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HARMONIC ANALYSIS OF ARITHMETICAL FUNCTIONS

by

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Introduction

The present article is largely an exposition of results treated more fully and generally in the author's paper [5], in Chapter 7 of his book [4], and in the paper of W. Schwarz and J. Spilker [8]. Some further developments of this subject are discussed in the articles of W. Schwarz and J. Spilker [9] and W. Schwarz [7].

The topic covered by the above heading is concerned with a kind of analysis of elementary arithmetical functions: $f: \mathbb{N} \to \mathbb{C}$ which is in many ways parallel to the classical analysis of certain ordinary, continuous real or complex functions in terms of periodic, or almost periodic, functions. In fact, it is possible to view it as one special aspect of the wide-ranging program of abstract harmonic analysis over general topological groups and semigroups. However, from this point of view, it is not *a priori* obvious that the application of such analysis to arithmetical functions should lead to conclusions of any great number-theoretical interest.

Although, with hindsight, it is to some extent possible to indicate intrinsic number-theoretical reasons for expecting or seeking some kind of "harmonic" analysis of arithmetical functions, the historical process of discovery occurred differently. In 1918, S. Ramanujan [6] published a paper containing a number of remarkable formulae expressing some particular arithmetical functions as the pointwise convergent sums of certain infinite trigonometrical series. For example, he proved that

$$\frac{\sigma(n)}{n} = \frac{1}{6}\pi^2 \left\{ 1 + \frac{(-1)^n}{2^2} + \frac{2\cos(\frac{2}{3}\pi n)}{3^2} + \frac{2\cos(\frac{1}{2}\pi n)}{4^2} \right.$$

$$\left. + \frac{2[\cos(\frac{2}{5}\pi n) + \cos(\frac{4}{5}\pi n)]}{5^2} + \frac{2\cos(\frac{1}{3}\pi n)}{6^2} + \ldots \right\},$$

where $n = 1,2,3,\ldots$ and $\sigma(n)$ is the sum of all the divisors of n. If one recalls that $\sigma(n)/n$ has the asymptotic mean-value $\frac{1}{6}\pi^2$, then the above formula provides a very striking indication of how the actual values of $\sigma(n)/n$ fluctuate "harmonically" about their mean-value. In briefer terms, this formula can be written as

$$\frac{\sigma(n)}{n} = \frac{1}{6}\pi^2 \sum_{r=1}^{\infty} \frac{c_r(n)}{r^2},$$

where the $c_r(n)$ are certain trigonometrical sums which later became known as *Ramanujan sums*.

By means of *ad hoc* methods, Ramanujan and (slightly thereafter) G.H. Hardy established certain further formulae for particular arithmetical functions in terms of the sums $c_r(n)$, some of which again exhibit the fluctuations of a function f with asymptotic mean value $m(f)$ about the value $m(f)$. Then, at a later stage, further authors began investigations into features of harmonic analysis which seemed to explain the existence of at least some of the special formulae; see particularly A. Wintner [10].

The paragraphs below outline some more recent results which help to explain the general significance of the Ramanujan sums in a different way. This way is one in which periodicity is replaced by a concept of *evenness*, which is perhaps more suitable for dealing with functions on a domain like $\mathbb{N}$ in which multiplication rather than addition often plays a dominant rôle. The latter concept is also valuable in extending this theory to functions on more general arithmetical semigroups than

$\mathbb{N}$, which arise elsewhere in algebraic number theory and in more abstract areas of mathematics. Some initial steps towards such a type of "multiplicative harmonic analysis" were first taken by E. Cohen (see e.g. Cohen [2]), and additional starting points were provided by W. Schwarz and J. Spilker [8], who introduced concepts of topology and functional analysis into the area.

1. Topological theory of Ramanujan sums.

By definition, any complex-valued function on the set $\mathbb{N}$ of natural numbers will be called an *arithmetical function* on $\mathbb{N}$. We shall let $\mathfrak{D}$ denote the set of all arithmetical functions. Under the point-wise operations, $\mathfrak{D}$ becomes a complex commutative algebra with identity, which may be viewed as the unrestricted direct product $\prod_{a \in \mathbb{N}} \mathbb{C}_a$ of copies $\mathbb{C}_a$ of the complex field $\mathbb{C}$. It will be convenient here to regard $\mathfrak{D}$ as a *topological* algebra relative to the cartesian *product* topology when each $\mathbb{C}_a$ has the usual topology of the complex numbers.

