数理解析研究所講究録
第 633 巻 1987 年 64-73

Review of A. Hatcher and J. Wagoner's paper 'Pseudo-Isotopies of Compact Manifolds' (Part 1)

大阪市立大学理学部

Marina Fienkoß

As this paper by A. Hatcher and J. Wagoner, published in 1973 (astérisque 6, Soc. Math. de France), seems to be little known, we felt that it might be of some significance to give a rough summary of the paper at the symposium on Whitehead groups and transformation groups.

<u>What is the problem to be solved in Hatcher-Wagoner's paper?</u>

Let M be a smooth compact manifold with boundary ∂M and dim M = n. A map f: $(M,\partial M)\times I \longrightarrow (M,\partial M)\times I$ is called a <u>pseudo-isotopy of $(M,\partial M)$</u>, if it satisfies the following two conditions

1. f is a diffeomorphism
2. $f|M\times 0 = id : M\times 0 \longrightarrow M\times 0$ and
   $f|\partial M\times I : \partial M\times I \longrightarrow \partial M\times I$ is an isotopy

Let P = $P(M,\partial M)$ denote the group of pseudo-isotopies of $(M,\partial M)$. The multiplication in P is the composition of maps and P is given the C^∞ topology.

The problem is to compute $\pi_0(P)$.

<u>The simply connected case</u>

If M is simply connected, we have the following result.

<u>Theorem:</u> If dim $M\geq 5$, $\partial M = \phi$ and $\pi_1(M) = 0$ then $\pi_0(P) = 0$

This Theorem was proven by Jean Cerf in his paper 'La Stratification Naturelle des Espaces de Fonctions Différentiables Réelles et le

Théorème de la Pseudo-Isotopie' (1970, IHES Nr. 39).

The non simply connected case

Hatcher-Wagoner try to solve the non simply connected case. They show that in this case $\pi_0(P)$ can be related to algebraic K-theory. The exact statement is as follows.

Theorem:	If $n \geq 7$ then it exists a surjective homomorphism $\Sigma : \pi_0(P) \longrightarrow Wh_2(\pi_1 M)$

What is Wh_2 ?

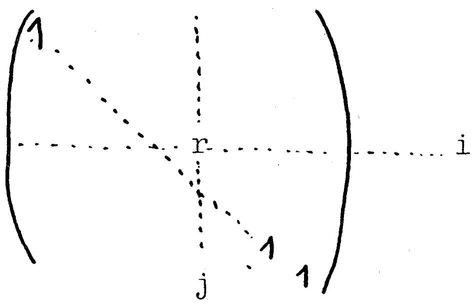
Let R denote an associative ring with 1. Then the Steinberg group St(of R is the group with the following presentation:

the generators of St(R) are symbols $x_{ij}(r)$, where $1 \leq i,j < \infty$, $i \neq j$, $r \in R$

these generators are subject to the following three relations

1. $x_{ij}(r)\, x_{ij}(s) = x_{ij}(r+s)$
2. commutator $[x_{ij}(r), x_{kl}(s)] = 1$ if $j \neq k$ and $i \neq l$
3. commutator $[x_{ij}(r), x_{jl}(s)] = x_{il}(rs)$ if $i \neq l$

Let $GL(R) = \varinjlim_{\infty} GL(n,R)$, where $GL(n,R)$ is the group of $n \times n$ invertib matrices over R, and let E(R) denote the subgroup of GL(R) generated by the elementary matrices

$e_{ij}(r) =$ 

Then we get a canonical surjective homomorphism

$$\pi : St(R) \longrightarrow E(R)$$
$$x_{ij}(r) \longmapsto e_{ij}(r)$$

$K_2(R)$ is defined to be the kernel of π:

$$K_2(R) \overset{\text{def}}{=} \operatorname{Ker} \pi \quad .$$

It is not too difficult to show that $\operatorname{Ker}\pi$ is precisely the center of $St(R)$. Thus $K_2(R)$ is an abelian group. If R is a group ring, $R = \mathbb{Z}[G]$, then we can define a subgroup $W(\pm G)$ of $St(R)$ as follows:

$W(\pm G)$ is the subgroup generated by the words

$w_{ij}(\pm g) = x_{ij}(\pm g)\, x_{ji}(\mp g^{-1})\, x_{ij}(\pm g)$, with $g \in G$ and $i \neq j$.

