsbaum R-modules such that

$$M_\lambda \not\cong M_\mu \quad \text{for} \quad \lambda \neq \mu .$$

We divide the proof of Theorem (3.2) into several steps. Let R be as in (3.2). Assume that $\overline{R}$ is module-finite over R and $e(R) \geqq 3$ . Let

$$A = \{ x \in \overline{R} \mid xm \subset m \},$$

which we shall identify with the endomorphism ring $\mathrm{Hom}_R(m,m)$ of m .

First of all we note

Lemma (3.3). R is a reduced ring and $l_R(A/R) \leqq 1$ .

Proof. Apply two functors $\mathrm{Hom}_R(m,.)$ and $\mathrm{Hom}_R(.,R)$ to the canonical exact sequence

$$0 \to m \to R \to R/m \to 0 .$$

Then we get a commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & & \mathrm{Ext}^1_R(R/m,R) & & \\
 & & & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & \mathrm{Hom}_R(m,R) & \longrightarrow & \mathrm{Hom}_R(m,R/m) \\
 & & \uparrow i & & \uparrow & & \\
 & & R & = & \mathrm{Hom}_R(R,R) & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}$$

with exact rows and columns, where $i : R \longrightarrow A$ denotes the inclusion map. As $R$ is Gorenstein, we have

$$\mathrm{Ext}^1_R(R/m,R) \cong R/m$$

whence the inequality $l_R(A/R) \leqq 1$ follows. See [13, Proof of Satz 3.6] for the second assertion.

Let $J$ (resp. $\underline{c} = R : \overline{R}$ ) denote the Jacobson radical (resp. the conductor ideal) of $\overline{R}$ . Since $\overline{R}$ is module-finite over $R$ , the ideal $\underline{c}$ contains some power $J^n$ of $J$ —— take $n$ as small as possible. Then $J^{n-1} \not\subset R$ and we may choose an element $h$ of $J^{n-1}$ so that $h \notin R$ . Because $hJ \subset J^n \subset \underline{c}$ , we have $hm \subset \underline{c} \subset m$ whence $h \in A$ . Thus by (3.3) we get the first part of the following

Proposition (3.4). $A = R + Rh$ and $hm \subset m^2$ .

Proof. As $hm \subset \underline{c}$ , to check that $hm \subset m^2$ it suffices to show that $\underline{c} \subset m^2$ . Assume $\underline{c} \not\subset m^2$ . Then as $v(R/\underline{c}) \leqq 1$ , $R/\underline{c}$ is a Gorenstein local ring of $\dim R/\underline{c} = 0$ . Accordingly, $\overline{R}/R$ contains $R/\underline{c}$ as a submodule and so we have an isomorphism

$$\overline{R}/R \cong R/\underline{c} ,$$

because

$$l_R(\overline{R}/R) = l_R(R/\underline{c})$$

(cf., e.g., [13, Korollar 3.5]). In particular $\mu_R(\overline{R}/R) = 1$ , which forces $\mu_R(\overline{R}) \leqq 2$ . However this contradicts our hypothesis that $e(R) \geqq 3$ , because $e(R) = \mu_R(\overline{R})$ (recall that $R$ is a reduced ring, cf. (3.3)). Thus $\underline{c} \subset m^2$ .

The proof of the next assertion is standard.

Corollary (3.5). Let $a$ , $b$ be elements of $R$ . Then $a + bh$ is a unit of $A$ if and only if $a$ is a unit of $R$ .

Let $L$ be the first syzygy module of $m$ . Then because $m$ is indecomposable (since the ring $A$ is local, cf. (3.5)), by (2.8) $L$ is indecomposable too. Furthermore we get again by (2.8) an exact sequence

$$0 \to m \to F \to L \to 0$$

of R-modules with $F$ free of rank 2 . Now let $x$ , $y$ be a (minimal) system of generators for $m$ . For each $\lambda \in R/m$ , choose $c_\lambda \in R$ so that $\lambda = c_\lambda \bmod m$ and put

$$N_\lambda = m^2 + Rz_\lambda ,$$

where $z_\lambda = x + c_\lambda y$ . Then by (2.3)(1) the R-modules $M_\lambda = F/N_\lambda$ are indecomposable maximal Buchsbaum modules.

Proposition (3.6). $\lambda = \mu$ , if $M_\lambda \cong M_\mu$ .

Proof. Assume that $M_\lambda \cong M_\mu$. Then we may choose, by (2.3) (2), an automorphism $\psi$ of m so that $\psi(N_\lambda) = N_\mu$. Write $\psi = a + bh$ with $a, b \in R$. Then a is a unit of R (cf. (3.5)). Furthermore as $hm \subset m^2$ by (3.4), we have

$$\psi(z_\lambda) \equiv az_\lambda \mod m^2 .$$

Hence $z_\lambda \in N_\mu$, which guarantees that $\lambda = \mu$ because x , y form a minimal system of generators for m . This completes the proof of Theorem (3.2).

4. Cohen-Macaulay rings of finite Buchsbaum-representation type, II.

Let R be a reduced complete local ring of $e(R) = 2$ and assume that $m^2 = ym$ for some $y \in m$. (Such an element y must exist, when the field R/m is infinite; see, e.g., [17].) The purpose of this section (and the next section too) is to prove the following

Theorem (4.1). R has finite Buchsbaum-representation type.

Notice that the last assertion in Theorem (1.1) follows from Theorems (3.1) and (4.1).

To begin with we note

Lemma (4.2). Let M be an indecomposable maximal Cohen-Macaulay R-module such that $M \ncong R$. Then for any $x \in m$ such that $m = (x, y)$, M can be regarded as an R[x/y]-module.

Proof. See the proof of [12, Satz 1.6, a)].

Now let $\overline{R}$ (resp. $J$ ) denote the normalization of R (resp. the Jacobson radical of $\overline{R}$ ). First we consider the case where R is an integral domain. Let v denote the discrete valuation of $\overline{R}$ . Then as

$$e(R) = v(y).l_R(\overline{R}/J) ,$$

we have the following two cases:

(I) $v(y) = 2$ and $R/m = \overline{R}/J$ ,

(II) $v(y) = 1$ and $l_R(\overline{R}/J) = 2$ .

In this section we mainly deal with the case (I) —— the case (II) and the case where R is not an integral domain shall be postponed to the next section.

Let us now assume that $v(y) = 2$ and $R/m = \overline{R}/J$ .

Lemma (4.3). The ring R contains an element x such that $m = (x, y)$ and $v(x) = 2n + 1$ ( $n \geqq 1$).

Proof. Suppose that $v(x)$ is even for any $x \in R$ such that $m = (x, y)$. Let $v(x) = 2n$ and write $x = zy^n$ with $z \in \overline{R}$ . Then as $z = c + z'$ ( $c \in R$ , $z' \in J$ ), letting $x' = z'y^n$ we get $x = cy^n + x'$ —— hence $x' \in R$ and $m = (x', y)$. Because

$$v(x') = 2n + v(z') > v(x) ,$$

repeating this argument we have a sequence $\{x_i\}_{i \geq 1}$ of elements in R that satisfies $m = (x_i, y)$ and $v(x_i) < v(x_{i+1})$ for any $i \geqq 1$ . Choose $n \geqq 1$ so that the conductor ideal $\underline{c} = R : \overline{R}$ contains $J^n$ . Then as $m \supset \underline{c}$ , we get $x_i \in m^2$ for any $i \geqq 1$ such that $v(x_i) \geqq 2n$ , whence $m = yR$ —— this is a contradiction.

Let $x \in R$ be as in (4.3) and write $v(x) = 2n + 1$ ( $n \geqq 1$ ).

Let $t = x/y^n$ (hence $J = t\overline{R}$ , as $v(t) = 1$ ) and

$$R_i = R[ty^i] \subset \overline{R}$$

for each $0 \leqq i \leqq n$ . Then $\overline{R} = R + Rt$ , $R_0 = \overline{R}$ , and $R_n = R$ . We write $t^2 = a + bt$ with $a$ , $b \in m$ and denote by $m_i$ the maximal ideal of $R_i$ ( $0 \leqq i \leqq n$ ).

Proposition (4.4). Let $1 \leqq i \leqq n$ . Then

(1) $m_i = (ty^i, y)R_i$ .

(2) $e(R_i) = 2$ and $m_i^2 = ym_i$ .

