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Fourier Eisenstein transform on the space of rational binary quadratic forms

Fumihiro SATO (Rikkyo Univ.)

§ 0. Introduction

Put $G = GL(2,\mathbf{Q})$ and $\Gamma = GL(2,\mathbf{Z})$. Let X be the set of nondegenerate rational symmetric matrices of size 2: $X = \{x \in M(2,\mathbf{Q});\ {}^t x = x,\ \det x \neq 0\}$. The group G acts on X via $x \longmapsto gx{}^tg$ $(x \in X,\ g \in G)$. We put

$$X_K = \{x \in X;\ (-\det x)^{1/2} \in K \setminus \mathbf{Q}\}$$

for a quadratic field K and

$$X_{\mathbf{Q}} = \{x \in X;\ (-\det x)^{1/2} \in \mathbf{Q}^{\times}\}.$$

Then X is decomposed into G-stable subsets as follows:

$$X = (\bigcup_{K:\text{quadratic}} X_K) \cup X_{\mathbf{Q}}.$$

Let $\mathcal{H}(G,\Gamma)$ be the Hecke algebra of G with respect to Γ and $\mathcal{S}(\Gamma\setminus X)$ (resp. $\mathcal{S}(\Gamma\setminus X_K)$, K = a quadratic field or $\mathbf{Q}$) the space of finite $\mathbf{C}$-linear combinations of characteristic functions of Γ-orbits in X (resp. X_K). The space $\mathcal{S}(\Gamma\setminus X)$ becomes an $\mathcal{H}(G,\Gamma)$-module and

$$\mathcal{S}(\Gamma\setminus X) = (\bigoplus_{K:\text{quadratic}} \mathcal{S}(\Gamma\setminus X_K)) \oplus \mathcal{S}(\Gamma\setminus X_{\mathbf{Q}})$$

is a direct sum of sub $\mathcal{H}(G,\Gamma)$-modules. Our aim is to describe the $\mathcal{H}(G,\Gamma)$-module structure of $\mathcal{S}(\Gamma\setminus X)$ explicitly. It is obvious that we need to consider the sub $\mathcal{H}(G,\Gamma)$-modules $\mathcal{S}(\Gamma\setminus X_K)$ (K = a quadratic field or $\mathbf{Q}$). We can see that each $\mathcal{S}(\Gamma\setminus X_K)$ is a free $\mathcal{H}(G,\Gamma)$-module of infinite rank. To see this, we introduce a kind of integral transform, which we call the *Fourier Eisenstein transform*, defined by taking the zeta functions of binary quadratic forms as a kernel function. Here the zeta functions of binary quadratic forms play a similar role to that of zonal spherical functions in the theory of the Satake transform of p-adic groups ([S1], [M1]).

In this note we consider only the case $K = \mathbf{Q}$ and at the end of the note we shall give a brief indication how the result should be modified in the case of quadratic fields as well as a discussion on a possible generalization to higher dimensional symmetric spaces.

Finally we note that an analogous problem has been investigated by Y.Hironaka in [H] for the space of nondegenerate symmetric matrices over a p-adic number field.

§ 1. Fourier Eisenstein transform

1.1. For an $x \in X_{\mathbf{Q}}$ we denote by $x[u] = {}^t uxu$ $(u \in \mathbf{Q}^2)$ the binary quadratic form corresponding to the matrix x. Put

$$Z_{\varepsilon,s}(x) = \sum_{\substack{u \in \mathbf{Z}^2 \\ x[u] \neq 0}} \operatorname{sgn}(x[u])^{\varepsilon} \, |x[u]|^{-s_1} \, |\det x|^{-s_2}$$

$(s = (s_1, s_2) \in \mathbf{C}^2,\ \varepsilon = 0, 1)$ and

$$\Xi_\lambda(x) = \Xi_{\lambda_1,\lambda_2}(x) = \pi^{-s_1}\,\Gamma(s_1)\begin{pmatrix} \cos(\pi s_1/2) Z_{0,s}(x) \\ \sin(\pi s_1/2) Z_{1,s}(x) \end{pmatrix},$$

where $\lambda = (\lambda_1, \lambda_2)$ is a variable which relates with s

by

$$\begin{cases} s_1 = \lambda_2 - \lambda_1 + 1/2 \\ s_2 = \quad\; - \lambda_2 + 1/2. \end{cases}$$

Fundamental properties of the zeta function $Z_{\varepsilon,s}(x)$ can be summarized as follows:

Proposition 1. (i) *$Z_{\varepsilon,s}(x)$ is absolutely convergent for* $\mathrm{Re}\, s_1 > 1$.