Finally $Wh_2(G)$ is defined to be

$$Wh_2(G) = K_2(\mathbb{Z}[G]) \Big/ K_2(\mathbb{Z}[G]) \cap W(\pm G) \, .$$

Cerf's functional approach to pseudo-isotopies

Hatcher-Wagoner make use of Cerf's functional approach to the pseudo-isotopy problem.

Let $F = \{ C^\infty$ function $f: M \times I \longrightarrow I \mid f(M\times 0) = 0,\ f(M\times 1) = 1$, f is the projection on $\partial M \times I$ and f has no critical points near $M\times 0$ and $M\times 1 \}$.

The base point in F is the standard projection $p: M\times I \longrightarrow I$.

By joining each function f in F linearly to the base point p

$$f + t(p - f)\ ,\quad 0 \leq t \leq 1\ ,$$

we see that F is contractible.

Let $E = \{ f: M\times I \longrightarrow I \mid f$ has no critical points$\} \subset F$.

From the homotopy exact sequence of the pair (F,E) we get

$$\pi_1(F,E,p) \underset{\partial}{\overset{\cong}{\longrightarrow}} \pi_0(E) \ .$$

Consider the following map π between P and E

$$\begin{aligned} \pi : P &\longrightarrow E \\ f &\longmapsto p \circ f^{-1} \end{aligned}$$

As f is a diffeomorphism this map is well defined. The inverse image of the standard projection p under this map is

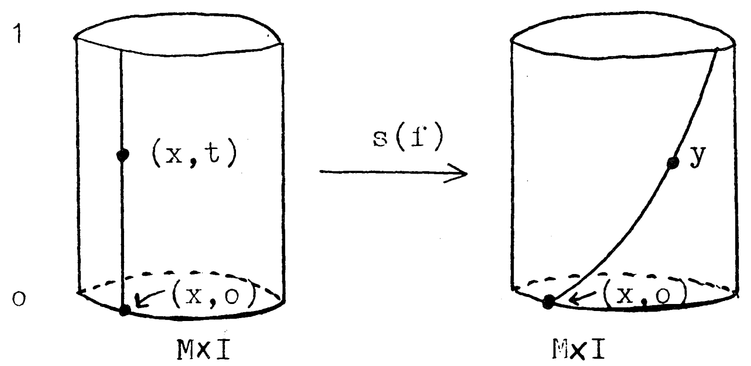
$\pi^{-1}(p)$ = the space of isotopies of M×I which are the identity on M×O

$\pi^{-1}(p)$ is contractible.

Lemma:	$\exists$ s: E $\longrightarrow$ P such that $\pi \circ s = id_E$ ($\forall f \in E$ $p \circ s(f)^{-1} = f$)

Idea of the proof:

Fix a Riemannian metric μ on M×I. Let f be an element of E and think of the vector field $grad_\mu f$ on M×I. As f has no critical points, the solution curves of this vector field run from M×O to M×1. For every $x \in M$, s(f) is defined to map the interval $\{x\} \times I$ to the solution curve of $grad_\mu f$ starting at (x,O) in the following way.



s(f) maps (x,t) to y, with f(y) = t

From this Lemma we get a homeomorphism between P and $\pi^{-1}(p) \times E$.

$$P \xleftrightarrow{\approx} \pi^{-1}(p) \times E$$
$$G \longmapsto (s(pG^{-1})^{-1} \circ G,\ pG^{-1})$$
$$s(f) \circ H \longleftarrow (H,\ f)$$

Thus we have the following bijections

$$\pi_0(P) \cong \pi_0(E) \cong \pi_1(F,E,p) .$$

As $\pi_0(P)$ is a group, these bijections induce a group structure on $\pi_0(E)$ and $\pi_1(F,E,p)$.

This is Cerf's functional approach to the pseudo-isotopy problem.

