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On the mixed Hodge structures in the normal crossing case

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§1. Introduction and the main result

1.0 The purpose of this note is to remark that the existence of natural mixed Hodge structures (abbreviated as MHS) on certain cohomology groups follows from the recent study of variation of MHS by Kashiwara [K1] (What he studied was named as the infinitesimal mixed Hodge module) as well as the purity theorem of the intersection cohomology in the normal crossing case by Kashiwara-Kawai [KK1] and Cattani-Kaplan-Schmid [CKS]. So this note may be regarded as an appendix to Kashiwara [K1].

1.1 To be precice, let X be a compact Kähler manifold, Y a hypersurface of X with normal crossings, and H a polarizable variation of Hodge structure of weight w on $X - Y =: U$. Then our result is the following:

Theorem 1.2 There exist MHS's on $H^i(U,H)$ and $H^i_c(U,H)$ in a functorial way.

Of course , this generalizes the classical case $H = \mathbb{Q}_U$ by Deligne [D,II] and the one-dimensional case by Zucker [Z,§13,14]. Special case was treated in Shimizu [Sh].

We can generalize Theorem 1.2 to the case allowing H to be an admissible variation of MHS (cf.4.6).

Remark 1.3 Theorem 1.2 can be proved in the frame work of mixed Hodge modules (cf. [Sa2]) at least when X is projective. This follows from the following facts : (i) j_*H, $j_!H$ are weakly mixed Hodge modules [Sa2], where j denotes the inclusion $j : U \hookrightarrow X$ and H is as above. (j_*, $j_!$ are taken in the sense of filtered $\mathcal{D}$-modules with $\mathbb{Q}$-structure.) (ii) For a weakly mixed Hodge module M on a projective manifold X, $R\Gamma(X,M)$ is a cohomological mixed Hodge complex [D,Ⅲ(8.1)].

Stronger fact holds indeed : (i') j_*H and $j_!H$ in (i) are mixed Hodge modules, and we can show directly that at least in the algebraic category, $R\Gamma(U,H)$ and $R\Gamma_c(U,H)$ are complexes of MHS. These facts are remarked in [Sa2] and Kashiwara's theory [K1] is vital for their proof.

1.4 The outline of the proof is as follows.

We will make use of the formalism of cohomological mixed Hodge complexes by Deligne [D,Ⅲ] to put MHS on the above cohomologies. In our setup, these cohomologies are the hypercohomologies of perverse sheaves Rj_*H, $j_!H$ (j denotes the inclusion $U \hookrightarrow X$). Then we can use the explicit description of perverse sheaves in the normal crossing case by Galligo-Granger-Maisonobe [GGM] and its counterpart in the mixed Hodge theory is more or less Kashiwara's theory [K1]. Using these, we can give the weight filtration on the above perverse sheaves. The procedure for giving the Hodge filtrations using the canonical extension of $\mathcal{O}_U \otimes H$ is well known. Finally results in [K1] and the purity theorem in the normal crossing case [KK1] or [CKS] imply that Rj_*H, $j_!H$ are graded-polarizable cohomological mixed Hodge complexes.

We remark that our method is "ad hoc" compared to the recent theory of mixed Hodge modules (cf. Remark 1.3).

1.5 The construction of this paper is as follows. In § 2,3, we recall the necessary facts on the description of perverse sheaves and infinitesimal mixed Hodge modules. Finally in §4, we construct filtrations, i.e., the data necessary for a cohomological mixed Hodge complex.

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§2. Infinitesimal mixed Hodge modules

2.0 Let H be a polarized variation of HS of weight w on Δ^{*n}, having unipotent monodromies T_j $(1 \le j \le n)$. According to Schmid [Sc], $\exp(-\sqrt{-1}\Sigma t_j N_j)\cdot F(t) \in D$ has a limit F in $D^{\vee}$ as t_j tends to infinity, where D denotes an appropriate period domain, $D^{\vee}$ its compact dual and t varies in the universal covering of Δ^{*n}. Moreover, $\exp(\sqrt{-1}\Sigma t_j N_j)\cdot F$ approximates $F(t)$ well with respect to an invariant metric on $D^{\vee}$. It is why one should study nilpotent orbits (cf.[CK]).

2.1 Recall that a nilpotent orbit of weight w $(H,F,S;N_1,\cdots,N_n)$ or $(H,F;N_1,\cdots,N_n)$ consists of a $\mathbf{R}$-vector space $H_{\mathbf{R}}$, a decreasing filtration F of $H := H_{\mathbf{R}}\otimes\mathbf{C}$, a $(-1)^w$-symmetric bilinear form on $H_{\mathbf{R}}$, and a set of mutually commuting nilpotent endomorphisms $(N_1,\cdots,N_n)$ of $H_{\mathbf{R}}$. These data should satisfy the following two conditions.