As indicated earlier, we shall be particularly interested in the *Ramanujan sums* c_r $(r \in \mathbb{N})$,

defined by

$$c_r(a) = \sum_{s \leq r, (r,s)=1} \exp(-2\pi i a s/r)$$

$$= \sum_{d|(r,a)} \mu\left(\frac{r}{d}\right) d \quad [a \in \mathbb{N}] ,$$

where $(r,s) = g.c.d.(r,s)$ and μ is the *Möbius* function. We shall let $\mathfrak{R}$ denote the vector subspace of $\mathfrak{D}$ spanned by all the Ramanujan sums c_r, and for a given element $k \in \mathbb{N}$ let $\mathfrak{R}_k$ denote the further subspace spanned by those functions c_r with $r|k$. If $r|k$ then $(r,a) = (r,(k,a))$, and so any function f on $\mathbb{N}$ which can be expressed in the form $f(a) = \sum_{d|(r,a)} g(d,r/d) [a \in \mathbb{N}]$ will have the property

(1.1) $\quad f(a) = f((k,a)) \quad$ for every $\quad a \in \mathbb{N}$;

following E. Cohen [2], functions with property (1.1) will be called *even* (mod k). Evidently the set $\mathfrak{D}_k$ of all even functions (mod k) is a subalgebra of $\mathfrak{D}$ containing $\mathfrak{R}_k$. In fact it can be proved that $\mathfrak{D}_k = \mathfrak{R}_k$.

It is easy to see that every even function f (mod k) over $\mathbb{N}$ is periodic (mod k); hence evenness may be viewed as a restricted form of periodicity. Nevertheless the next proposition shows that the set $\mathfrak{R} = \bigcup_{k \in \mathbb{N}} \mathfrak{R}_k$ of all even functions on $\mathbb{N}$ plays an important rôle in $\mathfrak{D}$.

(1.2) **Proposition.** The vector space $\mathfrak{R}$ constitutes a dense subalgebra of the topological algebra $\mathfrak{D}$ of all arithmetical functions on $\mathbb{N}$.

Sketch Proof. Let $C(X)$ denote the algebra of all continuous complex-valued functions on the given topological space X. If $\mathbb{N}$ is regarded as a *discrete* topological space, it may be deduced that the *compact-open* topology for $C(\mathbb{N}) = \mathfrak{D}$ coincides with the product topology considered earlier. Hence the *Stone-Weierstrass theorem* is applicable to $\mathfrak{D}$ and this was used by Schwarz and Spilker [8] to derive (1.2), with the aid of certain calculations of the values $c_r(a)$ for prime-powers r, as well.

(1.3) **Corollary.** Given any $f \in \mathfrak{D}$, any $\varepsilon > 0$ and any number $x > 0$, there exists a finite linear combination

$$g = g(f,\varepsilon,x) = \sum_i \alpha_i c_i \quad (\alpha_i \text{ complex})$$

such that $|f(a) - g(a)| < \varepsilon$ whenever $a \leq x$.

According to Proposition 1.7, every arithmetical function is "nearly" even in a certain sense, while the equation $\mathfrak{D}_k = \mathfrak{R}_k$ shows that every even function on $\mathbb{N}$ coincides with some "Ramanujan polynomial". If one follows the analogy with continuous functions of a real variable, it then becomes reasonable to investigate stronger forms of approximation by means of even functions.

In the first place, note that each function c_r is bounded on $\mathbb{N}$. Hence $\mathcal{R} \subseteq \mathcal{B}$, the Banach algebra of all *bounded* complex-valued functions on $\mathbb{N}$ under the *uniform* norm $\|f\|_\infty = \sup_{a\in\mathbb{N}} |f(a)|$. In analogy with the theory of Bohr almost periodic functions, it then appears reasonable to define an arithmetical function f to be (*uniformly*) *almost even* if and only if f belongs to the closure $\mathcal{D}^*$ of $\mathcal{R}$ in $\mathcal{B}$ relative to the uniform metric.

Since $\mathcal{R}$ is a subalgebra of $\mathcal{D}$, it follows from this definition that $\mathcal{D}^*$ is also a Banach algebra; in fact, it is a *commutative* B^*-*algebra*. Therefore the Gelfand-Naimark theorem implies that $\mathcal{D}^*$ is isometrically isomorphic to the Banach algebra $C(X)$ where X is a certain compact Hausdorff space - its maximal ideal space. The following theorem, which is independent of these remarks, gives more precise information as to how almost even functions may be represented by continuous functions.