Replacing the pair (F,E) by a homotopy equivalent pair $(\hat{F},\hat{E})$

To construct the desired homomorphism $\Sigma: \pi_1(F,E,p) \longrightarrow Wh_2(\pi_1 M)$ it is not enough to consider only smooth functions between M×I and I. The pair (F,E) is replaced by a homotopy equivalent pair $(\hat{F},\hat{E})$, where $\hat{F}$, $\hat{E}$ are defined to be

$$\hat{F} = \{(\eta, f, \mu) \mid f \in F, \mu \text{ is a Riemannian metric on } M\times I, \eta \text{ is a vector field on } M\times I \text{ which is gradient like for } f \text{ with respect to } \mu\}$$

$$\hat{E} = \{(\eta, f, \mu) \mid f \in E, \mu \text{ is a Riemannian metric on } M\times I, \eta \text{ is a vector field on } M\times I \text{ which is gradient like for } f \text{ with respect to } \mu\}$$

Remark: The vector field η is gradient like for f with respect to the metric μ if it satisfies the following two conditions:

1. If p is a critical point of f then $\exists$ nbd U of p such that $\forall$ $x \in U$ $\eta(x) = \text{grad}_\mu f(x)$

   (This implies that the vector field vanishes at the critical point p)

2. If x is not a critical point of f and if $\varphi_x: (-\varepsilon,\varepsilon) \longrightarrow M\times I$, $0 \longmapsto x$ is a solution curve of the vector field η through x, then we have $\frac{d(f\circ\varphi_x)}{dt}(0) > 0$.

   (This means that the solution curve does not run inside the level surface of f containing x, but is transverse to the level surface containing x)

If we fix a Riemannian metric μ_0 on M×I, we get a homotopy equivalence

$$(F,E,p) \xrightarrow{\text{h.e.}} (\hat{F},\hat{E},\hat{p}), \quad \hat{p} = (\text{grad}_{\mu_0} p, p, \mu_0)$$
$$f \longmapsto (\text{grad}_{\mu_0} f, f, \mu_0)$$

(Every triple (η, f, μ) in $(\hat{F},\hat{E})$ is joined to $(\text{grad}_{\mu_0} f, f, \mu_0)$ in two steps:

$$(\eta, f, \mu) \xrightarrow[0\le t\le 1]{(t\,\text{grad}_\mu f + (1-t)\eta,\, f,\, \mu)} (\text{grad}_\mu f, f, \mu) \xrightarrow[\text{where } \mu_t = t\mu_0 + (1-t)\mu,\ 0\le t\le 1]{(\text{grad}_{\mu_t} f,\, f,\, \mu_t)} (\text{grad}_{\mu_0}, f, \mu_0)\ )$$

Two concepts which play a central role in the construction of

$\Sigma : \pi_1(\hat{F},\hat{E},\hat{p}) \longrightarrow Wh_2(\pi_1 M)$

Let $\{(\eta_t, f_t, \mu_t)\}_{0\le t\le 1}$ represent an element in $\pi_1(\hat{F},\hat{E},\hat{p})$.

1. We look at the functions $\{f_t\}_{0\le t\le 1}$ and define the graphic of $\{f_t\} = \bigcup_{t\in[0,1]}$ critical values of $f_t \subset I\times I$
2. For each t in $[0,1]$ we look at the critical points of f_t and at the vector field η_t. Let p be a critical point of f_t. Then the stable manifold W(p) and the unstable manifold $W^*(p)$ of p are defined as follows:

$W(p) \overset{\text{def}}{=} \{x\in M\times I \mid$ the solution curve of η_t passing through x 'runs into' p$\}$

$W^*(p) \overset{\text{def}}{=} \{x\in M\times I \mid$ the solution curve of η_t passing through x 'comes out of' p$\}$

('runs into' p ('comes out of' p) means the following. If φ_x is a solution curve of η_t passing through x then $\lim_{s\to\infty}\varphi_x(s) = p$ ($\lim_{s\to-\infty}\varphi_x(s) = p$))