(i) $N_j F^p \subset F^{p-1}$ for all p,j.

(ii) There is a constant $C > 0$ such that $(H,\exp(\sqrt{-1}\Sigma t_j N_j)\cdot F,S)$ is a polarized HS of weight w for $t_j > C$.

2.2 Kashiwara introduced the notion of infinitesimal mixed Hodge module (IMHM for short) as an object arising from an admissible variation of MHS [SZ,§3] as in (2.0). We recall its definition.

Definition 2.3 1) A pre-IMHM $(H;W,F,\{S_k\};N_1,\cdots,N_n)$ consists of a $\mathbf{R}$-vector space $H_{\mathbf{R}}$, an increasing filtration W on $H_{\mathbf{R}}$, a decreasing filtration F on $H := H_{\mathbf{R}}\otimes\mathbf{C}$, a bilinear form S_k on $\mathrm{Gr}^W_k H_{\mathbf{R}}$ for each k, and mutually commuting nilpotent endomorphisms $(N_1,\cdots,N_n)$.

These data should satisfy the following conditions (i),(ii).

(i) $N_jF^p \subset F^{p-1}$, $N_jW_k \subset W_k$ for all p,j,k.

(ii) $\{Gr_k^W H, F(Gr_k^W H), S_k; N_1, \cdots, N_n\}$ is a nilpotent orbit of weight k for all k.

We don't often mention the bilinear forms $\{S_k\}$ explicitly.

2) A pre-IMHM $\{H,W,F;N_1,\cdots,N_n\}$ is called an IMHM if there exists a filtration $M(J)$ for each $J \subset I = \{1,\cdots,n\}$ such that

(iii) $N_jM_k(J) \subset M_{k-2}(J)$ for $j \in J$.

(iv) $M(J)$ is the relative monodromy filtration of $\sum_{j\in J} N_j$ with respect to W (For the definition, see (2.4) below).

It is shown in [K1] that IMHM has many nice properties concerning the relative monodromy filtration. We recall some of them, which will be used later.

2.4 Here, let H denote an object in an abelian category, W its increasing filtration and N a nilpotent endomorphism of H such that $NW_k \subset W_k$. The relative monodromy filtration M of N with respect to W is, by definition, the unique (increasing) filtration (if it exists) satisfying the conditions :

(i) $NM_k \subset M_{k-2}$, (ii) $N^\ell : Gr^M_{k+\ell}Gr^W_k \xrightarrow{\sim} Gr^M_{k-\ell}Gr^W_k$ for all $\ell \geq 1,k$.

We often denote it by $M(N,W)$. Several properties follow from

its existence (cf.[SZ],[K1]).

Assuming the existence of $M = M(N,W)$, we recall two filtrations N_*W and $N_!W$ related to the perverse sheaves Rj_*H, $j_!H$ (cf.[K1,(3.4)]). We put

$$(2.5)\quad (N_*W)_k := NW_{k+1} + M_k \cap W_k = NW_{k+1} + M_k \cap W_{k+1}\ ,$$
$$(N_!W)_k := W_{k-1} + M_k \cap N^{-1}W_{k-1} = W_{k-1} + M_k \cap N^{-1}W_{k-2}\ .$$

Lemma 2.6 ([K1,(3.4.2),(3.4.3)]) The following hold :

(i) $M(N,N_*W) = M(N,N_!W) = M(N,W)$.

(ii) $Gr^W_{k+1} \underset{\longleftarrow}{\longrightarrow} Gr^{N_*W}_k \rhd$ (resp. $Gr^W_{k-1} \underset{\longleftarrow}{\longrightarrow} Gr^{N_!W}_k \rhd$). (For the meaning of $\rhd$, see (3.4).)

(iii) $Gr^{N_*W}_k = \mathrm{Im}(N:Gr^W_{k+1} \longrightarrow Gr^W_{k+1}) \oplus \mathrm{Coker}(N:W_kGr^M_{k+2} \longrightarrow W_kGr^M_k)$ (resp. $Gr^{N_!W}_k = \mathrm{Coim}(N:Gr^W_{k-1} \longrightarrow Gr^W_{k-1}) \oplus \mathrm{Ker}(N:Gr^M_k(H/W_{k-1}) \longrightarrow Gr^M_{k-2}(H/W_{k-1}))$.

Proposition 2.7 ([K1,(5.5.1),(5.5.5)])

Let $\{H,W,F;N_1,\cdots,N_n\}$ be an IMHM. Then, for any $i,j \in I := \{1,\cdots,n\}$, the following hold.

(i) $N_{i*}(N_{j*}W) = N_{j*}(N_{i*}W)$.

(ii) $M(N_i,N_{j*}W) = N_{j*}M(N_i,W)$.