(1.4) **Theorem.** There exists a commutative compact topological semigroup $\mathbb{N}^*$ with identity, and a monomorphism $\tau: \mathbb{N} \to \mathbb{N}^*$ which sends $\mathbb{N}$ onto a dense subset of $\mathbb{N}^*$, such that the adjoint homomorphism $\tau^* : C(\mathbb{N}^*) \to \mathcal{B}$ maps $C(\mathbb{N}^*)$ isomorphically and isometrically onto $\mathcal{D}^*$.

Sketch Proof. We shall outline a topological construction introduced by Schwarz and Spilker [8]. For the construction, first consider any prime p and regard the set $\{p^r : r=0,1,2,...\}$ as a discrete topological space. Let $\mathbb{N}_{(p)}$ denote the one-point compactification of this space, and let p^∞ denote the extra point adjoined. Then $\mathbb{N}_{(p)}$ becomes a commutative compact topological semigroup with identity if one uses the multiplication of prime-powers in $\mathbb{N}$ and in addition lets $(p^\infty)^2 = p^\infty$ and $p^\infty p^r = p^r p^\infty = p^\infty$ for $r = 0,1,2,...$.

Now form the topological product $\mathbb{N}^* = \prod_{p\in P} \mathbb{N}_{(p)}$. Then $\mathbb{N}^*$ is a compact topological semigroup, and the existence of unique factorization into prime-powers for the elements of $\mathbb{N}$ gives rise to a semigroup monomorphism $\tau: \mathbb{N} \to \mathbb{N}^*$. By using the definition of the product topology and of the topology in each $\mathbb{N}_{(p)}$, it is easy to verify that $\tau(\mathbb{N})$ is dense in $\mathbb{N}^*$. It can then be shown that the adjoint algebra homomorphism $\tau^*: C(\mathbb{N}^*) \rightarrow \mathcal{B}$ is a norm-preserving algebra isomorphism.

The above property of $\mathbb{N}^*$ relative to the almost even functions on $\mathbb{N}$ is directly analogous to that of the *Bohr compactification* of a topological group, or the *almost periodic compactification* of a topological semigroup, relative to the almost periodic functions on the group or semigroup. Hence it is natural to call $\mathbb{N}^*$ the *almost even compactification* of $\mathbb{N}$. (It may be interesting to note in passing that *algebraically*, the elements of $\mathbb{N}^*$ occur also in the theory of fields, where they are sometimes called *Steinitz numbers*.) The following uniqueness property of $\mathbb{N}^*$ provides a further parallel to properties of the Bohr or almost periodic compactifications.

(1.5) **Proposition.** Let $\mathbb{N}$ denote a commutative compact topological semigroup with identity. Suppose that there exists a homomorphism $\rho: \mathbb{N} \rightarrow \mathbb{N}'$ of $\mathbb{N}$ onto a dense subset of $\mathbb{N}'$, such that the adjoint homomorphism $\rho: C(\mathbb{N}') \rightarrow \mathcal{B}$ gives an algebra isomorphism of $C(\mathbb{N}')$ onto $\mathcal{D}^*$. Then there exists a topological isomorphism $\alpha: \mathbb{N}' \rightarrow \mathbb{N}^*$ such that $\alpha\rho = \tau$.

2. A Fourier theory of almost even functions.

Recall that the *mean-value* of an arithmeical function f is defined to be

$$m(f) = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{a \leq x} f(a) ,$$

if the limit exists. We shall next indicate how the existence of mean-values in certain cases leads to the existence of a probability measure ν on $\mathbb{N}^*$, which can then be used to initiate a Fourier theory of almost even functions, involving a definition of inner product which involves *integration* over the compactification $\mathbb{N}^*$. In order to establish the existence of a suitable integral over $\mathbb{N}^*$, first consider

(2.1) **Theorem.** Every almost even function f possesses a finite asymptotic mean-value

$$m(f) = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{a \leq x} f(a) .$$

Schwarz and Spilker [8] deduced this statement from the fact that the classical Ramanujan sums are *periodic* functions of a positive integer, but one can also avoid periodicity and this fact is useful to know in certain situations as well.