Example of crucial importance for the construction of

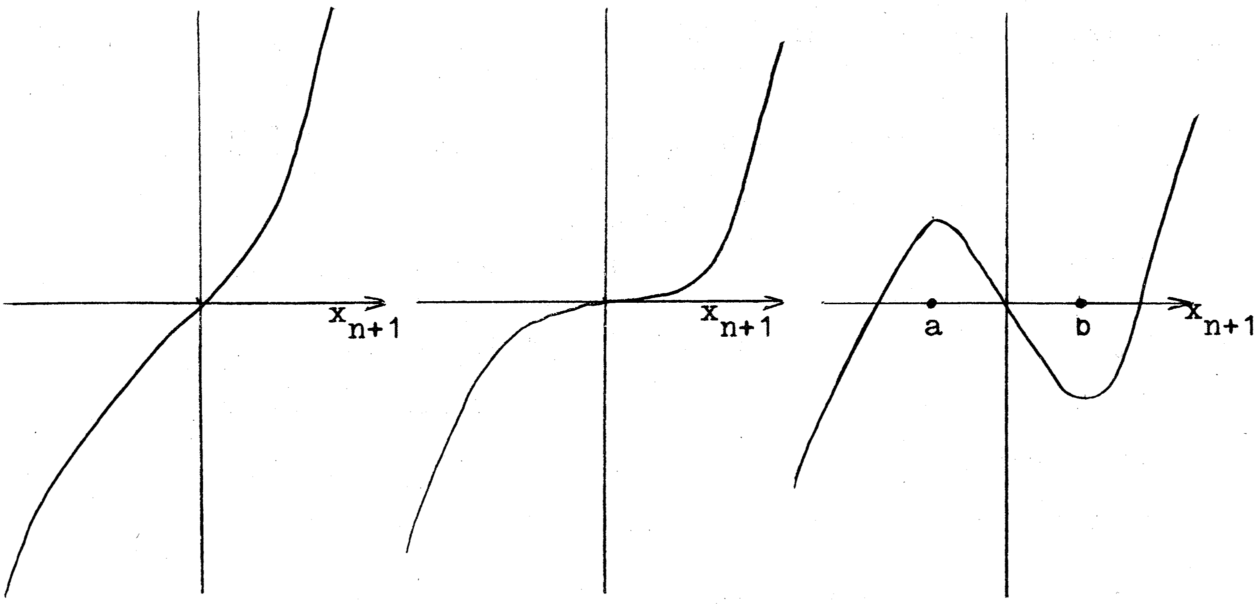
$\Sigma : \pi_1(\hat{F},\hat{E},\hat{p}) \longrightarrow Wh_2(\pi_1 M)$

Let $\mathbb{R}^{n+1}$ be the Euclidean space with the standard Riemannian metric. Consider the one-parameter family

$$f_t : \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \quad , \ (-1\le t\le 1)$$
$$(x_1,\ldots,x_{n+1}) \mapsto -x_1^2-x_2^2-\ldots-x_i^2+x_{i+1}^2+\ldots x_n^2+x_{n+1}^3-tx_{n+1}$$

and let $\eta_t = \operatorname{grad} f_t$.

If we omit the first n coordinates this family has the following shape:



$t<0$ $\quad$ $t = 0$ $\quad$ $t>0$

The origin $(o,\dots,o)$ is a degenerate critical point. In this case the origin is called <u>birth-point</u>. (If we change the direction of the parameter t, the origin is called <u>death-point</u>.)

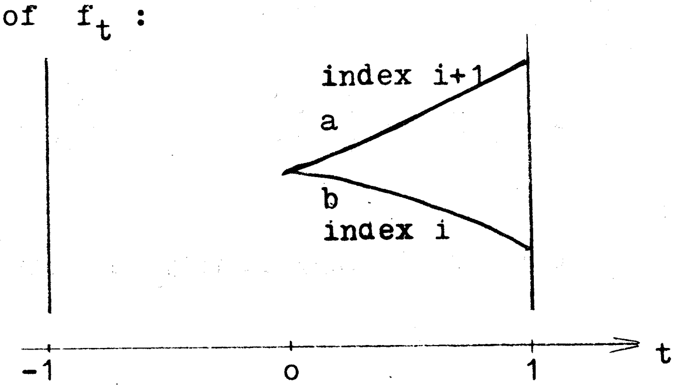
$a=(o,\dots,o,-\sqrt{\frac{t}{3}})$

$b=(o,\dots,o,\sqrt{\frac{t}{3}})$

a is a non-degenerate critical point of index i+1

b is a non-degenerate critical point of index i

The graphic of f_t :



The shape of the stable and unstable manifolds:

<u>case1:</u> $t = o$

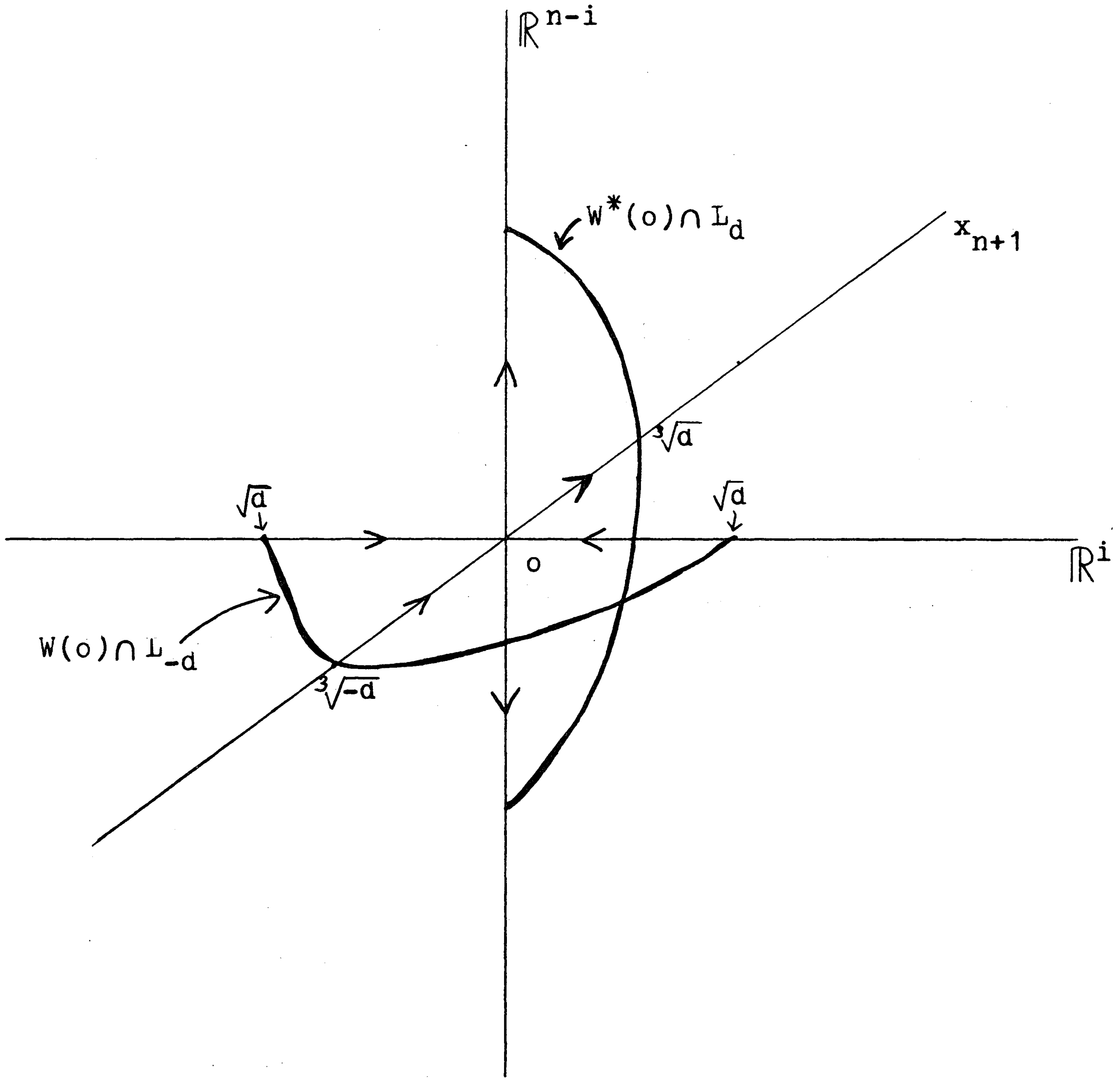
$W(o) = \mathbb{R}^i \times o \times \{x_{n+1} \leq o\}$

$W^*(o) = o \times \mathbb{R}^{n-i} \times \{x_{n+1} \geq o\}$

Let d be a positive real number, and let $L_d = f_o^{-1}(d)$ be a level surface above the critical point o and $L_{-d} = f_o^{-1}(-d)$ be a level surface below the critical point o. Then

$W(o) \cap L_{-d} \approx$ i-dimensional closed disc and

$W^*(o) \cap L_d \approx (n-i)$ - dimensional closed disc.