Thus we can define the following : For $J = \{j_1,\cdots,j_\ell\} \subset I$,

$$(2.8)\quad N_{J*}W := N_{j_1*}\cdots N_{j_\ell*}W\ ,\quad N_{J!}W := N_{j_1!}\cdots N_{j_\ell!}W \quad (N_J := \prod_{j\in J} N_j).$$

Then, putting $M(J,W) = M(\sum_{j\in J} N_j, W)$, we have

$$M(J_1, N_{J_2}{}^* W) = N_{J_2}{}^* M(J_1, W).$$

2.9 We recall an important property of the relative monodromy filtration ; There is a canonical decomposition [K1,(3.2.9)] :

$$Gr_k^M = \bigoplus_{\ell} Gr_k^M Gr_\ell^W .$$

Therefore, Gr_k^M is polarizable if so is Gr_ℓ^W (cf.[Sc,(6.16)]). In particular, for an IMHM, $Gr_k^{N_J*W}$ or $Gr_k^{N_J!W}$ is polarizable by (2.6).

§3. IMHM's and n-cubes

3.0 There is a description of perverse sheaves in the normal crossing case due to Galligo-Granger-Maisonobe [GGM], which we recall now.

In this section, we use the following notation :

(3.1) $X = \Delta^n \quad z = (z_1,\cdots,z_n), \quad Y = \{z_1\cdots z_n = 0\}, \quad \Delta = \{x\in\mathbf{C}; |x|<1\},$
$Y_J = \bigcap_{j\in J}\{z_j = 0\}, \quad Y_J^* = Y_J - \bigcap_{j\in J} Y_j \quad \text{for } J \subset I = \{1,\cdots,n\}.$
k ; a field.

Let $\mathcal{P}(n)$ denote the category of perverse sheaves F such that $F|_{Y_J^*}$ is a locally constant sheaf of k-modules. We refer the reader to [BBD] about perverse sheaves. Then the objects in $\mathcal{P}(n)$ can be described in the following way.

Proposition 3.2 ([GGM]) $\mathcal{P}(n)$ is equivalent to the category $\mathcal{C}(n)$ consisting of the data $\mathcal{H} = \{H_\alpha; f_{\alpha\beta}, g_{\beta\alpha}\}$ satisfying the following conditions (the morphisms in $\mathcal{C}(n)$ are obviously defined) :

(0) H_α is a k-vector space ($\alpha \subset I$), and $f_{\alpha\beta}: H_\beta \longrightarrow H_\alpha$, $g_{\beta\alpha}: H_\alpha \longrightarrow H_\beta$ are k-homomorphisms ($\beta \subset \alpha \subset I$).

(1) $f_{\alpha\beta}\cdot f_{\beta\gamma} = f_{\alpha\gamma}$, $g_{\gamma\beta}\cdot g_{\beta\alpha} = g_{\gamma\alpha}$ for $\gamma \subset \beta \subset \alpha$.

(2) $f_{\alpha\alpha} = g_{\alpha\alpha} = \mathrm{id}$.

(3) $g_{\gamma\alpha}\cdot f_{\alpha\beta} = f_{\gamma\delta}\cdot g_{\delta\beta}$ for $\delta \subset \beta \subset \alpha$ and $\delta \subset \gamma \subset \alpha$.

(4) $1 - g_{\beta\alpha}\cdot f_{\alpha\beta}$ is invertible for $\beta \subset \alpha$ and $|\alpha| = |\beta|+1$.

We call an object in $\mathcal{C}(n)$ an n-cube.

Example 3.3 Given a local system H on Δ^{*n} with unipotent monodromies. Then, if we denote a stalk of H by the same H,

1) the minimal extension ${}^{\pi}H$ of H to Δ^n corresponds to an n-cube $\langle\pi,H\rangle := \{H_\alpha; f_{\alpha\beta}, g_{\beta\alpha}\}$ defined by $H_\alpha = \mathrm{Im}\, N_J$, $f_{\alpha\beta} = N_{\alpha-\beta}$, $g_{\beta\alpha} = \mathrm{id}$.

2) the perverse sheaf Rj_*H corresponds to an n-cube $\langle *,H\rangle$ defined by $H_\alpha = H$, $f_{\alpha\beta} = N_{\alpha-\beta}$, $g_{\beta\alpha} = \mathrm{id}$.

3) the perverse sheaf $j_!H$ corresponds to an n-cube $\langle !,H\rangle$ defined by $H_\alpha = H$, $f_{\alpha\beta} = \mathrm{id}$, $g_{\beta\alpha} = N_{\alpha-\beta}$.