(ii) *$Z_{\varepsilon,s}(x)$ has an analytic continuation to a meromorphic function of s in $\mathbf{C}^2$.*

(iii) *The following functional equation holds for any $x \in X_{\mathbf{Q}}$:*

$$\Xi_{\lambda_2,\lambda_1}(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \Xi_{\lambda_1,\lambda_2}(x).$$

1.2. Let $\mathcal{H}(G,\Gamma)$ be the Hecke algebra of G with respect to Γ. The underlying vector space of $\mathcal{H}(G,\Gamma)$ is by definition the space of all finite $\mathbf{C}$-linear combinations of characteristic functions of double cosets in $\Gamma\backslash G/\Gamma$. Denote by $[g]$ $(g \in G)$ the characteristic function of the double coset $\Gamma g \Gamma$. The multiplication of $[g_1]$ and $[g_2]$ in $\mathcal{H}(G,\Gamma)$ is defined as follows: Decompose the double coset $\Gamma g_i \Gamma$ into left cosets

$$\Gamma g_i \Gamma = \bigcup_{j=1}^{k_i} g_{ij}\Gamma \quad (i = 1,2)$$

and put

$$m(g_1,g_2;h) = \#\{(j_1,j_2);\ g_{1j_1} g_{2j_2}\Gamma = h\Gamma\} \quad (h \in G).$$

Then

$$[g_1]\cdot[g_2] = \sum_{\Gamma h\Gamma\in\Gamma\backslash G/\Gamma} m(g_1,g_2;h)[h].$$

Let $C^\infty(\Gamma\backslash X_{\mathbf{Q}})$ be the space of all Γ-invariant $\mathbf{C}$-valued functions on $X_{\mathbf{Q}}$ and $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$ its subspace of functions which vanish outside some finite union of Γ-orbits in $X_{\mathbf{Q}}$. For $\Phi\in C^\infty(\Gamma\backslash X_{\mathbf{Q}})$ and $g\in G$, we define the action of $[g]$ on Φ by

$$([g]*\Phi)(x) = \sum_{i=1}^{k} \Phi(g_i^{-1}x\,{}^t g_i^{-1}),$$

where $\Gamma g\Gamma = \bigcup_{i=1}^{k} g_i\Gamma$ (disjoint union). Then it is easy to see that $C^\infty(\Gamma\backslash X_{\mathbf{Q}})$ becomes an $\mathcal{H}(G,\Gamma)$-module and $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$ is its $\mathcal{H}(G,\Gamma)$-submodule.

1.3. Since $Z_{\varepsilon,s}({}^t\gamma x\gamma) = Z_{\varepsilon,s}(x)$ for any $\gamma\in\Gamma$, we may regard $Z_{\varepsilon,s}(x)$ as a function in $C^\infty(\Gamma\backslash X_{\mathbf{Q}})$.

Proposition 2. *There exists a* $\mathbf{C}$-*algebra homomorphism* ω_λ *of* $\mathcal{H}(G,\Gamma)$ *into* $\mathbf{C}$ *satisfying*

$$(f*Z_{\varepsilon,s})(x) = \omega_\lambda(f)\cdot Z_{\varepsilon,s}(x)$$

for all $f\in\mathcal{H}(G,\Gamma)$. *For any prime number* p,

$$\omega_\lambda\left(\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\right]\right) = p^2(p^{-2\lambda_1}+p^{-2\lambda_2})$$

and for any non-zero rational number a,

$$\omega_\lambda\left(\left[\begin{pmatrix}a&0\\0&a\end{pmatrix}\right]\right) = |a|^{3-2(\lambda_1+\lambda_2)}.$$

Denote by $\mathfrak{R}$ the restricted tensor product $\otimes'_p\ \mathbf{C}[p^{-2\lambda_1}+p^{-2\lambda_2},\ p^{\pm 2(\lambda_1+\lambda_2)}]$, where p runs through all rational primes. Then it is known that ω_λ gives rise to an algebra isomorphism of $\mathcal{H}(G,\Gamma)$ onto $\mathcal{R}$.

For an $x \in X_{\mathbf{Q}}$ let ch_x $(\in \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}}))$ be the characteristic function of Γ-orbit containing x.