<u>case2:</u> $t>o$

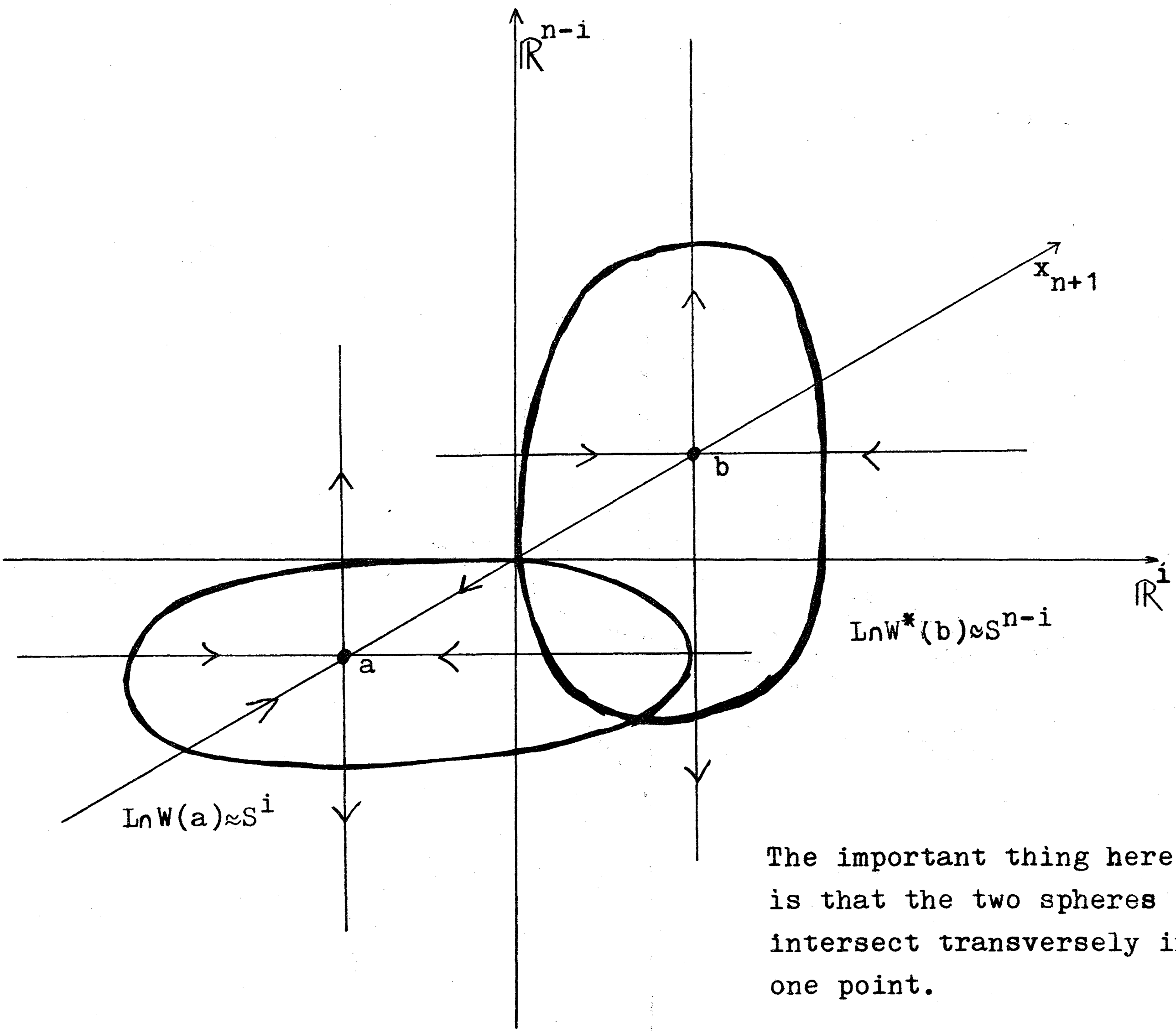
$W(b) = \mathbb{R}^i \times o \times \{x_{n+1} = \sqrt{\frac{t}{3}}\}$