For n = 1,

$$\langle\pi,H\rangle = [\ H \underset{1}{\overset{N}{\rightleftarrows}} \mathrm{Im}N\ ]\ ,\ \langle *,H\rangle = [\ H \underset{1}{\overset{N}{\rightleftarrows}} H\ ]\ ,\ \langle !,H\rangle = [\ H \underset{N}{\overset{1}{\rightleftarrows}} H\ ].$$

For n = 2,

$$\langle\pi,H\rangle = \begin{pmatrix} H & \underset{1}{\overset{N_1}{\rightleftarrows}} & \mathrm{Im}N_1 \\ {\scriptstyle N_2}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle N_2}\downarrow\uparrow{\scriptstyle 1} \\ \mathrm{Im}N_2 & \underset{1}{\overset{N_1}{\rightleftarrows}} & \mathrm{Im}N_1N_2 \end{pmatrix}\ ,\quad \langle *,H\rangle = \begin{pmatrix} H & \underset{1}{\overset{N_1}{\rightleftarrows}} & H \\ {\scriptstyle N_2}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle N_2}\downarrow\uparrow{\scriptstyle 1} \\ H & \underset{1}{\overset{N_1}{\rightleftarrows}} & H \end{pmatrix}.$$

3.4 An n-cube $\mathcal{H}$ is said to have the decomposition property if $H_\beta \underset{g_{\beta\alpha}}{\overset{f_{\alpha\beta}}{\rightleftarrows}} H_\alpha \rhd$ for all $\beta \subset \alpha$ (or equivalently, for all $\beta \subset \alpha$ such that $|\alpha| = |\beta|+1$). Here we write $A \underset{g}{\overset{f}{\rightleftarrows}} B \rhd$ when $B = \mathrm{Im}f \oplus \mathrm{Ker}g$ (cf.[K1,§2]).

The terminology is explained by the following lemma.

Lemma 3.5 ([K1,(2.3.1)]) The following statements are equivalent.

a) An n-cube $\mathcal{H}$ has the decomposition property.

b) $\mathcal{H}$ is a direct sum of n-cubes $\{M_\alpha^{(i)};f_{\alpha\beta}^{(i)},g_{\beta\alpha}^{(i)}\}$ such that for some $\gamma^{(i)}\subset I$, $M_\alpha^{(i)}=0$ if $\gamma^{(i)}\not\subset\alpha$, and $f_{\alpha\gamma}^{(i)}$ is surjective and $g_{\gamma\alpha}^{(i)}$ is injective if $\gamma=\gamma^{(i)}\subset\alpha$.

c) We have a decompsition $H_\alpha \xleftarrow{\sim} \bigoplus_{\beta\subset\alpha} f_{\alpha\beta}(P_\beta)$, where we put $P_\beta = \bigcap_{\beta\subsetneq\alpha} \mathrm{Ker}\, g_{\beta\alpha}$, and $g_{\beta\alpha}: f_{\alpha\beta}(P_\beta)\longrightarrow P_\beta$ is injective for $\alpha\supset\beta$.

3.6 We recalled the notion of IMHM in (2.3), which was motivated by the infinitesimal study of admissible VMHS. For further study, Kashiwara introduced an object, a mixture of a cube and an IMHM, which may be regarded as the infinitesimal version of a mixed Hodge module of M.Saito [Sa2].

Definition ([K1,(5.6)]) Let $\{H_\alpha;f_{\alpha\beta},g_{\beta\alpha}\}$ be an n-cube, W^α (resp. F_α) an increasing (resp. decreasing) filtration on H_α for $\alpha\subset I=\{1,\cdots,n\}$. Then we call $\mathcal{H}=\{H_\alpha,W^\alpha,F_\alpha;f_{\alpha\beta},g_{\beta\alpha}\}$ an MH-cube if the following conditions hold.

(1) $f_{\alpha\beta}F_\beta^p\subset F_\alpha^{p-|\alpha-\beta|}$, $f_{\alpha\beta}W_k^\beta\subset W_{k-|\alpha-\beta|}^\alpha$,
$g_{\beta\alpha}F_\alpha^p\subset F_\beta^p$, $g_{\beta\alpha}W_k^\alpha\subset W_{k+|\alpha-\beta|}^\beta$ for $\beta\subset\alpha$.

(2) $N_j\in\mathrm{End}(H_\alpha)$ is nilpotent for any j, where $N_j := g_{\alpha,\alpha\cup\{j\}}f_{\alpha\cup\{j\}}$ if $j\in\alpha$, $f_{\alpha,\alpha-\{j\}}g_{\alpha-\{j\},\alpha}$ if $j\in\alpha$.

(3) For each α, $\{H_\alpha,W^\alpha,F_\alpha;N_1,\cdots,N_n\}$ is an IMHM.

(4) The property $\mathrm{Gr}_k^{W^\beta}\rightleftarrows\mathrm{Gr}_{k-|\alpha-\beta|}^{W^\alpha}$ holds for $\beta\subset\alpha$.