Proof. (1) As $v(ty^i) \geqq 1$ , $ty^i \in m_i$ whence $m_i = mR_i + ty^iR_i$ . Since $x = ty^i.y^{n-i}$ , we get $m_i = (y, ty^i)R_i$ .

(2) Because $m_i\overline{R} = y\overline{R} = J^2$ by (1), we get $e(R_i) = \mu_{R_i}(\overline{R}) = 2$ . As $t^2 = a + bt$ , $(ty^i)^2 = ay^{2i} + ty^i.by^i$ , whence $(ty^i)^2 \in ym_i$ and so $m_i^2 = ym_i$ .

We get $mR_i = yR_i$ for each $0 < i < n$ , since $x = ty^i.y^{n-i}$ . Therefore by (4.4) we have

Corollary (4.5). Let $0 < i < n$ . Then $ty^i \notin mR_i$ but $(ty^i)^2 \in mR_i$ . Hence $\mu_R(R_i) = 2$ and $R_i = R + Rty^i$ .

Corollary (4.6). Let $0 \leqq i \leqq j \leqq n$ . Then the conductor ideal $R_j : R_i$ contains $y^{j-i}$ and $ty^j$ .

Proof. As $R_i = R_j + R_jty^i$ by (4.5), the assertion $y^{j-i} \in R_j : R_i$ is obvious. Because $ty^i.ty^j = ay^{i+j} + ty^j.by^i \in R_j$ , we get $ty^j$ is in the conductor too.

Proposition (4.7). The ring $R$ has finite CM-representation type and the R-modules $R_i$ $(0 \leq i \leq n)$ are the representatives of indecomposable maximal Cohen-Macaulay R-modules.

Proof (cf. [12, p. 26, Bemerkungen b)]). Let $M$ be an indecomposable maximal Cohen-Macaulay R-module such that $M \not\cong R_i$ for any $1 \leq i \leq n$. Then as $M \not\cong R$, by (4.2) we may consider $M$ to be an $R_{n-1}$-module. Notice that the $R_{n-1}$-module $M$ is again an indecomposable maximal Cohen-Macaulay module (cf. [11, Lemma 1]) and $M \not\cong R_i$ as $R_{n-1}$-modules for any $1 \leq i \leq n-1$. Therefore because by (4.4) the ring $R_{n-1}$ satisfies the same standard assumption as that of $R = R_n$, we can repeat the above argument to conclude that $M$ is an indecomposable maximal Cohen-Maculay $R_0$-module. Hence $M \cong R_0$, as $R_0 = \overline{R}$ is a discrete valuation ring.

To see that $R_i \not\cong R_j$ as R-modules for $i < j$, it suffices to check that $R_i \neq R_j$ if $i < j$ (cf. [11, Lemma 1]). Assume the contrary and we get $R_{j-1} = R_j$ for some $1 \leq j \leq n$. Then $m_j = yR_j$ by (4.4)(1), since $ty^{j-1} \in R_j$ —— this contrdicts (4.4)(2).

Corollary (to the proof) (4.8). $J^{2n-1} \not\subset m$ but $J^{2n} \subset m$.

Proof. As $ty^{n-1} \notin R = R_n$, we see $J^{2n-1} \not\subset m$. Because $J^{2n} = y^n\overline{R}$ and $y^n\overline{R} = y^nR + ty^nR$, we get $J^{2n} \subset (y^n, x)R$.

Lemma (4.9). Let $0 \leq i < n$. Then there exists an exact sequence $0 \longrightarrow R_i \xrightarrow{\sigma_i} R^2 \xrightarrow{\varepsilon_i} R_i \longrightarrow 0$ of R-modules such that

$$\sigma_i(1) = \begin{pmatrix} x \\ y^{n-i} \end{pmatrix} \quad , \quad \sigma_i(ty^i) = \begin{pmatrix} bxy^i + ay^{n+i} \\ x \end{pmatrix} ,$$

$$\varepsilon_i(e_1) = 1 \text{ , and } \varepsilon_i(e_2) = -\, ty^i ,$$

where $e_1$, $e_2$ denote the standard basis of $R^2$ .

Proof. As $\mu_R(\mathrm{Ker}\ \varepsilon_i) = 2$ (cf. (2.9) and (4.5)), we have $\mathrm{Ker}\ \varepsilon_i$ to be generated by $\begin{pmatrix} x \\ y^{n-i} \end{pmatrix}$ and $\begin{pmatrix} bxy^i + ay^{n+i} \\ x \end{pmatrix}$ . Because

$$\begin{pmatrix} x & -\, bxy^i - ay^{n+i} \\ y^{n-i} & -\, x \end{pmatrix}^2 = 0 ,$$

a monomorphism $\sigma_i : R_i \longrightarrow R^2$ is induced so that $\sigma_i(1) = \begin{pmatrix} x \\ y^{n-i} \end{pmatrix}$ and $\sigma_i(ty^i) = \begin{pmatrix} bxy^i + ay^{n+i} \\ x \end{pmatrix}$ .

For each $0 \leqq i < n$ we define

$$M_{i1} = R_i ,$$
$$M_{i2} = R^2/\sigma_i(mR_i + R) ,$$
$$M_{i3} = R^2/\sigma_i(m_i) ,$$
$$M_{i4} = R^2/\sigma_i(mR_i) .$$

Then by (2.3) $M_{ij}$'s are indecomposable maximal Buchsbaum R-modules with

$$l_R(H_m^0(M_{ij})) = 0 \quad ( j = 1 ),$$
$$= 1 \quad ( j = 2, 3 ),$$
$$= 2 \quad ( j = 4 ).$$

We furthermore have

Theorem (4.10). $R$ has finite Buchsbaum-representation type and the R-modules $M_{ij}$ $(0 \leq i < n \, , \, 1 \leq j \leq 4)$ and $R$ are the representatives of indecomposable maximal Buchsbaum modules.

To prove Theorem (4.10) we need one more lemma. Let $0 \leq i_1 \leq i_2 \leq \ldots \leq i_r < n$ be integers and $I = \{ i_\alpha \mid 1 \leq \alpha \leq r \}$. We write

$$I = \{j_1, j_2, \ldots , j_q\} \quad \text{with} \quad j_1 < j_2 < \ldots < j_q \, .$$

For each $1 \leq \beta \leq q$ , let $r_\beta = \#\{ \alpha \mid 1 \leq \alpha \leq r$ such that $i_\alpha = j_\beta\}$. We put

$$L = \bigoplus_{\alpha = 1}^{r} R_{i_\alpha} \quad \text{and} \quad \overline{L} = \bigoplus_{\alpha = 1}^{r} R_{i_\alpha}/mR_{i_\alpha}$$

and regard each element of $L$ (resp. $\overline{L}$ ) as a column vector with entries in $R_{i_\alpha}$ (resp. $R_{i_\alpha}/mR_{i_\alpha}$ ). Let $v_j$ $(1 \leq j \leq s)$ be elements of $\overline{L}$ and put $U = \sum_{j=1}^{s} kv_j$ , where $k = R/m$ .

Lemma (4.11). By some automorphism of $\overline{L}$ induced from that of $L$ , $U$ is mapped onto the k-subspace $U'$ of $\overline{L}$ which is spanned by the columns of an $r$ by $s$ matrix of the following form

$$\left(\begin{array}{ccc|c|c|c|c}
 & & & A_1 & 0 & 0 & 0 \\ \cline{4-7}
 & 0 & 1 & 0 & A_2 & 0 & 0 \\ \cline{4-7}
 & & \cdot & 0 & 0 & \ddots & 0 \\
 & \cdot & & & & & \\
1 & & 0 & & & & \\ \cline{4-7}
 & & & 0 & 0 & 0 & A_q
\end{array}\right)$$

where the submatrix $A_\beta$ $(1 \leq \beta \leq q)$ consists of $r_\beta$ rows and the entries of $A_\beta$ are non-units of $R_{j_\beta}/mR_{j_\beta}$ .

Proof. First of all let us introduce three kinds of permissible automorphisms of $\bar{L}$, which we utilize as elementary row transformations on the matrix $A = (v_1, v_2, \ldots, v_s)$.

