数理解析研究所講究録
第 632 巻 1987 年 42-54

Closure relations for orbits on affine symmetric spaces under the action of parabolic subgroups. Intersections of associated orbits

Toshihiko MATSUKI 松木敏彦
鳥取大教養

§1. Introduction

Let G be a connected Lie group, σ an involution of G and H an open subgroup of $G^{\sigma} = \{x \in G \mid \sigma x = x\}$. Then the G-homogeneous manifold $H\backslash G$ is called an affine symmetric space. Suppose that G is a real semisimple Lie group. Let P be a minimal parabolic subgroup of G and P' a parabolic subgroup of G containing P. Then the double coset decompostion $H\backslash G/P$ is studied in [2] and [5], the relation between $H\backslash G/P'$ and $H\backslash G/P$ is studied in [3], and the closure relation for $H\backslash G/P$ is studied in [4].

Let θ be a Cartan involution of G such that $\sigma\theta = \theta\sigma$. Put $K = G^{\theta}$ and let H^a be the open subgroup of $G^{\sigma\theta}$ such that $K \cap H = K \cap H^a$. Then $H^a\backslash G$ is called the affine symmetric space associated to $H\backslash G$. Let A be a θ-stable split component of P and put $U = \{x \in K \mid xAx^{-1}$ is σ-stable$\}$.

There exists a natural one-to-one correspondence between the double coset decompositions $H\backslash G/P'$ and $H^a\backslash G/P'$ given by $D \to D^a = H^a(D \cap U)P'$ for H-P' double cosets D in G ([2], [3]). Moreover it follows easily from Corollary of [4] Theorem that this correspondence reverses the closure relations for the double coset decompositions. In this paper we prove the following theorem.

Theorem. Let D_1 and D_2 be arbitrary H-P' double cosets in G.

(i) $D_1^{cl} \supset D_2 \iff D_1 \cap D_2^a \neq \phi$.

(ii) Let $I(D_1, D_2)$ be the set of all the H-P' double cosets D in G such that $D_1^{cl} \supset D^{cl} \supset D_2$. Then

$$(D_1 \cap D_2^a)^{cl} \cap D_2^a = \bigcup_{D \in I(D_1, D_2)} D \cap D_2^a.$$

(iii) Let x be an element of U. Then $HxP' \cap H^a xP' = (K \cap H)xP'$.

(iv) $D_1 \cap D_2^a$ is nonempty and closed in G $\iff D_1 = D_2$.

Example. Let G_1 be a connected semisimple Lie group, θ_1 a Cartan involution of G_1, $K_1 = \{x \in G_1 \mid \theta_1 x = x\}$, and P_1 a minimal parabolic subgroup of G_1 with a θ_1-stable split component A_1. Let P_1' and P_1'' be parabolic subgroups of G_1 containing P_1. Put $G = G_1 \times G_1$, $H = \{(x, x) \in G \mid x \in G_1\}$, $H^a = \{(\theta_1 x, x) \in G \mid x \in G_1\}$ and $P' = P_1' \times P_1''$. Then we have natural bijections

$$H\backslash G/P' \overset{\sim}{\to} P_1'\backslash G_1/P_1''$$

and

$$H^a\backslash G/P' \overset{\sim}{\to} \theta_1(P_1')\backslash G_1/P_1''$$

by the maps $(x, y) \to x^{-1}y$ and $(x, y) \to \theta_1(x^{-1})y$, respectively. Hence by the Bruhat decomposition of G_1, every H-P' double coset and H^a-P' double coset have representatives in $W(A_1) \times 1$.