The morphism between MH-cubes is obviously defined. The category of MH-cubes is an abelian category and is denoted by MH(I).

We have the operations like dual, Tate twist, neaby and vanishing cycles in the category MH(I).

3.7 Let $\{H,W,F;N_1,\cdots,N_n\}$ be an IMHM. Then we defined cubes $\langle\pi,H\rangle$, $\langle *,H\rangle$, $\langle !,H\rangle$ in (3.3). We recall the definition of filtrations W^α, F_α on H_α giving a structure of MH-cube, according to [K1,(5.8)].

Definition 1) For $\langle *,H\rangle$, put $W^\alpha := N_{\alpha *}W$, $F_\alpha := F$.

2) For $\langle !,H\rangle$, put $W^\alpha := N_{\alpha !}W(-|\alpha|)$, $F_\alpha := F$.

3) For $\langle \pi,H\rangle$, put $W^\alpha := N_{\alpha *}W \cap \mathrm{Im}N_\alpha$, $F_\alpha := F$.

(Here $W(\ell)$ is defined by $W(\ell)_k := W_{k-2\ell}$.)

These are MH-cubes by (2.6),(3.5).

Of course, $\langle *,H\rangle$ and $\langle !,H\rangle$ are dual to each other, and $\langle \pi,H\rangle$ ia a quotient of $\langle !,H\rangle$ as well as a subobject of $\langle *,H\rangle$.

We put

$$P_k(*\beta) := P_\beta(\mathrm{Gr}^W_k\langle *,H\rangle) = \bigcap_{\gamma\subsetneq\beta} \mathrm{Ker}(g_{\gamma\beta}:\mathrm{Gr}^{N_\beta *W}_{k-|\beta|} \longrightarrow \mathrm{Gr}^{N_\gamma *W}_{k-|\gamma|}) ,$$

$$P_k(!\beta) := P_\beta(\mathrm{Gr}^W_k\langle !,H\rangle) = \bigcap_{\gamma\subsetneq\beta} \mathrm{Ker}(g_{\gamma\beta}:\mathrm{Gr}^{N_\beta !W}_{k-|\beta|} \longrightarrow \mathrm{Gr}^{N_\gamma !W}_{k-|\gamma|}) ,$$

(the "primitive part" cf.(3.5,c)). Then, since an MH-cube has the decomposition property (3.4) by definition, we get the following :

Lemma 3.8 We have a decomposition :

$$\mathrm{Gr}^W_k\langle *,H\rangle = \bigoplus_J \langle \pi,P_{k-|J|}(*I-J)\rangle_{I-J}$$

$$(\text{resp. } \mathrm{Gr}^W_k\langle !,H\rangle = \bigoplus_J \langle \pi,P_{k-|J|}(!I-J)\rangle_{I-J}).$$

Here $\langle\ ,\ \rangle_{I-J}$ denote the object in $\mathcal{C}(I-J)$ regarded as in $\mathcal{C}(I)$.

Example 3.9 We illustrate the case n = 1.

$$W_k\langle *,H\rangle = [\ W_k \underset{1}{\overset{N}{\rightleftarrows}} (N_*W)_{k-1}\ ]$$

$$Gr_k^W\langle *,H\rangle = [\ Gr_k^W \underset{}{\overset{N}{\rightleftarrows}} Gr_{k-1}^{N_*W}\ ]$$

$$= [\ Gr_k^W \rightleftarrows NGr_k^W\ ] \oplus [\ 0 \rightleftarrows \mathrm{Coker}(N:W_{k-1}Gr_{k+1}^M \longrightarrow W_{k-1}Gr_{k-1}^M)]$$

$$W_k\langle !,H\rangle = [\ W_k \underset{N}{\overset{1}{\rightleftarrows}} (N_!W)_{k+1}\ ]$$

$$Gr_k^W\langle !,H\rangle = [\ Gr_k^W \rightleftarrows NGr_k^W\ ] \oplus [\ 0 \rightleftarrows \mathrm{Ker}(N:Gr_{k+1}^M(H/W_k) \longrightarrow Gr_{k-1}^M(H/W_k))\ ]$$

§4. Proof of Theorem 1.2 : Construction of CMHC

4.0 We give a concrete expression of cohomological MH complexes (CMHC) alluded to in §1 using the result of §3. See (4.8) for another and simpler expression due to Kashiwara.

We use the same notation as in §1, except for H denoting an admissible VMHS on U. Denote a general stalk of H by the same H.

We will use some notation related to the integrable connection $\mathcal{O}_U \otimes H$. Let $E_X(H)$ denote the (left) canonical extension of $\mathcal{O}_U \otimes H$ [KK2]. Perverse sheaves corresponding to regular holonomic $\mathcal{D}$-modules by the Riemann-Hilbert correspondence (see e.g.[K2]), we denote by $\mathcal{M}(F)$ the $\mathcal{D}$-module associated to a perverse sheaf F.