Let $1 \leq \alpha, \beta \leq r$ $(\alpha \neq \beta)$ be integers. For $h \in \mathrm{Hom}_R(R_{i_\alpha}, R_{i_\beta})$, we denote by $\psi(h)$ the automorphism of $L$ which sends each element ${}^t(x_1, \ldots, x_r)$ of $L$ to ${}^t(x_1, \ldots, x_\alpha, \ldots, x_\beta + h(x_\alpha), \ldots, x_r)$. For example assume that $\alpha < \beta$ and let $u \in R_{i_\beta} : R_{i_\alpha}$ and $v \in R_{i_\alpha}$. Then the element $u$ (resp. $v$) defines a homomorphism $\hat{u} : R_{i_\alpha} \to R_{i_\beta}$ (resp. $\hat{v} : R_{i_\beta} \to R_{i_\alpha}$) so that $\hat{u}(f) = u.f$ (resp. $\hat{v}(f) = v.f$) for each $f \in R_{i_\alpha}$ (resp. $f \in R_{i_\beta}$). We denote by $\xi(u)$ (resp. $\eta(v)$) the automorphism of $\bar{L}$ induced from $\psi(\hat{u})$ (resp. $\psi(\hat{v})$). For each unit $\varepsilon$ of $R_{i_\alpha}$, let $\hat{\varepsilon}$ be the automorphism of $L$ which sends each ${}^t(x_1, \ldots, x_r) \in L$ to ${}^t(x_1, \ldots, \varepsilon x_\alpha, \ldots, x_r)$. The automorphism $\rho(\varepsilon)$ of $\bar{L}$ induced from $\hat{\varepsilon}$ is permissible too.

In what follows, we will show that by a successive application of the elementary row transformations $\xi(u)$, $\eta(v)$ and $\rho(\varepsilon)$ together with elementary column transformations with coefficients in $k$, the $r$ by $s$ matrix $A$ can be transformed into a matrix of the above form. Let $a_{\alpha j}$ denote the $(\alpha, j)$-entry of $A$.

First let $1 \leq \alpha < \beta \leq r$ and $1 \leq j \leq s$. Assume that $a_{\alpha j} = 1$, while $a_{\beta j}$ is a non-unit of $R_{i_\beta}/mR_{i_\beta}$. Then since

$$a_{\beta j} = cty^{i_\beta} \mod mR_{i_\beta}$$

for some $c \in R$ (cf. (4.5)) and since $u = -cty^{i_\beta} \in R_{i_\beta} : R_{i_\alpha}$ (cf. (4.6)), by $\xi(u)$ we may reduce $a_{\beta j}$ to $0$. Similarly, applying $\eta(v)$ where $v \in R_{i_\gamma}$ such that

$$a_{\gamma j} = -v \mod mR_{i_\gamma} ,$$

we may assume $a_{\gamma j} = 0$ for any $1 \leqq \gamma < \alpha$ too. Consequently starting from the lower rows, our matrix $A$ can be transformed so that it has the form

$$\left(\begin{array}{ccc|c} 0 & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & B \\ 1 & & 0 & \end{array}\right),$$

where all the entries of $B$ are non-units.

Now apply column operations to $B$, until $A$ has the form

$$\left(\begin{array}{ccc|c|c} 0 & & 1 & A_1 & 0 \\ \hline & \cdot^{\cdot^{\cdot}} & & B_1 & B_2 \\ 1 & & 0 & & \end{array}\right),$$

where $A_1$ consists of $r_1$ rows and the columns of $A_1$ are linearly independent over $k$. Then because any non-unit of $R_{j_1}/mR_{j_1}$ has the form

$$cty^{j_1} \mod mR_{j_1}$$

with $c \in R$, only making use of elementary column transformations with coefficients in $k$, the matrix $A_1$ is transformed so that each column of $A_1$ crosses a row (of $A_1$) whose unique non-zero entry lies in the column and equals $ty^{j_1} \mod mR_{j_1}$

Let $r_1 < \beta \leqq r$ and choose an entry $a_{\beta j}$ of $B_1$. Let us write $a_{\beta j} = cty^{i_\beta} \bmod mR_{i_\beta}$ with $c \in R$ and consider the row of $A_1$ whose unique non-zero entry lies in the j-th column (of A ) and equals $ty^{j_1} \bmod mR_{j_1}$ . Then because

$$y^{i_\beta - j_1} \in R_{i_\beta} : R_{j_1} ,$$

by virtue of the row transformation $\xi(-cy^{i_\beta - j_1})$ we may reduce $a_{\beta j}$ to 0 . Since $y^{i_\beta - j_1} \in mR_{i_\beta}$ and since the row of $A_1$ is chosen to have a unique non-zero entry, the operation $\xi(-cy^{i_\beta - j_1})$ causes no change on the other entries of A . Thus the matrix $B_1$ may be assumed to be 0 and repeating this procedure, we reach the required normal form

$$\left(\begin{array}{cccc|cccc} & & & 1 & A_1 & & & \\ & & \cdot & & & A_2 & & \\ & \cdot & & & & & \ddots & \\ 1 & & & & & & & A_q \end{array}\right) .$$

Proof of Theorem (4.10) (cf. Proof of [10, Theorem (5.3)]).

The R-modules $M_{ij}$ $(0 \leqq i < n , 1 \leqq j \leqq 4)$ and R are not isomorphic to each other. In fact, since $M_{ij}/H_m^0(M_{ij}) \cong R_i$ , it is enough to check that $M_{i2} \not\cong M_{i3}$ for each $0 \leqq i < n$ . Assume the contrary. Then by (2.3)(2) we may choose an automorphism $\phi$ of the R-module $R_i$ so that $\phi(mR_i + R) \subset m_i$ —— this is impossible, because $\phi = \varepsilon 1_{R_i}$ with a unit $\varepsilon$ of $R_i$ .

Now let M be an indecomposable maximal Buchsbaum R-module such that $M \not\cong R$ . Let $V = H_m^0(M)$ . Then since M/V is a maximal

Cohen-Macaulay R-module, by (4.7) we get an isomorphism

$$M/V \cong \bigoplus_{\alpha=1}^{r} R_{i_\alpha}$$

with $0 \leq i_1 \leq i_2 \leq \dots \leq i_r < n$ . Let

$$L = \bigoplus_{\alpha=1}^{r} R_{i_\alpha} \quad \text{and} \quad F = (R^2)^r \ .$$

Then as $V \subset mM$ (cf. (2.1)(2)), we have by (4.9) a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & L & \xrightarrow{\xi} & F & \xrightarrow{\eta} & L & \longrightarrow & 0 \\
 & & \uparrow i & & \| & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & V & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns, where $\xi$ (resp. $\eta$ ) denotes the direct sum $\bigoplus_{\alpha=1}^{r} \sigma_{i_\alpha}$ (resp. $\bigoplus_{\alpha=1}^{r} \varepsilon_{i_\alpha}$ ) and the homomorphism $i : N \to L$ is considered to be the inclusion map. Notice that $mL \subset N$ , as $V \cong L/N$ (cf. (2.1)(1)). Let

$$\overline{L} = \bigoplus_{\alpha=1}^{r} R_{i_\alpha}/mR_{i_\alpha}$$

and let $\tau : L \to \overline{L}$ denote the canonical epimorphism. We put $U = \tau(N)$ and $s = \dim_k U$ (here $k = R/m$ ). If $s = 0$ , then $N = mL$ and so $M \cong \bigoplus_{\alpha=1}^{r} R^2/\sigma_{i_\alpha}(mR_{i_\alpha})$ . Hence we get $r = 1$ and $M \cong M_{i4}$ with $0 \leq i < n$ .

Suppose that $s \geqq 1$ and apply Lemma (4.11) to the k-subspace $U$ of $\overline{L}$ . Then we find an automorphism of $\overline{L}$ , induced from an automorphism $\phi$ of $L$ , that maps $U$ onto the k-subspace $U'$ spanned by the columns of an $r$ by $s$ matrix $A$ of the following form

$$\left(\begin{array}{ccc|c|c|c|c} 0 & & 1 & A_1 & 0 & 0 & 0 \\ \hline & & & 0 & A_2 & 0 & 0 \\ \hline & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ \hline 1 & & 0 & 0 & 0 & 0 & A_q \end{array}\right) ,$$

where each submatrix $A_\beta$ consists of $r_\beta$ rows. Let $N' = \tau^{-1}(U')$. Then $N' = \phi(N)$ clearly and so, by (2.3)(2) and (2.9), we get an isomorphism

$$M \cong F/\xi(N')$$

— hence we may assume $N = N'$ . Then the condition that $M$ is indecomposable now causes a restriction on the form of the above matrix $A$ so that $q = 1$ , whence $i_1 = i_\alpha$ for any $1 \leqq \alpha \leqq r$ . Thus $L = {R_i}^r$ with $0 \leqq i < n$ .