Definition. For a $\Phi = \sum_i c_i \; ch_{x_i} \in \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$ $(x_i \in X_{\mathbf{Q}})$ we call

$$F_\varepsilon(\Phi)(\lambda) = \sum_i c_i \, Z_{\varepsilon,s}(x_i) \qquad (\varepsilon = 0,1)$$

the *Fourier Eisenstein transform* of Φ.

Proposition 3.

(i) $F_\varepsilon(f*\Phi) = \omega_\lambda(f)\cdot F_\varepsilon(\Phi)$ $\quad (f \in \mathcal{H}(G,\Gamma),\ \Phi \in \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}}))$.

(ii) $F_\varepsilon(\Phi_1) = F_\varepsilon(\Phi_2)$ *if and only if* $\Phi_1 = \Phi_2$

$$(\Phi_1,\ \Phi_2 \in \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})).$$

1.4. For positive rational numbers r and q, we denote by $ch_{q,r}$ the characteristic function of the Γ-orbit containing $r\cdot\begin{pmatrix} q & 1 \\ 1 & 0 \end{pmatrix}$. Let N be a positive integer. We define an equivalence relation on $(\mathbf{Z}/(N))^\times$ as follows:

$$x \sim y \iff x = y \text{ or } y^{-1} \quad (x,\ y \in (\mathbf{Z}/(N))^\times).$$

Then the set

$$\left\{ch_{m/N,r};\ \begin{array}{l} r \in \mathbf{Q},\ r > 0,\ N \in \mathbf{Z},\ N \geq 1 \\ m = 0, \quad \text{if } N = 1, \\ m \in (\mathbf{Z}/(N))^\times/\sim, \quad \text{if } N \geq 2 \end{array}\right\}$$

forms a $\mathbf{C}$-basis of $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$.

For a primitive Dirichlet character χ with conductor f_χ, we define a linear mapping

$$T_\chi : \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}}) \longrightarrow \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$$

by

$$T_\chi(ch_{m/N,r}) = \varphi(f_\chi N)^{-1} \sum_{\alpha\in(\mathbb{Z}/(f_\chi N))^\times} \chi(\alpha)\ ch_{\alpha m/N,r},$$

where $\varphi(\ )$ stands for the Euler function.

Proposition 4.

(i) $T_\chi(ch_{m/N,r}) = 0$ *unless* f_χ *divides* N.

(ii) $T_\chi = T_{\bar\chi}$ *and* $T_\chi \circ T_\chi = T_\chi$.

(iii) *For any primitive Dirichlet characters* χ *and* ψ $T_\chi \circ T_\psi = T_\psi \circ T_\chi = 0$ *unless* $\chi = \psi$ *or* $\bar\psi$.

(iv) T_χ *commutes with the action of* $\mathcal{H}(G,\Gamma)$ *on* $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$.

For a square free positive integer m, we put

$$\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m} = \mathcal{H}(G,\Gamma)\cdot T_\chi(ch_{1/f_\chi,m}).$$

It is obvious that $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m} = \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\bar\chi,m}$.

Theorem. (i) *For a primitive Dirichlet character* χ, *set* $\varepsilon_\chi = 0$ *or* 1 *according as* $\chi(-1) = 1$ *or* -1. *For* $\Phi \in \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m}$, *set*

$$F_{\chi,m}(\Phi) = F_{\varepsilon_\chi}(\Phi)\cdot 2^{s_1-2}\ f_\chi^{-s_1}\ m^{\lambda_1+\lambda_2}\ \varphi(f_\chi)/L(s_1,\chi)L(s_1,\bar\chi),$$

where $L(s,\chi)$ *is the Dirichlet L-function. Then* $F_{\chi,m}(\Phi)$ *is in* $\mathfrak{R}$ *and the mapping*

$$F_{\chi,m} : \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m} \longrightarrow \mathfrak{R}$$

is a linear isomorphism and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{H}(G,\Gamma) \times \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m} & \xrightarrow{\ \ *\ \ } & \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m} \\ {\scriptstyle \omega_\lambda \times F_{\chi,m}} \Big\downarrow \simeq & & {\scriptstyle F_{\chi,m}} \Big\downarrow \simeq \\ \mathfrak{R} \times \mathfrak{R} & \xrightarrow{\ \ \textit{multiplication}\ \ } & \mathfrak{R} \end{array} .$$

In particular $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m}$ *is a free* $\mathcal{H}(G,\Gamma)$*-module of rank* 1.