(2.2) **Corollary.** There exists a regular and complete probability measure ν on a σ-algebra of subsets of $\mathbb{N}^*$ containing all the open sets, such that for every almost even function f

$$\int f^* d\nu = m(f)$$

where $f^* = \tau^{*-1}(f) \in C(\mathbb{N}^*)$.

Sketch Proof. This corollary is due to Schwarz and Spilker [8]. The proof amounts to an application of the *Riesz Representation Theorem,* since the rule $f \to m(f)$ and the isomorphism $\mathcal{D}^* \cong C(\mathbb{N}^*)$ clearly define a non-negative complex-linear functional on $C(\mathbb{N}^*)$. The resulting measure ν then satisfies $\nu(\mathbb{N}^*) = \int c_1^* \, d\nu = m(c_1) = 1$. (It can also be proved that $\tau(\mathbb{N})$ *has measure zero* in $\mathbb{N}^*$.)

For almost even functions, Corollary 2.2 leads to

(2.3) **Theorem.** The rule $f \to f^* = \tau^{*-1}(f)$ defines an embedding of the Banach algebra $\mathcal{D}^*$ of all almost even functions on $\mathbb{N}$ into the Hilbert space $L^2(\nu)$ of all square-integrable functions on $\mathbb{N}^*$ relative to the measure ν. The (new) inner product on $\mathcal{D}^*$ is given by $\langle f,g \rangle = m(f\cdot\bar{g}) = \int f^* \, \overline{g^*} d\nu$. A complete orthonormal subset of $L^2(\nu)$ is provided by all the functions C_r^* where

$$C_r = \frac{1}{\sqrt{\phi(r)}} c_r \quad (r \in \mathbb{N}).$$

In terms of the above pre-Hilbert space structure on $\mathcal{D}^*$, it now follows that every almost even function f has a unique L^2-convergent *Ramanujan expansion*

$$f = \sum_{r \in \mathbb{N}} F(r) c_r, \quad \text{where} \quad F(r) = \frac{1}{\phi(r)} \cdot \langle f, c_r \rangle$$

may be called the r-th *Ramanujan coefficient* of f. In terms of this notation, the corresponding Parseval identity takes the form

$$<f_1,f_2> = \sum_{r\in\mathbb{N}} \phi(r)F_1(r)\overline{F_2(r)}\ .$$

In the next sections we shall consider some wider classes of arithmetical functions f such that all the corresponding Ramanujan coefficients $F(r)$ exist, and relate these to the Lebesgue spaces $L^q(\nu)[q\geq 1]$ over $\mathbb{N}^*$. Some more specifically number-theoretical questions about which functions lie in these classses, and when the Ramanujan expansion $\Sigma F(r)c_r$ converges *pointwise* to f, will be discussed in the final section.

3. A Besicovitch-type Fourier theory.

Following the idea of the Besicovitch theory of ordinary almost periodic functions, we now consider some weaker modes of approximation by means of even functions. For this purpose, given a real-valued arithmetical function g, first consider the *upper mean-value* $\overline{m}(g) = \limsup_{x\to\infty} m(g,x)$ where $m(g,x) = \frac{1}{x}\sum_{a\leq x} g(a)$. For $q\geq 1$, let $\mathcal{B}^q$ denote the set of all arithmetical functions f such that $|f|^q$ has a finite upper mean-value. Then it follows from Minkowski's inequality that $\mathcal{B}^q$ is a *vector subspace* of $\mathcal{D}$, on which $\|f\|_q = [\overline{m}(|f|^q)]^{1/q}$ behaves as a *seminorm*. As in ordinary function theory, it is easy to deduce that $\|f\|_q \leq \|f\|_{q'} \leq \|f\|_\infty$ and $\mathcal{B}\subseteq \mathcal{B}^{q'}\subseteq \mathcal{B}^q \subseteq \mathcal{B}^1$ for $q'\geq q\geq 1$.

Now define an arithmetical function f to be *almost even* (B^q) if and only if f belongs to the closure B^q, of the vector space $\mathcal{R}$ in $\mathcal{B}^q$ relative to the topology defined by $\|f\|_q$; if $q=1$, f will simply be called *almost even* (B).