In this case because our matrix $A$ has the normal form

$$(\#) \qquad \left(\begin{array}{c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 & \begin{matrix} ty^i & & \\ & \ddots & \\ & & ty^i \\ \hline & 0 & \end{matrix} \\ \hline 0 & \begin{matrix} ty^i & & \\ & \ddots & \\ & & ty^i \end{matrix} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \quad \text{mod } mR_i$$

(cf. [10, Lemma (5.4)]), we may assume that $N = mL + W$ where $W$ is the R-submodule of $L$ generated by the columns of the matrix (#). Since $M$ is indecomposable, we conclude that $r = 1$ and the matrix (#) must be one of $(1\ ty^i)$, $(1)$, and $(ty^i)$. Hence $M \cong M_{i1}$, $M \cong M_{i2}$, or $M \cong M_{i3}$ as required.

## 5. Cohen-Macaulay rings of finite Buchsbaum-representation type, II (continued).

Let us consider the case (II) where $v(y) = 1$ and $l_R(\overline{R}/J) = 2$. We note

Lemma (5.1). The ring $R$ contains an element $x$ such that $m = (x, y)$ and $\overline{R} = R + R(x/y^n)$, where $n = v(x)$.

Proof. Assume that $\overline{R} \neq R + R(x/y^n)$ for any $x \in R$ such that $m = (x, y)$. Let $t = x/y^n$ ( $n = v(x)$ ). Then $t \in R + m\overline{R}$, as $\overline{R} \neq R + Rt$. We write $t = c + z$ with $c \in R$ and $z \in m\overline{R}$. Then $x = cy^n + zy^n$ and so we get $x' = zy^n \in R$ and $m = (x', y)$. Because $v(x) = n < v(x')$, repeating this argument we have a sequence $\{x_i\}_{i \geq 1}$ of elements in $R$ that satisfies $m = (x_i, y)$ and $v(x_i) < v(x_{i+1})$ for any $i \geq 1$. This sequence $\{x_i\}_{i \geq 1}$ forces $m = yR$ (cf. Proof of (4.3)), which is a required contradiction.

Let $x$ be as in (5.1) and put $t = x/y^n$. For each $0 \leq i \leq n$, we define

$$R_i = R[ty^i] \subset \overline{R} .$$

Clearly $R_0 = \overline{R}$ and $R_n = R$. We write $t^2 = a + bt$ with $a, b \in R$ and denote by $m_i$ the maximal ideal of $R_i$. Then $R$ has finite CM-representation type and the R-modules $R_i$ $(0 \leqq i \leqq n)$ are the representatives of indecomposable maximal Cohen-Macaulay modules. (Notice that the assertions (4.4), (4.5) and (4.6) hold in the case (II) too. The number $n$ is characterized by the condition that $J^{n-1} \not\subset m$ but $J^n \subset m$.) For each $0 \leqq i < n$, the R-module $R_i$ has a presentation

$$0 \longrightarrow R_i \xrightarrow{\sigma_i} R^2 \xrightarrow{\varepsilon_i} R_i \longrightarrow 0$$

such that

$$\sigma_i(1) = \begin{pmatrix} x \\ y^{n-i} \end{pmatrix}, \quad \sigma_i(ty^i) = \begin{pmatrix} bxy^i + ay^{n+i} \\ x \end{pmatrix},$$

$$\varepsilon_i(e_1) = 1 \text{ and } \varepsilon_i(e_2) = -ty^i,$$

where $e_1, e_2$ denote the standard basis of $R^2$ (cf. (4.9)). Similarly as in the case (I), we define

$$\begin{aligned} M_{i1} &= R_i, \\ M_{i2} &= R^2/\sigma_i(mR_i + R), \\ M_{i3} &= R^2/\sigma_i(m_i), \\ M_{i4} &= R^2/\sigma_i(mR_i). \end{aligned}$$

Then $M_{ij}$'s are indecomposable maximal Buchsbaum R-modules with

$$\begin{aligned} l_R(H_m^0(M_{ij})) &= 0 \quad (j = 1), \\ &= 1 \quad (j = 2), \end{aligned}$$

$$= 1 \quad ( j = 3,\ i \geqq 1 ) ,$$
$$= 2 \quad ( j = 4 ) .$$

(Notice that $M_{03} = M_{04}$ , as $J = m\overline{R}$ .) Furthermore we have the following

Theorem (5.2). The ring $R$ has finite Buchsbaum-representation type and the R-modules

$M_{ij}$ $(1 \leqq i < n,\ 1 \leqq j \leqq 4)$, $M_{0j}$ $(1 \leqq j \leqq 3)$, and $R$

are the representatives of indecomposable maximal Buchsbaum modules.

The proof of (5.2) is the same as that of (4.10). The lemma corresponding to (4.11) is

Lemma (5.3) (stated with the same notation as in (4.11)). By some automorphism of $\overline{L}$ induced from that of $L$ , $U$ is mapped onto the k-subspace $U'$ of $\overline{L}$ spanned by the columns of an $r$ by $s$ matrix of the following form

$$\left(\begin{array}{c|c|c|c|c}
0 & A_1 & 0 & & 0 \\
\hline
\begin{matrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{matrix} & \begin{matrix} 0 \\ \hline \\ \hline 0 \end{matrix} & \begin{matrix} A_2 \\ \hline \\ \hline 0 \end{matrix} & \begin{matrix} \\ \hline \ddots \\ \hline \\ \end{matrix} & \begin{matrix} 0 \\ \hline \\ \hline A_q \end{matrix}
\end{array}\right) ,$$

where the submatrix $A_\beta$ $(1 \leqq \beta \leqq q)$ consists of $r_\beta$ rows, the entries of $A_\beta$ $(2 \leqq \beta \leqq q)$ are non-units of $R_{j_\beta}/mR_{j_\beta}$ , and $A_1$ has the form

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{array}{c} \begin{matrix} t & & \\ & \ddots & \\ & & t \end{matrix} \\ \hline 0 \end{array} \\ \hline 0 & 0 \end{array}\right) \quad \bmod m\overline{R} .$$

Proof. Let us maintain the notation in the proof of (4.11). First, starting from the lower rows, we transform our matrix $A$ into

$$\left(\begin{array}{c|c} 0 & B \\ \hline \begin{matrix} & & 1 \\ & ⋰ & \\ 1 & & \end{matrix} & C \end{array}\right) ,$$

where $B$ consists of $r_1$ rows and the entries of $C$ are non-units. Because $\overline{R}/m\overline{R}$ is a field with the k-basis $1, t \bmod m\overline{R}$, the matrix $B$ is subsequently transformed into

$$\left(\begin{array}{c|c|c} \begin{array}{c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline 0 \end{array} & \begin{array}{c} \begin{matrix} \bar{t} & & \\ & \ddots & \\ & & \bar{t} \end{matrix} \\ \hline 0 \end{array} & 0 \end{array}\right)$$

(here $\bar{t} = t \bmod m\overline{R}$ ), whence $A$ has the form

$$\left(\begin{array}{c|c|c|c} 0 & \begin{array}{c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline 0 \end{array} & \begin{array}{c} \begin{matrix} \bar{t} & & \\ & \ddots & \\ & & \bar{t} \end{matrix} \\ \hline 0 \end{array} & 0 \\ \hline \begin{matrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} & C_1 & C_2 & C_3 \end{array}\right) .$$

Let $r_1 < \beta \leq r$ and choose an entry $a_{\beta j}$ of $C_1$. We write $a_{\beta j} = cty^{i_\beta} \bmod mR_{i_\beta}$ with $c \in R$. Then as $t^2y^{i_\beta} = ay^{i_\beta} + b.ty^{i_\beta}$, the element $-ct^2y^{i_\beta}$ is still in $m_{i_\beta}$ and therefore by $\xi(-cty^{i_\beta})$ we can safely reduce $a_{\beta j}$ to $0$ —— hence $C_1$ may be assumed to be $0$. Now consider an entry $a_{\beta j}$ of $C_2$ and write $a_{\beta j} = c.ty^{i_\beta} \bmod mR_{i_\beta}$. Then by the row operation $\xi(-cy^{i_\beta})$ we may reduce $a_{\beta j}$ to $0$, while $C_1$ remains $0$ as $y^{i_\beta} \in mR_{i_\beta}$. Consequently, we may assume the matrix $A$ to have the form

$$\left(\begin{array}{c|c|c|c} 0 & \begin{array}{c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline 0 \end{array} & \begin{array}{c} \begin{matrix} \bar{t} & & \\ & \ddots & \\ & & \bar{t} \end{matrix} \\ \hline 0 \end{array} & 0 \\ \hline \begin{matrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} & 0 & 0 & C \end{array}\right) .$$

So the process in the proof of (4.11) still works to get the required normal form.