(ii) *Let* $\mathfrak{X}$ *be the set of all Dirichlet characters. Then as* $\mathcal{H}(G,\Gamma)$*-modules, we have the decomposition*

$$\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}}) = \bigoplus_{\chi\in\mathfrak{X}/\sim} \ \bigoplus_{m:\text{square free}} \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m},$$

where the equivalence relation $\sim$ *on* $\mathfrak{X}$ *is defined by*

$$\chi \sim \psi \iff \chi = \psi \ \text{ or } \ \overline{\psi}.$$

Thus $\mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})$ *is a free* $\mathcal{H}(G,\Gamma)$*-module of infinite rank.*

Remark. For a $\Phi \in \mathcal{S}(\Gamma\backslash X_{\mathbf{Q}})_{\chi,m}$, we have $F_\varepsilon(\Phi) = 0$, if $\varepsilon \neq \varepsilon_\chi$.

The proof is based on the following explicit calculation of the Fourier Eisenstein transform, namely the zeta functions of binary quadratic forms.

Proposition 5. *For a positive rational number* r, *a positive integer* N *and an integer* m *with* $(N,m) = 1$, *we have*

$$F_\varepsilon(ch_{m/N,r})(\lambda) = 2^{2-s_1}\, r^{-s_1-2s_2}\, N^{-s_1} \times \sum_{M|N} \varphi(m)^{-1} M^{2s_1} \sum_{\substack{\chi:\text{mod } M \\ \chi(-1)=(-1)^\varepsilon}} \chi(m) L(s_1,\chi) L(s_1,\overline{\chi}).$$

§ 2. Remarks

2.1. For a quadratic field K, we can define a Fourier Eisenstein transform of $\mathcal{G}(\Gamma\backslash X_K)$ by using zeta functions of binary quadratic forms (irreducible over $\mathbf{Q}$). An analogue of the intertwining operator T_χ is obtained from characters of the class groups of orders of K and we can get a structure theorem of $\mathcal{G}(\Gamma\backslash X_K)$ quite similar to the theorem above.

If K is an imaginary quadratic field, Mautner [M2] obtained essentially the same result, though he employed the formulation in terms of $PGL(2)$ rather than $GL(2)$. However, since he reduced the things to the local case, the role played by the zeta functions of binary quadratic forms is not clear in his work.

2.2. It is likely that the result of this paper can be generalized to higher dimensional (not necessarily Riemannian) reductive symmetric spaces defined over $\mathbf{Q}$. In a general case a kernel function of the Fourier Eisenstein transform will be Eisenstein series of the type introduced in [S3] (see also [S2], [S4]). In particular, if Eisenstein series has an Euler product expansion, then the theory of the Fourier Eisenstein transform will be a synthesis of local theories (theory of spherical functions on symmetric spaces over p-adic number fields). Such examples are supplied by

(i) $G = GL(2n)$, $X = GL(2n)/Sp(n)$

= the space of nondegenerate alternating forms.

(ii) $G = G_0 \times G_0$, $X = G_0 \times G_0/\Delta(G_0) \simeq G_0$,

where G_0 is a simply connected Chevalley group over $\mathbf{Q}$. The corresponding local theory is given by [HS] for the first example and by the theory of zonal sphericasl functions on p-adic reductive groups ([S1], [M1]) for the second example.

References

[H] Y.Hironaka, Spherical functions of hermitian and symmetric forms I, II, III, Preprint 1987.

[HS] Y.Hironaka and F.Sato, Spherical functions and local densities of alternating forms, to appear in Amer. J. Math.

[M1] I.G.Macdonald, Spherical functions on a group of p-adic type, Publ. Ramanujan Institute No.2, Madras, 1971.

[M2] F.I.Mautner, Fonctions propres des operateurs de Hecke, C. R. Acad. Sci. Paris, Serie A 269(1969), 940-943; 270(1970), 89-92.

[S1] I.Satake, Theory of spherical functions on reductive algebraic groups over p-adic fields, Publ. Math. I.H.E.S. 18(1963), 5-70.

[S2] F.Sato, Zeta functions in several variables associated with prehomogeneous vector spaces III: Eisenstein series for indefinite quadratic forms, Ann. of Math., 116(1982), 177-212.

[S3] F.Sato, Eisenstein series on semisimple symmetric spaces of Chevalley groups, Adv. Studies in pure Math. 7(1985), 295-332.

[S4] F.Sato, Eisenstein series on the Siegel half space of signature (1,1), Preprint 1986.