(3.1) **Theorem.** Every almost even (B) function f possesses a finite asymptotic mean-value $m(f)$. In particular, $\|f\|_1 = m(|f|)$ and all the Ramanujan coefficients $F(r) = (1/\phi(r))m(f\cdot c_r)[r\in\mathbb{N}]$ exist.

If f and g are almost even (B) functions such that $\|f - g\|_1 = 0$ it is easy to see that f and g have the same $\mathcal{B}^1$-seminorm, and same Ramanujan coefficients. For such and other reasons, as in the study of Lebesque spaces L^q, we sometimes slur over the distinction between $\mathcal{B}^q$ (or B^q) and: $\mathcal{B}^q$ (or B^q) *modulo the "$\mathcal{B}^q$-null" functions* (i.e. functions h such that $\|h\|_q = 0$). In this sense, we now state

(3.2) **Theorem.** For any $q \geqq 1$, $\mathcal{B}^q$ and B^q are complex Banach spaces.

Sketch Proof. Since B^q is closed in $\mathcal{B}^q$, it is sufficient to consider $\mathcal{B}^q$ here. This case can then be treated by an argument which is analogous to the proof of Bohr and Følner [1] that the classical Besicovitch almost periodic spaces are complete.

Now consider the following analogue of Følner's *Correspondence Theorem* [3] connecting the classical Besicovitch almost periodic spaces with the Lebesque spaces over the Bohr compactification of the real line:

(3.3) **Theorem.** For $q \geqq 1$, consider the Lebesque space $L^q(\nu)$ over the almost even compactification $\mathbb{N}^*$ of $\mathbb{N}$ There exists a norm-preserving isomorphism $*: B^q \to L^q(\nu)$ such that $f^* = \tau^{*-1}(f)$ for every $f \in \mathcal{D}^*$.

Sketch Proof. Convergence in $\mathcal{B}^q$ or L^q will be indicated by the symbol $\overset{q}{\to}$. Given $f \in B^q$, there exists a sequence of even functions g_n such that $g_n \overset{q}{\to} f$. Then (g_n) is a Cauchy sequence in $\mathcal{B}^q$, and so inside $L^q(\nu)$ the corresponding functions $g^*_n = \tau^{*-1}(g_n) \in C(\mathbb{N}^*)$ satisfy

$$\|g^*_i - g^*_j\|_q = \left[\int |g^*_i - g^*_j|^q d\nu\right]^{1/q} = [m(|g_i - g_j|^q)]^{1/q} = \|g_i - g_j\|_q ,$$

since $\tau^*(|g^*_i - g^*_j|^q) = |g_i - g_j|^q$. Therefore (g^*_n) is a Cauchy sequence in $L^q(\nu)$, and so it must have some limit f^* say. Also, it $h_n \overset{q}{\to} f$ where h_n is even, then the sequence (h^*_n) must similarly have a limit $f' \in L^q(\nu)$; then one may verify that $\|f^* - f\|_q = 0$. Therefore, by reducing modulo "null" functions, one obtains a well-defined linear map $*: B^q \to L^q(\nu)$ whose restriction to $\mathcal{D}^*$ is given by τ^{*-1}. This map can then be shown to be a norm-preserving isomorphism.

(3.4) **Corollary.** For $q > 1$, the dual space of B^q is isomorphic to $B^{q'}$, where $q' = q/(q-1)$.

(3.5) **Corollary.** The space $B^2 \cong L^2(\nu)$ is the Hilbert space completion of the algebra $\mathcal{D}^*$ of all uniformly almost even functions on $\mathbb{N}$. The inner product on B^2 is given by

$$< f, g > = m(f \cdot \bar{g}) = \int f^* \overline{g^*} d\nu .$$

In order to prove the above statement about the inner product on B^2 one first needs the conclusion that, if $f \in B^1$, then

$$m(f) = \int f^* \, d\nu .$$

(3.6) **Corollary.** The functions

$$C_r = \frac{1}{\sqrt{\phi(r)}} c_r \quad (r \in \mathbb{N})$$

make up an orthonormal basis for the Hilbert space B^2. Hence:

(i) The Ramanujan expansion $\Sigma F(r) c_r$ of every almost even (B^2) function f is B^2-convergent to f;

(ii) if $F_1(r)$, $F_2(r)$ denote the Ramanujan coefficients of $f_1, f_2 \in B^2$ then (*Parseval identity*)

$$m(f_1 \cdot \overline{f_2}) = \sum_{r \in \mathbb{N}} \phi(r) F_1(r) \overline{F_2(r)};$$

(iii) every series $\sum_{r \in \mathbb{N}} F(r) c_r$ such that $\sum_{r \in \mathbb{N}} \phi(r) |F(r)|^2 < \infty$ is the Ramanujan expansion of some $f \in B^2$ (*Riesz-Fischer theorem*).