Now we consider the case (III) where $R$ is not an integral domain. Recall that as $e(R) = 2$, the ring $R$ contains precisely two minimal prime ideals $p_i$ $(i = 1, 2)$ (cf. [16, (24.7]). $R/p_i$ is a discrete valuation ring with the regular parameter $y$ mod $p_i$ .

Lemma (5.4). The ideal $p_1$ contains an element $x$ such that $m = (x, y)$ and $x - y^n \in p_2$ for some $n \geqq 1$ .

Proof. We write $m = (x, y)$ . If $x \in p_1$ , then $x \equiv \varepsilon y^n$ mod $p_2$ for some unit $\varepsilon$ of $R$ and $n \geqq 1$ ; so the element $x/\varepsilon$ satisfies the requirements. Assume $x \notin p_1$ and write $x \equiv cy$ mod $p_1$ with $c \in R$ . Let $x' = x - cy$ . Then $x' \in p_1$ and $m = (x', y)$ clearly, whence the problem is reduced to the case where $x \in p_1$ .

Let $x$ be as in (5.4). Then we have $p_1 = (x)$ and $p_2 = (x - y^n)$ . (Hence the relation $x^2 = xy^n$ defines the ring $R$ .) We put $t = x/y^n$ and define

$$R_i = R[ty^i] \subset \overline{R}$$

for $0 \leqq i \leqq n$ . Then $R_0 = \overline{R}$ , as $\overline{R} = R + Rt$ . (Clearly $R_n = R$ .) For each $1 \leqq i \leqq n$ we denote by $m_i$ the maximal ideal of the <u>local</u> ring $R_i$ . Then the assertions (4.4), (4.5) and (4.6) hold in the case (III) too. Consequently, $R$ has finite CM-representation type and the R-modules $R_i$ $(1 \leqq i \leqq n)$ and $R/p_i$ $(i = 1, 2)$ are the representatives of indecomposable maximal Cohen-Macaulay modules (cf. Proof of (4.7). Notice that $\overline{R} = R/p_1 \times R/p_2$ .). The number $n$ is characterized also by the condition that $J^{n-1} \not\subset m$ but $J^n \subset m$ .

For each $1 \leq i < n$ , the R-module $R_i$ has a presentation

$$0 \longrightarrow R_i \xrightarrow{\sigma_i} R^2 \xrightarrow{\varepsilon_i} R_i \longrightarrow 0$$

with $\sigma_i(1) = \begin{pmatrix} x \\ y^{n-i} \end{pmatrix}$, $\sigma_i(ty^i) = \begin{pmatrix} xy^i \\ x \end{pmatrix}$ , $\varepsilon_i(e_1) = 1$ and $\varepsilon_i(e_2) = - ty^i$ . We put

$$\begin{aligned} M_{i1} &= R_i , \\ M_{i2} &= R^2/\sigma_i(mR_i + R) , \\ M_{i3} &= R^2/\sigma_i(m_i) , \\ M_{i4} &= R^2/\sigma_i(mR_i) . \end{aligned}$$

Then $M_{ij}$'s are non-isomorphic indecomposable maximal Buchsbaum R-modules with

$$\begin{aligned} l_R(H_m^0(M_{ij})) &= 1 \qquad (j = 1) , \\ &= 2 \qquad (j = 2,3) , \\ &= 3 \qquad (j = 4) . \end{aligned}$$

For $R/p_1$ and $R/p_2$ , we have the canonical exact sequences

$$0 \longrightarrow R/p_1 \xrightarrow{\xi_1} R \longrightarrow R/p_2 \longrightarrow 0 ,$$

$$0 \longrightarrow R/p_2 \xrightarrow{\xi_2} R \longrightarrow R/p_1 \longrightarrow 0$$

with $\xi_1(1) = x - y^n$ and $\xi_2(1) = x$ . Let $\sigma = \xi_1 \oplus \xi_2$ , the direct sum of $\xi_1$ and $\xi_2$ , and $N = m.(R/p_1 \oplus R/p_2) + R\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . We put

$$M = R^2/\sigma(N) .$$

Then

Proposition (5.5). $M$ is an indecomposable maximal Buchsbaum R-module with $l_R(H_m^0(M)) = 1$ .

Proof. By (2.3)(1) it suffices to check that $M$ is indecomposable. Assume that $M \cong M_1 \oplus M_2$ with non-zero R-modules $M_i$ . Then $\mu_R(M_i) = 1$ $(i = 1,2)$, whence the isomorphisms

$$R/p_1 \oplus R/p_2 \cong M/H_m^0(M)$$

claim that $\dim_R M_i = 1$ $(i = 1,2)$. Also these isomorphisms allow us to write

$$M_1/H_m^0(M_1) = R/p_2 \quad \text{and} \quad M_2/H_m^0(M_2) = R/p_1 .$$

Consequently, as $M_i$'s are indecomposable maximal Buchsbaum R-modules (cf. (2.1)), we have an isomorphism

$$M_i \cong R/\xi_i(N_i)$$

with an R-submodule $N_i$ of $R/p_i$ containing $m.(R/p_i)$ . Therefore

$$M \cong R^2/\sigma(N_1 \oplus N_2) ,$$

whence by (2.3)(2) we have

$$\phi(N) = N_1 \oplus N_2$$

for some automorphism $\phi$ of the R-module $R/p_1 \oplus R/p_2$ . Because $\mathrm{Hom}_R(R/p_i, R/p_j) = (0)$ if $i \neq j$, the automorphism $\phi$ is diagonal, say

$$\phi = \begin{pmatrix} f \mod p_1 & 0 \\ 0 & g \mod p_2 \end{pmatrix}$$

with $f$, $g$ units of $R$ . Consequently as $\phi(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} f \mod p_1 \\ g \mod p_2 \end{pmatrix}$ , we get $N_i = R/p_i$ $(i = 1,2)$ whence $N = R/p_1 \oplus R/p_2$ —— this is a contradiction.

We close the main part of this section with the following

Theorem (5.6). The ring $R$ has finite Buchsbaum-representation type and the R-modules $R/p_i$ $(i = 1, 2)$ , $R/(xy)$ , $R/(xy - y^{n+1})$ , $M$ , $M_{ij}$ $(1 \leqq i < n,\ 1 \leqq j \leqq 4)$ , and $R$ are the representatives of indecomposable maximal Buchsbaum modules.

The proof of (5.6) is the same as that of (4.10), which we leave to the readers. To state the lemma corresponding to (4.11), let $m \geqq 0$, $n \geqq 0$, and $1 \leqq i_1 \leqq i_2 \leqq \dots \leqq i_r < n$ be integers with $m + n + r \geqq 1$ . Let $I = \{ i_\alpha \mid 1 \leqq \alpha \leqq r \}$ and write

$$I = \{ j_2, j_3, \dots , j_q \} \text{ with } j_2 < j_3 < \dots < j_q .$$

We put $r_1 = m + n$ and $r_\beta = \#\{ \alpha \mid 1 \leqq \alpha \leqq r \text{ such that } i_\alpha = j_\beta \}$ for each $2 \leqq \beta \leqq q$ . Let

$$L = (R/p_1)^m \oplus (R/p_2)^n \oplus ( \bigoplus_{\alpha = 1}^{r} R_{i_\alpha} )$$

and

$$\overline{L} = k^m \oplus k^n \oplus ( \bigoplus_{\alpha = 1}^{r} R_{i_\alpha}/mR_{i_\alpha} ) ,$$

where $k = R/m$ . Finally, let $v_j$ $(1 \leqq j \leqq s)$ be elements of $\overline{L}$ and put $U = \sum_{j=1}^{s} kv_j$ . Then we have

Lemma (5.7). By some automorphism of $\overline{L}$ induced from that of $L$ , $U$ is mapped onto the k-subspace $U'$ of $\overline{L}$ spanned by the columns of an $r$ by $s$ matrix of the form

$$\left( \begin{array}{c|c|c|c|c} 0 & A_1 & 0 & & 0 \\ \hline & 0 & A_2 & & 0 \\ \cline{2-5} \begin{matrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} & & & \ddots & \\ \cline{2-5} & 0 & 0 & & A_q \end{array} \right) ,$$