4.1 First of all, note that the existence of MHS is automatic, if we give a structure of CMHC upon Rj_*H (resp. $j_!H$). This is due to the formalism by Deligne [D,III(8.1)].

We don't recall the precise definition of a CMHM here, but only mention that the data needed are :

(1) a filtration W on $K_{*\mathbf{Q}}$ (resp. $K_{!\mathbf{Q}}$),

(2) filtrations W, F on $K_{*\mathbf{C}}$ (resp. $K_{!\mathbf{C}}$).

Here K_{*A} (resp. $K_{!A}$) denotes a complex quasi-isomorphic to Rj_*H_A (resp. $j_!H_A$), $A = \mathbf{Q}, \mathbf{C}$, and we assume that these are compatible between $\mathbf{Q}$ and $\mathbf{C}$.

These data should satisfy the condition :

(3) The CMHC $\{Gr_k^W K_{\mathbf{Q}}, (Gr_k^W K_{\mathbf{C}}, F)\}$ are pure for all p, i.e., the spectral sequence associated to the filtered complex $\{R\Gamma(X, Gr_k^W K_{\mathbf{C}}), F\}$ degenerates at E_1 and gives rise to the Hodge filtration on the abutements ($K = K_*$ or $K_!$).

4.2 We will define the filtration W using MH-cubes in §3. So we take Rj_*H (resp. $j_!H$) itself as $K_{*\mathbb{Q}}$ (resp. $K_{!\mathbb{Q}}$). To realize the filtration F, we need a complex of $\mathcal{O}$-coherent modules relevant to the canonical extension $E_X(H)$. Thus we take the logarithmic de Rham complex $\Omega_X(\log Y)\otimes E_X(H)$ as $K_{*\mathbb{C}}$.

Lemma 4.2.1 To a subobject F of $Rj_*H_{\mathbb{C}}$ in the category of perverse sheaves $\mathcal{C}(n)$ (3.1), there is a subcomplex of $\Omega_X(\log Y)\otimes E_X(H)$ quasi-isomorphic to F. Denote it by $\Omega_{X,\log}(F)$.

Proof Take $\Omega_{X,\log}(F) := \Omega_X\otimes\mathcal{M}(F)\cap \Omega_X(\log Y)\otimes E_X(H)$. We have only to remark that the argument in [KK,§4] (case $F = {}^{\pi}H$) is applicable to F.

4.3 As for $K_{!\mathbb{C}}$, we use the following simplicial construction. The construction being local, we may assume $(X,U) = (\Delta^n,\Delta^{*n})$ and put $Y_j = \{z_j = 0\}$ (cf.(2.0). Let N_j be the logarithm of the monodromy T_j along Y_j.

Lemma 4.3.1 i) The cube $\langle !,H\rangle$ is quasi-isomorphic to the simple complex associated to

$$\{\bigoplus_{|J|=p} \langle \pi,\mathrm{Ker}N_J\rangle_{I-J}[p]\}_{0\le p\le n}.$$

Here $\langle \pi,\mathrm{Ker}N_J\rangle_{I-J}$ denotes the object (3.3,1) in $\mathcal{C}(I-J)$ regarded as in $\mathcal{C}(I)$, and $\mathrm{Ker}N_J := \bigcap_{j\in J}\mathrm{Ker}N_j$.

ii) $j_!H_{\mathbb{C}}$ is quasi-isomorphic to the simplicial complex

$$\{\bigoplus_{|J|=p} \Omega_{Y_J,\log}({}^{\pi}\mathrm{Ker}N_J)\}_{0\le p\le n}. \qquad (Y_J = \bigcap_{j\in J} Y_j)$$

Proof (ii) is an immediate consequence of (i). For (i), consider the case $n = 1$. The following is an exact sequence in $\mathcal{C}(n)$.

$$\begin{bmatrix} 0 \\ \downarrow\uparrow \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \downarrow\uparrow \\ \mathrm{Ker}N \end{bmatrix} \longrightarrow \begin{bmatrix} H \\ 1\downarrow\uparrow N \\ H \end{bmatrix} \begin{matrix} \xrightarrow{1} \\ \xrightarrow{N} \end{matrix} \begin{bmatrix} H \\ N\downarrow\uparrow 1 \\ \mathrm{Im}N \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \downarrow\uparrow \\ 0 \end{bmatrix}$$

Thus we get $\langle !,H\rangle = [\ \langle \pi,H\rangle_I \longrightarrow \langle \pi,\mathrm{Ker}N\rangle_\phi[1]\ ]$. The general case is obtained similarly.

Variant 4.3.2 There is a construction analogous to (4.3.1). To a subobject F of $j_!H$ in $\mathcal{P}(n)$, we associate a subcomplex of the complex in (4.3.1,ii), quasi-isomorphic to F. Denote it by $\Omega_{spl}(F)$. The procedure is the same as (4.3.1).