Consider the intersection $I = H(w, 1)P' \cap H^a(w', 1)P'$ for $w, w' \in W(A_1)$. Since $H \cap H^a = \{(x, x) \mid x \in K_1\}$ and since $G_1 = K_1P_1$ by the Iwasawa decomposition of G_1, I contains elements of the form $(x, 1)$ with $x \in G_1$ if I is nonempty. We have easily

$$(x, 1) \in I \iff x \in P_1'wP_1'' \cap \theta_1(P_1')w'P_1''.$$

Thus we have as a corollary of Theorem (i),

$$(1.1) \qquad (P_1'wP_1'')^{cl} \supset P_1'w'P_1'' \iff P_1'wP_1'' \cap \theta_1(P_1')w'P_1'' \neq \phi.$$

Especially we have

$$(1.2) \qquad (P_1wP_1)^{cl} \supset P_1w'P_1 \iff P_1wP_1 \cap \theta_1(P_1)w'P_1 \neq \phi.$$

Remark. In [1], V. V. Deodhar studied explicitly the above type of intesection $P_1wP_1 \cap \overline{P}_1w'P_1$ when G_1 is a semisimple algebraic group over an algebraically closed field. Here $\overline{P}_1 = w_0P_1w_0^{-1}$ with the longest element w_0 of the Weyl group. He gave (1.2) as a corollary of his results in this case. (Replace $\theta_1(P_1)$ by $\overline{P}_1$.)

The author is grateful to J. A. Wolf who suggested him the importance of the intersections of H-orbits and H^a-orbits on G/P. In fact, Theorem (iv) was conjectured by him.

§2. Notations and elementary lemmas

Let $\underline{g}$ be the Lie algebra of G. Let σ and θ be the involutions of $\underline{g}$ induced from the involutions σ and θ of G, respectively. Let $\underline{g} = \underline{k} + \underline{p}$, $\underline{g} = \underline{h} + \underline{q}$ and $\underline{g} = \underline{h}^a + \underline{q}^a$ be the decompositions of $\underline{g}$ into the $+1$ and -1 eigenspaces for σ, θ and $\sigma\theta$, respectively.

Let $\underline{a}$ be the Lie algebra of A. Then $\underline{a}$ is a maximal abelian subspace of $\underline{p}$. Let Σ denote the root system of the pair $(\underline{g}, \underline{a})$. Then P can be written as

$$P = P(\underline{a}, \Sigma^+) = Z_G(\underline{a})\exp \underline{n}$$

with a positive system Σ^+ of Σ. Here $Z_G(\underline{a})$ is the centralizer of $\underline{a}$ in G and $\underline{n} = \sum_{\alpha\in\Sigma^+} \underline{g}(\underline{a};\ \alpha)$ $(\underline{g}(\underline{a};\ \alpha) = \{X \in \underline{g} \mid [Y, X] = \alpha(Y)X$ for all $Y \in \underline{a}\})$.

Lemma 1. Let $\underline{P}'$ be a parabolic subalgebra of $\underline{g}$. Then $\underline{h} + \underline{h}^a + \underline{P}' = \underline{g}$.

Proof. Let X be an element of $\underline{P}'$. Then

$$\theta X = X - (X + \sigma X) + (\sigma X + \theta X) \subset \underline{P}' + \underline{h} + \underline{h}^a.$$

Hence $\theta\underline{P}' \subset \underline{h} + \underline{h}^a + \underline{P}'$. Since $\underline{P}' + \theta\underline{P}' = \underline{g}$, we have $\underline{h} + \underline{h}^a + \underline{P}' \supset \underline{g}$. Q.E.D.

Lemma 2. Let D_1 and D_2 be arbitrary H-P' double cosets in G.

(i) $$(D_1 \cap D_2^a)^{cl} \cap D_2^a = D_1^{cl} \cap D_2^a.$$

(ii) $$D_1^{cl} \supset D_2 \implies D_1 \cap D_2^a \neq \phi.$$

4

Proof. (i) It is clear that $(D_1 \cap D_2^a)^{cl} \cap D_2^a \subset D_1^{cl} \cap D_2^a$. Let x be an element of $D_1^{cl} \cap D_2^a$. Then we have only to show that $x \in (D_1 \cap D_2^a)^{cl}$. For any neighborhood V of the identity in H^a, the set $HVxP'$ contains a neighborhood of x in G by Lemma 1. Hence $D_1 \cap HVxP' \neq \phi$. Since $HD_1P' = D_1$, we have

$$D_1 \cap Vx \neq \phi.$$

On the other hand, $Vx \subset D_2^a$. Hence $(D_1 \cap D_2^a) \cap Vx \neq \phi$ and we have proved that $x \in (D_1 \cap D_2^a)^{cl}$.