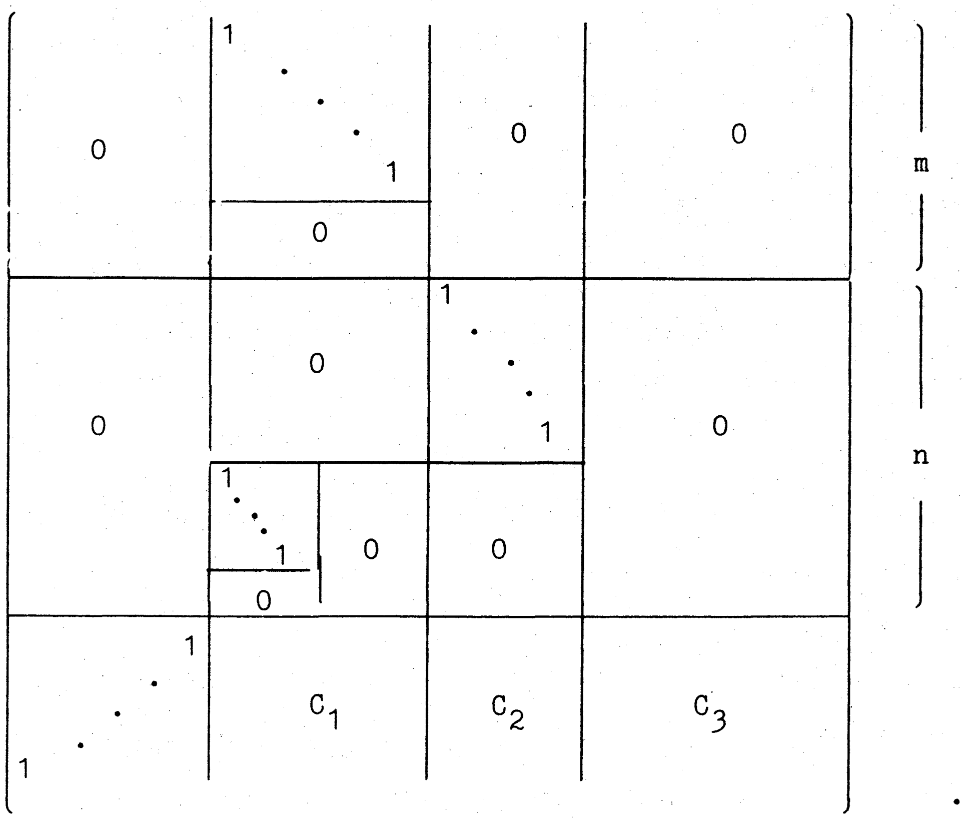
where the submatrix $A_\beta$ $(1 \leq \beta \leq q)$ consists of $r_\beta$ rows, the entries of $A_\beta$ $(2 \leq \beta \leq q)$ are non-units, and the matrix $A_1$ has the following form

$$(*)\quad \left(\begin{array}{c|c|c}
\begin{array}{c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline 0 \end{array} & 0 & 0 \\ \hline
0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \\ \cline{1-2}
\begin{array}{c|c} \begin{array}{c} \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} \\ \hline 0 \end{array} & 0 \end{array} & 0 & 0
\end{array}\right)
\begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} n \end{matrix}.$$

Proof. First of all, let us transform our matrix $A = (v_1, v_2, \ldots, v_s)$ into the form

$$\left(\begin{array}{c|c}
0 & B \\ \hline
\begin{matrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{matrix} & C
\end{array}\right)$$

where $B$ consists of $r_1$ rows and the entries of $C$ are non-units. Then as $B$ has its entries in the field $k$, subsequently we can transform $B$ into the above form (*) whence we may assume $A$ to have the form




Consequently by the same manner as in the proof of (5.3), we can reduce both the matrices $C_1$ and $C_2$ to $0$ and the process in the proof of (4.11) works to get the required normal form.

We shall use the rest of this section to give some examples. Let $n_B(R)$ denote the number of the isomorphism classes of indecomposable maximal Buchsbaum R-modules.

Example (5.8). Let $S = k[[t]]$ be a formal power series ring over a field $k$. Let $n \geq 1$ be an integer and $y \in t^2S$ that is not contained in $t^3S$. We put

$$R = k[[t^{2n+1}, y]] .$$

Then R has the normalization S , $e(R) = 2$ , and $m^2 = ym$ . This ring R is of type (I) and $n_B(R) = 4n + 1$ .

Example (5.9). Let K/k be an extension of fields with degree 2 and $S = K[[y]]$ a formal power series ring over K . Let $n \geqq 1$ be an integer and $t \in K$ that is not contained in k . We put

$$R = k[[\, ty^n, y]] .$$

Then R has the normalization S , $e(R) = 2$ , and $m^2 = ym$ . R is of type (II) and $n_B(R) = 4n$ .

Example (5.10). Let $k[[X,Y]]$ be a formal power series ring over a field k . Let $n \geqq 1$ be an integer. We put

$$R = k[[\, X,Y \,]]/(X^2 - XY^n) .$$

Then $e(R) = 2$ and $m^2 = ym$ , where $y = Y \mod (X^2 - XY^n)$ . This ring R is of type (III) and $n_B(R) = 4n + 2$ .

As is well-known, any equicharacteristic local ring of type (I) (resp. (III)) arises like (5.8) (resp. (5.10)). It is standard to check that if R contains a coefficient field k of ch $k \neq 2$ , any local ring of type (II) arises like (5.9). If R contains an algebraically closed coefficient field, the normal form of rings R of type (I) is known by [14] and listed in our corollary (1.2).

6. Proof of the implication (1) $\Rightarrow$ (2) in Theorem (1.1).

Let R be a complete local ring with infinite residue class field and assume that R has finite Buchsbaum-representation type.

The aim of this section is to prove the implication (1) => (2) in Theorem (1.1) and the next result (6.1) is the key.

Proposition (6.1). $v(R) \leq 2$ and $e(R) \leq 2$ .

Proof. Let $I = H_m^0(R)$ . Then because $R/I$ is a Cohen-Macaulay ring of finite Buchsbaum-representation type, we get by (3.1) that $R/I$ is a reduced ring of $e(R/I) = e(R) \leq 2$ . As $\mu_R(I) \leq 1$ by (2.4) and as $v(R/I) \leq 2$ (cf. (2.7)), we have $v(R) \leq 3$ too.

Now let us assume that $v(R) = 3$ . Then $v(R/I) = 2$ and $\mu_R(I) = 1$ —— hence $e(R/I) = 2$ .

Claim 1. $R/I$ is an integral domain.

Proof of Claim 1. Suppose that $R/I$ is not an integral domain. Then as $R/I$ is a reduced ring of $e(R/I) = 2$ , it contains a minimal prime ideal $p$ such that $(R/I)/p$ is regular (cf. (5.4)) —— so $R$ has a minimal prime ideal $P$ with $R/P$ regular, too. However for this prime ideal $P$ , since $v(R) = 3$ , we must have $\mu_R(P) = 2$ while $\mu_R(P) \leq 1$ by (2.4) —— this is a contradiction.

By this claim we reach, for the ring $R/I$ , the two cases (I) and (II) explored in Sections 4 and 5. Let us write $I = zR$ with $z \in R$ . We choose elements $x$ , $y$ of $m$ so that $\overline{x}$ and $\overline{y}$ (here $\overline{\cdot}$ denotes the reduction mod $I$ ) satisfy the requirements in (4.3) (resp. (5.1)), if we have the case (I) (resp. (II)). Then $m = (x, y, z)$ clearly. Let $S$ denote the normalization of $R/I$ .

First we consider the case (I). Let $t = \overline{y}/\overline{x}^n$ (here $n$ is the integer given in (4.3)). Write $t^2 = \overline{a} + \overline{b}t$ with $a, b \in m$. Then by (4.9) we get an exact sequence

$$(\#) \qquad R^4 \xrightarrow{\begin{pmatrix} z & 0 & x & bx + ay^n \\ 0 & z & y^n & x \end{pmatrix}} R^2 \xrightarrow{\varepsilon} S \to 0$$

of R-modules with $\varepsilon(e_1) = 1$ and $\varepsilon(e_2) = -t$. Let

$$F = R^2 \text{ and } K = \operatorname{Ker} \varepsilon .$$

For each $\lambda \in R/m$, we choose $c_\lambda \in R$ so that $\lambda = c_\lambda \bmod m$ and put

$$N_\lambda = mK + Rh_\lambda ,$$

where

$$h_\lambda = \begin{pmatrix} 0 \\ z \end{pmatrix} + c_\lambda \begin{pmatrix} bx + ay^n \\ x \end{pmatrix} .$$

Then by (2.3)(1), the R-modules $M_\lambda = F/N_\lambda$ are indecomposable maximal Buchsbaum modules. Accordingly, to get a contradiction, it suffices to check that

Claim 2. $M_\lambda \not\cong M_\mu$, if $\lambda \neq \mu$.