$W^*(b) = o \times \mathbb{R}^{n-i} \times \{x_{n+1} \geq -\sqrt{\frac{t}{3}}\}$

$w(a) = \mathbb{R}^i \times o \times \{x_{n+1} \leq \frac{t}{3}\}$

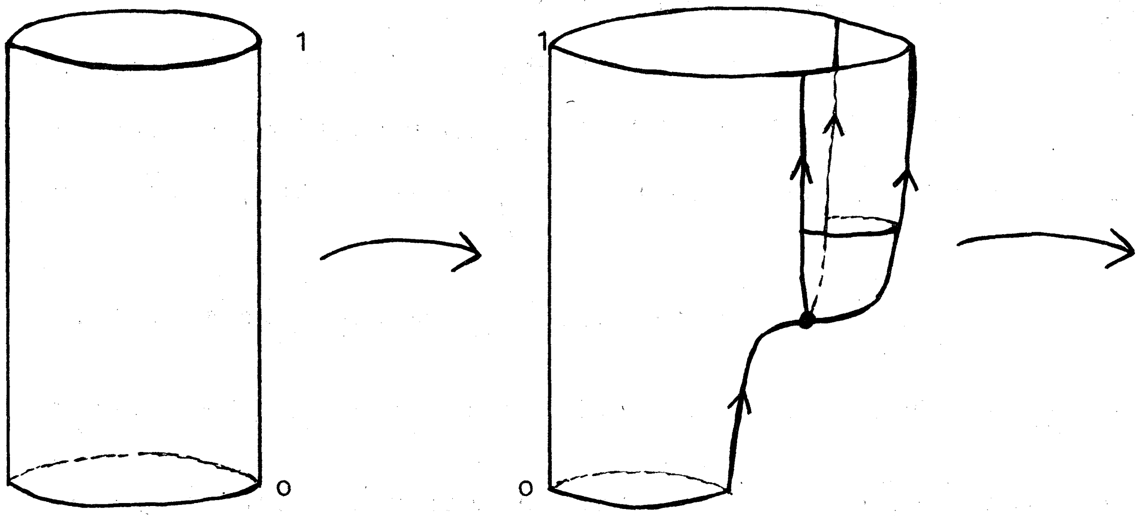
$W^*(a) = o \times \mathbb{R}^{n-i} \times \{x_{n+1} = -\sqrt{\frac{t}{3}}\}$

If $L = f_t^{-1}(o)$ is a level surface between a and b, then $L \cap W^*(b) \approx S^{n-i}$ and $L \cap W(a) \approx S^i$.

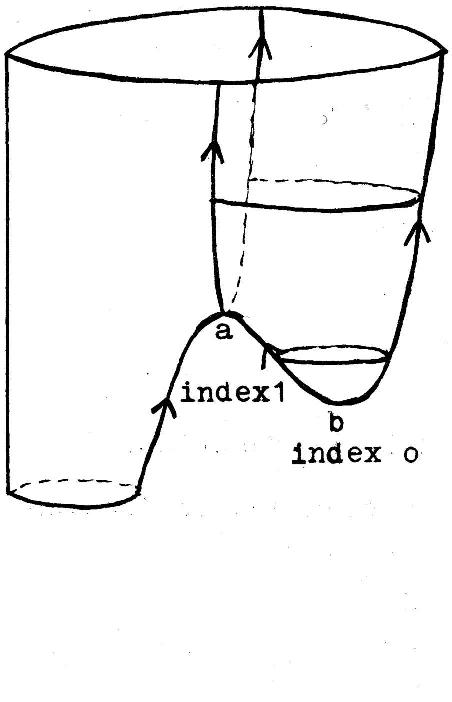


The important thing here is that the two spheres intersect transversely in one point.

In the picture of case2 it is difficult to see what happens in the level surface above a and the level surface below b. Here is another picture which makes it easier to understand the situation. Think of $S^1 \times I$ with the height function. Then gradually deform $S^1 \times I$, but still think of the height function.



$S^1 \times I$, height function



1. If L is a level above a, then
   $\partial(\overline{W^*(b) \cap L}) = W^*(a) \cap L$
2. If L is a level below b, then
   $\partial(\overline{W(a) \cap L}) = W(b) \cap L$
3. If L is a level between a and b, then
   $W(a) \cap L = S^i$, $W^*(b) \cap L = S^{n-i}$,
   $W(a) \cap W^*(b) \cap L = \{1 \text{ point}\}$
   transverse intersection