(ii) is clear from (i) since $D_2 \cap D_2^a \neq \phi$. Q.E.D.

§3. Proof of Theorem (i) and (ii)

By Lemma 2 (i), Theorem (ii) follows from Theorem (i).

Proof of Theorem (i). By [2] Theorem 1, we can write $D_1 = HxP' \supset HxP$ with $x \in U$. Considering xPx^{-1} and $xP'x^{-1}$ as P and P', respectively, we may assume that $D_1 = HP'$ and that $\underline{a}$ is σ-stable. By Lemma 2 (ii), we have only to prove the following.

$$D_1 \cap D_2^a \neq \phi \implies D_1^{cl} \supset D_2.$$

Suppose that $D_1 \cap D_2^a \neq \phi$. Then $HP \cap D_2^a \neq \phi$ since $D_2^aP' = D_2^a$. Hence there exists an element y of $D_2^a \cap U = D_2 \cap U$ such that $HP \cap H^ayP \neq \phi$. On the other hand, if $(HP)^{cl} \supset HyP$ for some $y \in D_2$, then it is clear that $D_1^{cl} \supset D_2$. Thus we have only to prove the following.

(3.1) If $HP \cap H^a yP \neq \phi$ for $y \in U$, then $(HP)^{cl} \supset HyP$.

We will prove (3.1) by induction on the real rank of G (= dim $\underline{a}$). Suppose that $\underline{a} \subset \underline{q}$ and that $Ad(y)\underline{a} \subset \underline{h}$. Then by [2] Proposition 1 and Proposition 2, HP is open in G and HyP is closed in G. By [4] Proposition, we have always $(HP)^{cl} \supset HyP$. Hence we may assume that

(3.2) $$\underline{a} \cap \underline{h} \neq \{0\}$$

or that

(3.3) $$Ad(y)\underline{a} \cap \underline{q} \neq \{0\}.$$

We first show that the case (3.3) is reduced to the case (3.2). Assume the condition (3.3). Then $Ad(y)\underline{a} \cap \underline{h}^a \neq \{0\}$ since $Ad(y)\underline{a} \subset \underline{p}$. Consider $Ad(y)\underline{a}$, $\underline{h}^a$ and yPy^{-1} as $\underline{a}$, $\underline{h}$ and P in the case (3.2), respectively. Then we have in the proof of the case (3.2) that

$$H^a yPy^{-1} \cap H^a Py^{-1} \neq \phi \implies (H^a yPy^{-1})^{cl} \supset H^a Py^{-1}.$$

Hence

$$HP \cap H^a yP \neq \phi \implies (H^a yP)^{cl} \supset H^a P.$$

On the other hand, we have

$$(HP)^{cl} \supset HyP \iff (H^a yP)^{cl} \supset H^a P$$

for $y \in U$ by Corollary of [4] Theorem. Thus the case (3.3) is reduced to the case (3.2) and so we may assume (3.2) in the followings.

By [4] Theorem (iv), there exists a sequence $\alpha_1, \ldots, \alpha_n$ of simple roots in Σ^+ such that

$$(3.4) \qquad (HP)^{cl} = H((L \cap H)(L \cap wPw^{-1}))^{cl} wPL_{\alpha_n} \ldots L_{\alpha_1}.$$

Here $w = w_{\alpha_1} \ldots w_{\alpha_n}$, L is the analytic subgroup of G for $\underline{l} = [\underline{z}_{\underline{g}}(\underline{a} \cap \underline{h}), \underline{z}_{\underline{g}}(\underline{a} \cap \underline{h})]$ and $L_\alpha = Z_G(\underline{a}^\alpha)$, $\underline{a}^\alpha = \{Y \in \underline{a} \mid \alpha(Y) = 0\}$ for $\alpha \in \Sigma$.