Proof of Claim 2. Suppose that $M_\lambda \cong M_\mu$ with $\lambda, \mu \in R/m$. Then we have, by (2.3)(2), an automorphism $\phi$ of $F$ that satisfies

$$\phi(N_\lambda) = N_\mu \text{ and } \phi(K) = K .$$

Let $\overline{\phi}$ be the automorphism of $S$ induced from $\phi$. We write

$$\overline{\phi} = (\overline{\alpha} - \overline{\beta}t)1_S$$

with $\alpha, \beta \in R$ (hence $\alpha \notin m$ ) and put $\rho = \begin{pmatrix} 0 & a \\ 1 & -b \end{pmatrix}$. Then as both the endomorphisms $\phi$ and $\alpha 1_F + \beta\rho$ of $F$ lift $\overline{\Phi}$, we get a homomorphism $\delta : F \to K$ such that

$$\phi = \alpha 1_F + \beta\rho + i \circ \delta$$

(here $i : K \to F$ denotes the inclusion map). Consequently, because

$$\rho(h_\lambda) = a(\begin{pmatrix} z \\ 0 \end{pmatrix} + c_\lambda \begin{pmatrix} x \\ y^n \end{pmatrix}) - b \begin{pmatrix} 0 \\ z \end{pmatrix} \in mK$$

(recall $a, b \in m$ ) and $\delta(K) \subset mK$ (recall $K \subset mF$ ), we see

$$\alpha h_\lambda \in N_\mu$$

—— hence $h_\lambda \in N_\mu$ as $\alpha \notin m$ . Thus $\lambda = \mu$, since $\begin{pmatrix} 0 \\ z \end{pmatrix}$ and $\begin{pmatrix} bx + ay^n \\ x \end{pmatrix}$ are part of a minimal system of generators for $K$ .

So we have the case (II). Similarly as in the case (I), let $t = \overline{x}/\overline{y}^n$ (here $n$ is the integer given in (5.1)) and write $t^2 = \overline{a} + \overline{b}t$ with $a, b \in R$ . Then as is noted in Section 5, we have the same exact sequence (#) above. Let $F$ and $K$ be as before. But take $h_\lambda = \begin{pmatrix} z \\ 0 \end{pmatrix} + c_\lambda \begin{pmatrix} x \\ y^n \end{pmatrix}$ in our case. Let $N_\lambda = mK + Rh_\lambda$ . Then $M_\lambda = F/N_\lambda$ is again an indecomposable maximal Buchsbaum R-module. Let us check that $\lambda = \mu$ , if $M_\lambda \cong M_\mu$ . First, take an automorphism $\phi$ of $F$ so that $\phi(N_\lambda) = N_\mu$ and $\phi(K) = K$ . Lift the automorphism $\overline{\Phi} = (\overline{\alpha} - \overline{\beta}t)1_S$ of $S$ (induced from $\phi$) by means of $\alpha 1_F + \beta\rho$ , where $\rho = \begin{pmatrix} 0 & a \\ 1 & -b \end{pmatrix}$. Choose a homomorphism $\delta : F \to K$ so that

$$\phi = \alpha 1_F + \beta\rho + i \circ \delta$$

(here $i : K \to F$ denotes the inclusion map). Then as $\phi(h_\lambda) \in N_\mu$ and $\delta(K) \subset mK$ , we get

$$\phi(h_\lambda) \equiv \alpha h_\lambda + \beta\rho(h_\lambda)$$
$$\equiv \alpha' h_\mu \mod mK$$

(here $\alpha' \in R$ ), from which we have two equalities:

$$(1) \quad \alpha z + \alpha c_\lambda x + \beta c_\lambda(ay^n) \equiv \alpha' z + \alpha' c_\mu x \mod m^2 ,$$

$$(2) \quad \alpha c_\lambda y^n + \beta z + \beta c_\lambda(x - by^n) \equiv \alpha' c_\mu y^n \mod m^2 .$$

Now recall that $m = (x, y, z)$ and $\mu_R(m) = 3$ . Then we find by (2) that $\beta \in m$ and so we get $\alpha \notin m$ , since $\overline{\alpha} - \overline{\beta}t$ is a unit of $S$ by our choice. Consequently because

$$\alpha \equiv \alpha' \text{ and } \alpha c_\lambda \equiv \alpha' c_\mu \mod m$$

by (1), we have $c_\lambda \equiv c_\mu \mod m$ . Thus $\lambda = \mu$ , which completes the proof of (6.1).

Now let us quickly finish the proof of the implication (1) => (2) in Theorem (1.1). By (6.1) we may assume that $v(R) = 2$ . Let

$$R = P/J$$

for some ideal $J$ in a complete regular local ring $P$ of $\dim P = 2$ . Then as the ideal $J$ is of height 1, we may write

$$J = fI$$

with $f \in P$ and $I$ an ideal of $P$ which contains some power of the maximal ideal $n$ in $P$ . Notice that $P/fP$ has finite Buchsbaum-representation type (as it is a homomorphic image of $R$ ) and we get by (3.1) that $P/fP$ is a reduced ring of $e(P/fP) \leq 2$ —— hence $f \notin n^3$ .

7. Proof of the implication (2) => (1) in Theorem (1.1).

Let $n_B(R)$ denote the number of the isomorphism classes of indecomposable maximal Buchsbaum R-modules. In this section we shall prove the implication (2) => (1) in Theorem (1.1), which now readily follows from the next

Theorem (7.1). Let $P$ be a regular local ring with maximal ideal $n$ and $\dim P = 2$. Let $0 \neq f \in n$ and $I$ an $n$-primary ideal of $P$. Then

$$n_B(P/fI) = n_B(P/fP) + 1 .$$

We divide the proof of Theorem (7.1) into several steps. Let $P$, $f$, and $I$ be as in (7.1). First we note

Lemma (7.2). $n_B(P/fI) = n_B(P/fn)$ .

Proof. Any maximal Buchsbaum $P/fn$-module is naturally a maximal Buchsbaum $P/fI$-module. Conversely, let $M$ be a maximal Buchsbaum $P/fI$-module. Then as $n.H_n^0(M) = (0)$ (cf. (2.1)(1)) and $H_n^0(P/fI).M \subset H_n^0(M)$ , we get the ideal $n.H_n^0(P/fI) = fn/fI$ annihilates $M$ so that $M$ may be regarded as a $P/fn$-module too.

By (7.2) we may assume $I = n$ . Let $R = P/fn$ and $S = P/fP$ . For a while we fix a maximal Cohen-Macaulay $S$-module $L$ and assume that $L$ doesn't contain $S$ as a direct summand. Let

$$0 \longrightarrow P^n \xrightarrow{\Phi} P^n \xrightarrow{\tau} L \longrightarrow 0$$

denote a minimal free resolution of the $P$-module $L$ . Then we have a (unique) endomorphism $\Psi$ of $P^n$ that satisfies

$$\Phi \circ \Psi = \Psi \circ \Phi = f 1_{P^n}$$

(cf. [6] or Proof of (2.9)). Choose a homomorphism $\varepsilon : R^n \to L$ so that the diagram

$$\begin{array}{ccc} P^n & \xrightarrow{\tau} & L \\ \sigma \downarrow & & \| \\ R^n & \xrightarrow{\varepsilon} & L \end{array}$$

is commutative, where $\sigma$ denotes the canonical map. We put

$$M = \operatorname{Ker} \varepsilon \ , \quad \phi = \Phi \mod fn \ , \text{ and } \quad \psi = \Psi \mod fn$$

and identify $\phi$ (resp. $\psi$ ) with an $n$ by $n$ matrix with entries $a_{ij}$ (resp. $b_{ij}$ ) in $R$ .

Lemma (7.3). $\mu_R(M) = n$ and $b_{ij} \in m$ for any $1 \leqq i , j \leqq n$ .