Example 4.3.4 We illustrate the case n = 1 for $W_k\langle !,H\rangle$.

$$W_k\langle !,H\rangle = [\begin{bmatrix} W_k \\ \downarrow\uparrow \\ N(N_!W)_{k+1} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \downarrow\uparrow \\ (N_!W)_{k+1} \cap \mathrm{Ker}N \end{bmatrix}[1]\ ]$$

$$\mathrm{Gr}^W_k\langle !,H\rangle = [\begin{bmatrix} \mathrm{Gr}^W_k \\ \downarrow\uparrow \\ \mathrm{Gr}^{N_*W}_{k-1}(\mathrm{Im}N) \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ \downarrow\uparrow \\ \mathrm{Gr}^{N_!W}_{k+1}(\mathrm{Ker}N) \end{bmatrix}[1]\ ]$$

4.4 CMHC on Rj_*H (resp. $j_!H$)

In (3.7), we defined a filtration W on $\langle *,H\rangle$ (resp. $\langle !,H\rangle$). By the construction (4.2) (resp. (4.3)), we get a filtration W on Rj_*H (resp. $j_!H$) as well as $\Omega_{log}(Rj_*H)$ (resp. $\Omega_{spl}(j_!H)$) :

$$W_k\Omega_{log}(Rj_*H) := \Omega_{log}(W_k(Rj_*H)) ,$$

$$W_k\Omega_{spl}(j_!H) := \Omega_{spl}(W_k(j_!H)) .$$

By Schmid's theorem [Sc,§6], the canonical extension $E_X(H)$ (or $E_{Y_J}(\mathrm{Ker}N_J)$) has a filtration F which prolongs the Hodge filtration of $\mathcal{O}_U\otimes H$. Therefore, $\Omega_{\log}(Rj_*H)$ (resp. $\Omega_{spl}(j_!H)$) has a filtration F :

$$F^p\Omega_{\log}(Rj_*H) := F^pE_X(H) \longrightarrow \Omega^1_X(\log Y)\otimes F^{p-1}E_X(H) \longrightarrow \cdots$$

$$F^p\Omega_{spl}(j_!H) := [\ \bigoplus_{|J|=q} F^p\Omega_{Y_J,\log}(\pi\mathrm{Ker}N\ )\ ]_{0\le q\le n} .$$

We can write down expliocitly W on $\Omega_{\log}(Rj_*H)$ locally. If $X = \Delta^n$, $Y_j = \{z_j = 0\}$, then

$$(4.4.1)\qquad W_k\Omega_{\log}(Rj_*H)^p = \sum_{|J|\le p} \frac{dz_J}{z_J}\wedge \Omega_X^{p-|J|}(\log Y)\otimes E_X((N_{J*}W)_{k-|J|}) .$$

Here $dz_J = \bigwedge_{j\in J} dz_j$, $z_J = \prod_{j\in J} z_j$. For $N_{J*}W$, see (2.8).

As for $W_k\Omega_{spl}(j_!H)$, we have an expression as

$$W_k\Omega_{spl}(j_!H) = [\ \bigoplus_{|J|=q} \Omega_{Y_J,\log}(W_{k-q}\langle\pi,\mathrm{Ker}N_J\rangle)(-q)\ ]_{0\le q\le n} ,$$

(cf.(3.7)). To get it, we must use the equality :

$$N_J(N_{J!}W)_{k+|J|} = (N_{J*}W)_{k-|J|}\cap \mathrm{Im}N_J .$$

To see this, the author have to use [Sa2] (the case $|J| = 1$ is trivial). But we won't use it later.

Remark 4.4.2 We defined W,F using §2,3, so that *a priori* they have meaning only locally around Y. But on U, Rj_*H and $j_!H$ reduce to H itself. Hence there is no problem. ($\Omega_U\otimes H$ patches to $\Omega_{\log}(Rj_*H)$ and $\Omega_{spl}(j_!H)$.)

Lemma 4.5 For each k, $\mathrm{Gr}^W_k\Omega_{\log}(Rj_*H)$ (resp. $\mathrm{Gr}^W_k\Omega_{spl}(j_!H)$) is a direct sum of pure CMHC (4.1,3).