Lemma 3. $HwP = (K \cap H)(L \cap H)_0 wP$. ($(L \cap H)_0$ is the connected component of $L \cap H$ containing the identity.)

Proof. Put $L_1 = Z_G(\underline{a} \cap \underline{h})$ and define a parabolic subgroup P_1 of G by $P_1 = L_1 wPw^{-1}$ as in [4] §1. Then $P_1 \cap H_0$ is a parabolic subgroup of H_0 and we have $H_0 = (K \cap H)_0 (P_1 \cap H)_0$ by the Iwasawa decomposition of H_0. On the other hand, $K \cap H$ intersects with every connected component of H since $H = (K \cap H)\exp(\underline{p} \cap \underline{h})$. Hence

$$(3.5) \qquad H = (K \cap H)(P_1 \cap H)_0.$$

Let $\underline{n}_1$ be the nilpotent radical of the Lie algebra of P_1. Then $P_1 = L_1 \exp \underline{n}_1$ is a Langlands decomposition of P_1. Since L_1 and $\underline{n}_1$ are σ-stable, we have

$$(3.6) \qquad (P_1 \cap H)_0 = (L_1 \cap H)_0 \exp(\underline{n}_1 \cap \underline{h}).$$

Let $\underline{z}$ be the center of the Lie algebra $\underline{l}_1$ of L_1. Then $\underline{l}_1 = \underline{z} + \underline{l}$. Since $\underline{z}$ and $\underline{l}$ are σ-stable, we have $\underline{l}_1 \cap \underline{h} = \underline{z} \cap \underline{h} +$

$\underline{l} \cap \underline{h}$ and therefore

(3.7) $$(L_1 \cap H)_0 = (L \cap H)_0 \exp(\underline{z} \cap \underline{h}).$$

We get the desired formula from (3.5), (3.6) and (3.7) since $\exp \underline{n}_1 \subset wPw^{-1}$ and $\exp \underline{z} \subset wPw^{-1}$. Q.E.D.

Now we will continue the proof of Theorem (i). Suppose that $HP \cap H^a yP \neq \phi$. Since $HP \subset HwPL_{\alpha_n} \ldots L_{\alpha_1}$, we have

$$HwP \cap H^a yPL_{\alpha_1} \ldots L_{\alpha_n} \neq \phi$$

By Lemma 3, we have

(3.8) $$(L \cap H)_0 \cap H^a yPL_{\alpha_1} \ldots L_{\alpha_n} w^{-1} \neq \phi.$$

Let y' be an element of the left hand side of (3.8) and y'' an element of $(L \cap H^a)_0 y'(L \cap wPw^{-1}) \cap U$. Then

(3.9) $$H^a y'' wP \subset H^a yPL_{\alpha_1} \ldots L_{\alpha_n}$$

and

(3.10) $$(L \cap H)_0 (L \cap wPw^{-1}) \cap (L \cap H^a)_0 y''(L \cap wPw^{-1}) \neq \phi.$$

Since $\sigma L = \theta L = L$ and $\dim(\underline{l} \cap \underline{a}) < \dim \underline{a}$, we have

$$((L \cap H)_0 (L \cap wPw^{-1}))^{cl} \supset (L \cap H)_0 y''(L \cap wPw^{-1})$$

by the assumption of induction. By (3.4), we have

(3.11) $$(HP)^{cl} \supset H(L \cap H)_0 y''(L \cap wPw^{-1}) wPL_{\alpha_n} \ldots L_{\alpha_1}$$
$$\supset Hy'' wPL_{\alpha_n} \ldots L_{\alpha_1}.$$