Proof. Let $\overline{\Phi} = \Phi \mod fP$ and $\overline{\Psi} = \Psi \mod fP$ . We choose a homomorphism $\overline{\tau} : S^n \to L$ making the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\varepsilon} & L \\ \xi \downarrow & & \| \\ S^n & \xrightarrow{\overline{\tau}} & L \end{array}$$

commutative (here $\xi$ denotes the canonical epimorphism). Then by the proof of (2.9) we have an exact sequence

$$0 \to L \xrightarrow{\partial} S^n \xrightarrow{\overline{\Phi}} S^n \xrightarrow{\overline{\tau}} L \to 0$$

with Image $\overline{\Psi} = \partial(L) \subset n.S^n$ . Hence $b_{ij} \in m$ for all $1 \leqq i , j \leqq n$ and $\mu_S(\operatorname{Ker} \overline{\tau}) = n$ . As $\xi(M) = \operatorname{Ker} \overline{\tau}$ and $\mu_R(M) \leqq n$ , we get $\mu_R(M) = n$ too.

Let $\overline{f}$ denote the reduction of $f$ mod $fn$ .

Proposition (7.4). (1) $\overline{f}.R^n \subset mM$ .
(2) Let $u_1, u_2, \ldots, u_n$ be a system of generators for $M$ and $z \in \overline{f}R$ . Then there is a homomorphism $\eta : R^n \to R$ such that $\eta(u_i) = \delta_{i1}z$ for any $1 \leq i \leq n$ .

Proof. (1) Since $\phi\psi = \overline{f}1_{R^n}$ and $b_{ij} \in m$ $(1 \leq i , j \leq n)$, we get $\overline{f}R^n \subset mM$ .

(2) Let $\eta_i : R^n \to R$ denote the homomorphism defined by the i-th row of $\psi$ and put $\underline{a}_j = {}^t(a_{1j}, a_{2j}, \ldots, a_{nj})$, the j-th column of $\phi$ . Then as $\psi\phi = \overline{f}1_{R^n}$ , we have $\eta_i(\underline{a}_j) = \delta_{ij}\overline{f}$ for any $1 \leq i , j \leq n$ . We write for $1 \leq i \leq n$

$$u_i = \sum_{j=1}^{n} c_{ij}\underline{a}_j$$

with $c_{ij} \in R$ . Then since $\{u_i\}_{1 \leq i \leq n}$ and $\{\underline{a}_i\}_{1 \leq i \leq n}$ are both minimal systems of generators for $M$ (cf. (7.3)), we have the matrix $[c_{ij}]$ to be invertible. Let $z = c\overline{f}$ $(c \in R)$ and solve the equations

$$[c_{ij}]\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $x_i \in R$ $(1 \leq i \leq n)$. Then the homomorphism $\eta = \sum_{i=1}^{n} x_i\eta_i$ : $R^n \longrightarrow R$ has the required property.

Now choose an indecomposable maximal Buchsbaum R-module N and write

$$N/H_m^0(N) \cong \bigoplus_{i=1}^{r} L_i$$

with $L_i$ indecomposable maximal Cohen-Macaulay R-modules. Then as $\overline{f}R = H_m^0(R)$ and as $H_m^0(R).N \subset H_m^0(N)$ , each $L_i$ is annihilated by $\overline{f}R$ and may be considered to be an S-module. We claim

Lemma (7.5). Suppose $L_i \cong S$ for some $1 \leqq i \leqq r$ . Then $L_i \cong S$ for any $1 \leqq i \leqq r$ .

Proof. Assume the contrary and write

$$N/H_m^0(N) \cong S^m \oplus L$$

with L a maximal Cohen-Macaulay S-module which doesn't contain S as a direct summand. Let

$$0 \longrightarrow M \longrightarrow R^n \xrightarrow{\varepsilon} L \longrightarrow 0$$

be the initial part of a minimal free resolution of L . Then as $H_m^0(N) \subset mN$ (cf. (2.1)(2)), similarly as in the proof of (4.10) we get an R-submodule W of $(\overline{f}R)^m \oplus M$ so that

$$W \supset m.((\overline{f}R)^m \oplus M) \quad \text{and} \quad N \cong (R^m \oplus R^n)/W$$

(here $(\overline{f}R)^m$ denotes the direct sum of m copies of $\overline{f}R$ ). Let

$$V = (\overline{f}R)^m \oplus M$$

and let $\rho : V \to \overline{V} = V/mV$ denote the canonical epimorphism. We regard each element of $\overline{V}$ as a column vector with entries in $\overline{f}R$'s and M/mM . Let $U = \rho(W)$ and $r = \dim_k U$ (here $k = R/m$ ). Then $r \geqq 1$ , as N is indecomposable.

Let us take a k-basis $v_1, v_2, \ldots, v_r$ of U and consider the m + 1 by r matrix $C = (v_1, v_2, \ldots, v_r)$. Let

$(\xi_1, \xi_2, \ldots, \xi_r)$ be the $(m+1)$-th row of $C$ and put $t = \dim_k \sum_{i=1}^{r} k\xi_i$. Then after elementary column operations with coefficients in $k$, the matrix $C$ is transformed into

$$\left(\begin{array}{c|c} C_1 & C_2 \\ \hline \bar{u}_1\ \bar{u}_2 \ldots \bar{u}_t & 0 \end{array}\right) \begin{matrix} \Big\} m \\ \phantom{x} \end{matrix}$$

where $u_1, u_2, \ldots, u_t$ are part of a minimal system of generators for $M$ and $\bar{u}_i = u_i \bmod mM$ for any $1 \leq i \leq t$.

Now let $1 \leq i \leq m$ and $1 \leq j \leq t$. Let $(Z_1, z_2, \ldots, z_t)$ denote the i-th row of $C_1$. Then by (7.4) we may choose a homomorphism $\eta_j : R^n \longrightarrow R$ so that

$$\eta_j(u_\alpha) = -\delta_{\alpha j} z_j$$

for all $1 \leq \alpha \leq t$. Let $\psi_j$ denote the automorphism of $R^m \oplus R^n$ that sends each element ${}^t(x_1, \ldots, x_m, y)$ of $R^m \oplus R^n$ to ${}^t(x_1, \ldots, x_i + \eta_j(y), \ldots, x_m, y)$. Then the restriction $\psi_j|V$ is an automorphism of $V$ too, whence it induces an automorphism, say $\phi_j$, of $\bar{V}$. Because $\eta_j(u_\alpha) = -\delta_{\alpha j} z_j$ for $1 \leq \alpha \leq t$, the <u>row operation</u> $\phi_j$ reduces $z_j$ to $0$, while the other entries of the i-th row of the matrix $C_1$ doesn't change at all. Thus we know that by some automorphism of $\bar{V}$ induced from an automorphism $\delta$ of $R^m \oplus R^n$ with $\delta(V) = V$, $U$ is mapped onto the k-subspace $U'$ of $\bar{V}$ spanned by the columns of an $m+1$ by $r$ matrix of the following form

$$\left(\begin{array}{c|c} 0 & C_2 \\ \hline \bar{u}_1\ \bar{u}_2 \ldots \bar{u}_t & 0 \end{array}\right).$$

Let $W' = \rho^{-1}(U')$. Then as $\delta(W) = W'$, by (2.3)(2) we get an isomorphism

$$N \cong (R^m \oplus R^n)/W' ,$$

which forces N to be decomposable —— this is a contradiction.

Let us finish the proof of (7.1). Suppose first that $L_i \ncong S$ for any $1 \leq i \leq r$. Then we get, with the same notation (but $m = 0$) as in Proof of (7.5), that

$$N \cong R^n/W .$$

Because $\overline{f}.R^n \subset mM$ by (7.4)(1) and because $mM \subset W$, we find that the ideal $\overline{f}R$ annihilates N whence N is an S-module. If $L_i \cong S$ for any $1 \leq i \leq r$, we see with the same notation (but $L = (0)$) as in Proof of (7.5) that

$$N \cong R^m/W$$

for some R-submodule W of $(\overline{f}R)^m$. As $m.(\overline{f}R) = (0)$, it is now standard to check that $m = 1$ and

$$N \cong R \quad \text{or} \quad N \cong S$$

in this case. Thus any indecomposable maximal Buchsbaum R-module N is isomorphic to either R or an S-module. Hence

$$n_B(R) = 1 + n_B(S) ,$$

which completes the proof of (7.1).

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