4. Pointwise ordinary convergence of expansions

We now associate with any arithmetical function f on G the Dirichlet series $\tilde{f}(z) = \sum_{a \in \mathbb{N}} f(a) a^{-z}$, where z is regarded as a complex variable. In particular, if $\zeta(a) = 1 \; [a \in \mathbb{N}]$ then $\zeta(z) = \tilde{\zeta}(z)$ is the *Riemann zeta* function.

(4.1) **Theorem.** Let f and g denote arithmetical functions such that $f(a) = \sum_{d|a} g(d)[a \in G]$, and $\tilde{g}(z)$ is absolutely convergent whenever $Re\ z > \alpha$, for some $\alpha < 1$. Then f is almost even (B) and its r-th Ramanujan coefficient is

$$F(r) = \frac{1}{\phi(r)} m(f \cdot c_r) = \sum_{\substack{b \in \mathbb{N} \\ r|b}} g(b)\ b^{-\delta} .$$

Further, in terms of ordinary convergence of complex numbers,

$$f(a) = \sum_{r \in \mathbb{N}} F(r) c_r(a) ,$$

the right-hand series being absolutely convergent, for each $a \in \mathbb{N}$.

The following functions provide examples to which Theorem 4.1 may be applied:

(i) The functions $a(n)$ and $S(n)$ such that $a(n)$ is the total number of isomorphism classes of *finite abelian groups* of order n, and $S(n)$ is the total number of isomorphism classes of *semisimple finite rings* of cardinal n;

(ii) the quotients d/d_* and d_*/d, where we consider the *divisor* function $d(a)$ of $a \in \mathbb{N}$, and the *unitary-divisor* function d_* such that $d_*(a)$ is the total number of *unitary* divisors of $a \in \mathbb{N}$ (i.e. divisors b with $(b,a/b) = 1$) ;

(iii) the function β on $\mathbb{N}$ such that $\beta(a)$ is the number of *square-full* divisors of $a \in \mathbb{N}$ (i.e. divisors b such that $p^2 | b$ whenever a prime $p | b$);

(iv) the characteristic function q_k of the set $\mathbb{N}_k$ of all k-*free* numbers (i.e. numbers b which are divisible by no k-th power $c^k \neq 1$ in $\mathbb{N}$); and lastly

(v) the functions $\sigma(a)/a$ and $\phi(a)/a$, where $\sigma(a) = \sum_{d|a} d$ and ϕ is Euler's function, lead back to two of the *original* trigonometrical expansions of Ramanujan referred to earlier.

More delicate number-theoretical theorems about Ramanujan coefficients and the pointwise ordinary convergence of Ramanujan expansions for special classes of arithmetical functions are given in the references quoted in the introduction. It should however be mentioned that there also exist examples of pointwise convergent Ramanujan expansions for particular functions of interest, which do *not* fit within the previous framework.

For example, in his original paper, Ramanujan [6] established the following formulae for the divisor function d, and for the function $r(n)$ giving the number of lattice points (a,b) with $a^2 + b^2 = n$:-

$$d(n) = - \sum_{k=1}^{\infty} \frac{\log k}{k} c_k(n); \tag{4.2}$$

$$r(n) = \pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} c_{2k-1}(n) . \tag{4.3}$$

The proofs of these formulae are *ad hoc* ones depending on results of roughly the same order of difficulty as the Prime Number Theorem. The fact that the formulae do not fit within the previous scheme may be seen in the first case from the fact that d does not have a finite asymptotic mean value, and so is certainly not almost even (B). In the second case, although $r(n)$ does have the mean-value π, so that (4.3) has the same general appearance as the formulae considered earlier, it turns out that $r(n)$ is nevertheless not even almost periodic (B).

Perhaps, as in the general theory of trigonometrical series, there exist "pathological" functions and expansions in the present arithmetical setting which cannot readily be covered by a systematic theory. In any case, it remains to be seen whether or not there may be some deeper conceptual explanation for such formulae as (4.2) and (4.3).

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