Now consider the formula (3.9) which can be rewritten as

$$y \in H^a y'' w P L_{\alpha_n} \cdots L_{\alpha_1}.$$

As in the proof of [4] Theorem (vi), we can chooses a $y_1 \in y''wPL_{\alpha_n} \ldots L_{\alpha_1} \cap U$ so that $y \in H^a y_1 P$. Since $y \in U$, it follows from [2] Theorem 1 that $y \in (K \cap H) y_1 P$. Hence

(3.12) $$y \in (K \cap H) y'' w P L_{\alpha_n} \cdots L_{\alpha_1}.$$

From (3.11) and (3.12), we have $(HP)^{cl} \supset HyP$ as desired.

Q.E.D.

§4. Proof of Theorem (iii) and (iv)

Theorem (iv) follows from (ii) and (iii). So we have only to prove (iii) in this section. Recall the definition of $P = P(\underline{a}, \Sigma^+)$ in §2 and let Ψ denote the set of all the simple roots in Σ^+.

Lemma 4. Suppose that $H^a P$ is not open in G. Then there exists an $\alpha \in \Psi$ such that $\dim H^a P_\alpha > \dim H^a P$.

Proof. By [2] Theorem 1, we may assume that $\sigma\underline{a} = \underline{a}$. By [2] Proposition 1, Σ^+ is not σ-compatible or $\underline{a} \cap \underline{h}$ is not maximal abelian in $\underline{p} \cap \underline{h}$. First suppose that Σ^+ is not σ-compatible. Then by [4] Lemma 4 and Lemma 5, there exists an $\alpha \in \Psi$ such that $H^a P_\alpha = H^a P \cup H^a w_\alpha P$ and that $\dim H^a w_\alpha P > \dim H^a P$. Hence we may assume that Σ^+ is σ-compatible and that $\underline{a} \cap \underline{h}$ is not maximal abelian in $\underline{p} \cap \underline{h}$.

Put $\underline{l}_1 = z_{\underline{g}}(\underline{a} \cap \underline{h})$. Suppose that there exists an $\alpha \in \Psi \cap \Sigma(\underline{l}_1; \underline{a})$ such that $\underline{g}(\underline{a}; \alpha) \cap \underline{g}^a \neq \{0\}$. Here $\Sigma(\underline{l}_1; \underline{a})$ is the root system of the pair $(\underline{l}_1, \underline{a})$, and it is clear that $\alpha \in \Sigma(\underline{l}_1; \underline{a})$ if and only if $\alpha \in \Sigma$, $\sigma\alpha = -\alpha$. Then by [4] Lemma 3 (F), $\dim H^a P_\alpha > \dim H^a P$. Hence we may assume that

(4.1) $\quad \underline{g}(\underline{a}; \alpha) \cap \underline{g}^a = \{0\}$ for all $\alpha \in \Psi \cap \Sigma(\underline{l}_1; \underline{a})$.

Let β be a root in $\Sigma(\underline{l}_1; \underline{a}) \cap \Sigma^+$ and write $\beta = \sum_{\alpha \in \Psi} n_\alpha \alpha$. Choose an element $Y \in \underline{a} \cap \underline{h}$ such that $\alpha(Y) > 0$ for all $\alpha \in \Sigma^+ - \Sigma(\underline{l}_1; \underline{a})$ by [4] Lemma 4. If $n_\alpha > 0$ for some $\alpha \in \Psi - \Sigma(\underline{l}_1; \underline{a})$, then $\beta(Y) > 0$. But since $\beta(Y) = 0$, we have proved that β is written as a linear combination of roots in $\Psi \cap \Sigma(\underline{l}_1; \underline{a})$. By (4.1) and [4] Lemma 10, we have $\underline{g}(\underline{a}; \beta) \subset \underline{h}^a$. Hence

$$\underline{a} \cap \underline{g} + \textstyle\sum_{\beta \in \Sigma(\underline{l}_1;\, \underline{a})} \underline{g}(\underline{a}; \beta) \subset \underline{h}^a.$$

Since $z_{\underline{g}}(\underline{a} \cap \underline{h}) = \underline{l}_1 = \underline{a} + \sum_{\beta \in \Sigma(\underline{l}_1;\, \underline{a})} \underline{g}(\underline{a}; \beta)$, $\underline{a} \cap \underline{h}$ is a maximal abelian subspace of $\underline{p} \cap \underline{h} = \underline{p} \cap \underline{g}^a$. But this is a contradiction to the assumption on $\underline{a} \cap \underline{h}$. Q.E.D.