Proof We calculate Gr^W_k using the local expression (4.4.1). Obviously $\mathrm{Gr}^W_k\Omega_{\log}(Rj_*H)$ is mapped into

$$\bigoplus_{|J|\le p} \Omega_{Y_J}^{p-|J|}(\log\, Y\cap Y_J)\otimes\mathbb{Q}(-|J|)\otimes E_X(\mathrm{Gr}^{N_J*W}_{k-|J|}) ,$$

where $Y\cap Y_J$ means the divisor on Y_J induced by Y, and $\mathbb{Q}(-|J|)$ is the Tate twist. Calculate the stalk of Gr^W_k in the same way as in [KK1,§3] (using the homotopy formula for the Euler operator). Then the calculation reduces to that of $\mathrm{Gr}^W_k\langle *,H\rangle$. But we see by (3.8) that

$$\mathrm{Gr}^W_k\Omega_{\log}(Rj_*H) = \bigoplus_J \Omega_{Y_J,\log}(\pi P_{k-|J|}(*I-J))(-|J|)[-|J|] .$$

Since $\mathrm{Gr}^{N_J*W}_{k-|J|}$ is polarizable by (2.9), $\mathrm{Gr}^{N_J*W}_{k-|J|}$ is a PVHS associated to a nilpotent orbit (i.e. pure IMHM) of weight $k-|J|$. Thus the second term is a pure complex of weight k by the main theorem of [KK1] or [CKS].

Similar but more complicated reasoning shows that $\mathrm{Gr}^W_k\Omega_{spl}(j_!H)$ is a pure complex of weight k :

$$\mathrm{Gr}^W_k\Omega_{spl}(j_!H) = \bigoplus_J \Omega_{Y_J,\log}(\pi P_{k-|J|}(!I-J))(-|J|)[-|J|] .$$

Corollary 4.6 (including Theorem 1.2) Let X, Y be as in (1.1), but H be an admissible VMHS. Then there exist MHS's on $H^i(U,H)$ and $H^i_c(U,H)$. If H is pure of weight w, then the weights of $H^i(U,H)$ (resp. $H^i_c(U,H)$) are $\ge w+i$ (resp. $\le w+i$).

Remark 4.7 1) The MHS constructed above is independent of the choice of compactification X. We can see this using the language of filtered $\mathcal{D}$-modules [Sa1,§2].

2) The case $\dim X = 1$ was treated in [SZ,§4]. The description by 1-cubes is essentially used there.

3) The MHS's on $H^i(U,H)$ and $H^{2d-i}_c(U,H^*)$ are dual to each other

($d = \dim X$, $H^* = \mathcal{H}om_{\mathbb{C}}(H, \mathbb{C}_X)$). This can be proven by observing the natural pairing between $\Omega_{log}(Rj_*H)$ and $\Omega_{spl}(j_!H)$ (cf.[SZ,(4.30)], [Sh,(3.3)]).

4.8 We present here a more direct way of expressing Rj_*H or $j_!H$ as CMHC due to Kashiwara.

In general, let g be a holomorphic function on a complex manifold X, $Y = \{g = 0\}$. Then, for a regular holonomic $\mathcal{D}$-module $\mathcal{M}$ on X, there is a canonical filtration $\{V_\lambda \mathcal{M}\}$ on $\mathcal{M}$ ("V-filtration"), which we don't recall here (cf.[K3],[Sa1,§3]). According to Kashiwara, we have a quasi-isomorphism :

$$DR_X\mathcal{M} := \mathcal{M} \longrightarrow \Omega^1 \otimes \mathcal{M} \longrightarrow \Omega^2 \otimes \mathcal{M} \longrightarrow \cdots$$
$$\xleftarrow{\sim} \quad V_0 DR_X\mathcal{M} := V_0\mathcal{M} \longrightarrow \Omega^1 \otimes V_1\mathcal{M} \longrightarrow \Omega^2 \otimes V_2\mathcal{M} \longrightarrow \cdots$$

Each $V_\lambda \mathcal{M}$ is coherent over $V_0\mathcal{D}_X$. Moreover, it is $\mathcal{O}_Y$-coherent if $\mathcal{M}$ has support in Y.

In our setup (1.1), we use the graph construction : take $X \times \mathbb{C}$ and pr_2 in place of X and g above. For $\mathcal{M}$, we take the direct image of a regular holonomic $\mathcal{D}_X$-module $\mathcal{N}$, $\mathcal{O}_X$-coherent over a dense open subset X^* of X, by the graph map $\iota_g : X \longrightarrow X \times \mathbb{C}$. Then $V_\lambda \mathcal{M}$ has support in $\iota_g(X)$, is coherent over $\mathcal{O}_X$ and is related to the canonical extension of $\mathcal{M}|_{X^*}$. Thus we get a complex of $\mathcal{O}_X$-coherent modules $V_0 DR_{X\times\mathbb{C}}(\int_{\iota_g} \mathcal{N})$. Since the construction is functorial, we get a representative of the filtered complex $DR_X\mathcal{M}(Rj_*H)$ or $DR_X\mathcal{M}(j_!H)$ (4.0). (Take $\mathcal{M}(W_k Rj_*H)$ or $\mathcal{M}(W_k j_!H)$ as $\mathcal{N}$.)

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