Lemma 5. If HP is closed in G, then $HP = (K \cap H)P$.

Proof. If $HP = (K \cap H)xP$ for some $x \in HP$, then $HP = (K \cap H)P$. So taking a conjugate of P, we may assume that $\sigma\underline{a} = \underline{a}$. Since Σ^+ is σ-compatible, we can apply Lemma 3 for $w = 1$ to get

$$HP = (K \cap H)(L \cap H)_0 P.$$

Since $\underline{a} \cap \underline{h}$ is maximal abelian in $\underline{p} \cap \underline{h}$, we have $L \subset H^a$ by the argument in [4] Proof of Lemma 6 (i). Since $H \cap H^a = K \cap H$, we have $HP = (K \cap H)(L \cap H^a \cap H)_0 P = (K \cap H)P$ as desired. Q.E.D.

Proof of Theorem (iii). Choose $x' \in xP' \cap U$ so that $Hx'P$ has the minimum dimension among the H-P double cosets contained in HxP'. Clearly $HxP' \cap H^a xP' = (HxP' \cap H^a x'P)P'$. Since $(Hx'P)^{cl} \cap HxP' = Hx'P$, it follows from Theorem (i) that $HxP' \cap H^a x'P = Hx'P \cap H^a x'P$. So we have only to prove that

$$(4.2) \qquad Hx'P \cap H^a x'P = (K \cap H)x'P \quad \text{for} \quad x' \in U.$$

We will prove (4.2) by induction on the codimension of $H^a x'P$. Rewriting $x'Px'^{-1}$ by P, we may assume that $x' = 1$ and that $\sigma\underline{a} = \underline{a}$.

Suppose that $H^a P$ is open in G. Then HP is closed in G by [2] §3 Corollary and $HP = (K \cap H)P \subset H^a P$ by Lemma 5. Hence we may assume that $H^a P$ is not open in G.

By Lemma 4, there exists an $\alpha \in \Psi$ such that $\dim H^a P_\alpha > \dim H^a P$. Then by [4] Lemma 3, there are two cases (B^a) : $\sigma\theta\alpha \neq \pm\alpha$, $\sigma\theta\alpha \in \Sigma^+$ and (D^a) : $\sigma\theta\alpha = \alpha$, $\underline{g}(\underline{a}; \alpha) \cap \underline{g}^a \neq \{0\}$. Put $z = w_\alpha$ in the case (B^a) and put $z = c_\alpha$ in the case (D^a). Then we have $(HzP)^{cl} \cap HP_\alpha = HzP$ by [4] Lemma 3 (A) and (F). (Since $\theta|_{\underline{a}} = -1$, we have $(B^a) = (A)$: $\sigma\alpha \neq \pm\alpha$, $\sigma\alpha \notin \Sigma^+$ and $(D^a) = (F)$: $\sigma\alpha = -\alpha$, $\underline{g}(\underline{a}; \alpha) \cap \underline{g}^a \neq \{0\}$.) Apllying Theorem (i), we have

$$(4.3) \qquad HzP \cap H^a P_\alpha = HzP \cap H^a zP.$$

Let y be an element of $HP \cap H^a P$. Then we have only to show that $y \in (K \cap H)P$ since it is clear that $(K \cap H)P \subset HP \cap H^a P$. Let y' be an element of $HzP \cap yP_\alpha$. Then by (4.3) and the assumption of induction, we have

$$y' \in HzP \cap H^a P_\alpha \cap yP_\alpha = HzP \cap H^a zP \cap yP_\alpha = (K \cap H)zP \cap yP_\alpha$$

and therefore

$$y \in (K \cap H)zP_\alpha = (K \cap H)P_\alpha.$$

Since $y \in H^a P$, we have

$$y \in (K \cap H)P_\alpha \cap H^a P = (K \cap H)(P_\alpha \cap H^a)P = (K \cap H)JP.$$

Here J is the image of $P_\alpha \cap H^a$ under the projection $P_\alpha \to L_\alpha$ with respect of the Langlands decomposition $P_\alpha = L_\alpha \exp \underline{n}_\alpha$. We consider the two cases (B^a) and (D^a) separately.

First consider the case (B^a). We have only to show that $J \subset L_\alpha \cap P$. Let L_α^s denote the identity component of the semisimple part of L_α as in [4] §3. Since $L_\alpha^s \cap J \supset \exp(\underline{g}(\underline{a};\ \alpha)+\underline{g}(\underline{a};\ 2\alpha))$, we have $L_\alpha^s - (L_\alpha^s \cap J)w_\alpha(L_\alpha^s \cap P) \subset L_\alpha^s \cap P$ by the Bruhat decomposition of L_α^s. Since $L_\alpha/L_\alpha \cap P \simeq L_\alpha^s/L_\alpha^s \cap P$, we have $L_\alpha - Jw_\alpha(L_\alpha \cap P) \subset L_\alpha \cap P$. On the other hand, we have $J(L_\alpha \cap P) \cap Jw_\alpha(L_\alpha \cap P) = \phi$ since $H^a P \cap H^a w_\alpha P \neq \phi$. Hence $J \subset L_\alpha \cap P$.

Next consider the case (D^a). We have only to show that $J \subset (K \cap H)(L_\alpha \cap P)$. In this case, $J \supset L_\alpha^s \cap H^a$ and it follows easily from the proof of [4] Lemma 3 (D) that

$$L_\alpha = D(1) \cup D(w_\alpha) \cup D(c_\alpha) \cup D(c_\alpha^{-1}).$$

Here $D(x) = (L_\alpha^s \cap H^a)x(L_\alpha \cap P)$ for $x \in L_\alpha$. We also have

$$(4.4) \qquad J(L_\alpha \cap P) = \begin{cases} D(1) & \text{if } w_\alpha \notin N_{K\cap H}(\underline{a})Z_K(\underline{a}) \\ D(1) \cup D(w_\alpha) & \text{if } w_\alpha \in N_{K\cap H}(\underline{a})Z_K(\underline{a}) \end{cases}$$

since $(H^aP \cup H^a w_\alpha P) \cap (H^a c_\alpha P \cup H^a c_\alpha^{-1} P) = \phi$. Since $D(1)$ and $D(w_\alpha)$ are closed in L_α, we have

$$(4.5) \qquad D(x) = (L_\alpha^s \cap K \cap H)x(L_\alpha \cap P) \quad \text{for } x = 1 \text{ and } w_\alpha$$

by Lemma 5. (Note that $L_\alpha/L_\alpha \cap P \simeq L_\alpha^s/L_\alpha^s \cap P$.) From (4.4) and (4.5), we get

$$J(L_\alpha \cap P) \subset (K \cap H)(L_\alpha \cap P)$$

as desired. Q.E.D.

References

[1] V. V. Deodhar, On some geometric aspects of Bruhat orderings I. A finer decomposition of Bruhat cells, Invent. Math. 79(1985), 499-511

[2] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan 31(1979), 331-357

[3] T. Matsuki, Orbits on affine symmetric spaces under the action of parabolic subgroups, Hiroshima Math. J. 12(1983), 307-320

[4] T. Matsuki, Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups, to appear

[5] W. Rossmann, The structure of semisimple symmetric spaces, Canad. J. Math. 31(1979), 157-180