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The number of periodic points of smooth maps

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1. Introduction

In [12], Levinson obtained a result on the number of periodic points of diffeomorphisms of a disk. He gave a classification of periodic points and proved certain equalities between the cardinal numbers of the classes of periodic points. His equalities were improved by Massera[13] and extended to compact manifolds by Shiraiwa[16]. Recently, Dold obtained an extension of these equalities to smooth maps on manifolds in [4] (Theorem 1 below).

In this paper, we give two applications of the Dold's equalities to the study of periodic points of smooth maps. The first application is concerned with the problem of whether the number $N(m)$ of periodic points of a given odd period m is even or odd. We give a condition on m and on the homotopy class of the map under which $N(m)$ is even. This generalizes results in [12], [13], [16].

In the second application, we generalize some results of Franks[7],[8] on the existence of infinitely many periodic points.

In the last section, we generalize the Dold's equalities by making use of the notion of a Reidemeister class.

2. The Dold's equalities

Let M be a C^∞ manifold (possibly with boundary), U an open set of M, and let $f : U \to M$ be a continuous map. For positive integers m, define the iterates $f^m : U_m \to M$ inductively by $f^1 = f$, $U_m = f^{-1}(U_{m-1})$, $f^m(x) = f^{m-1}(f(x))$ for $m > 1$. Denote by $\mathrm{Fix}(f^m)$ the fixed point set of f^m, i.e, the set of points $x \in U_m$ with $f^m(x) = x$.

Suppose $\mathrm{Fix}(f^m)$ is compact. Then $\mathrm{Fix}(f^k)$ is also compact for any divisor k of m because $\mathrm{Fix}(f^k)$ is a closed subset of $\mathrm{Fix}(f^m)$, and hence the fixed point index $I(f^k)$ of f^k is defined (cf. [3]). Define an integer $I_m(f)$ by

$$I_m(f) = \sum_{d|m} \mu(d)\, I(f^{m/d}),$$

where the sum is taken over all divisors d of m and $\mu(d)$ denotes the Moebius function, i.e., $\mu(1) = 1$, $\mu(d) = (-1)^k$ if d is a product of k distinct primes, and $\mu(d) = 0$ otherwise.

Definition 1. Set

$$P(m) = \mathrm{Fix}(f^m) - \bigcup_{d < m} \mathrm{Fix}(f^d),$$

and call an element of $P(m)$ an <u>m-periodic point</u> , or simply a <u>periodic point</u>, of f.

Now assume that f is a C^1 map. For $x \in P(d)$, $d \geq 1$, let $Df^d(x)$ denote the derivative of f^d at x, and let $a_+(x)$ (resp. $a_-(x)$) be the number of real eigenvalues λ of $Df^d(x)$ with $\lambda > 1$ (resp. $\lambda < -1$) (counting multiplicity). Set

$P_{EE}(d) = \{\ x \in P(d) \mid a_+(x), a_-(x)$ are even $\}$,

$P_{EO}(d) = \{\ x \in P(d) \mid a_+(x)$ is even, $a_-(x)$ is odd $\}$,

$P_{OE}(d) = \{\ x \in P(d) \mid a_+(x)$ is odd, $a_-(x)$ is even $\}$,

$P_{OO}(d) = \{\ x \in P(d) \mid a_+(x), a_-(x)$ are odd $\}$,

$N_{EE}(d) = \# P_{EE}(d)$, $\qquad N_{EO}(d) = \# P_{EO}(d)$

$N_{OE}(d) = \# P_{OE}(d)$, $\qquad N_{OO}(d) = \# P_{OO}(d)$,

$N_E(d) = N_{EE}(d) + N_{EO}(d)$, $\qquad N_O(d) = N_{OE}(d) + N_{OO}(d)$.

A fixed point x of f^m is <u>simple</u> if the derivative $Df^m(x)$ does not have 1 as an eigenvalue.

In [4, pages 431-432], Dold proved the following result (in a somewhat implicit form) :

Theorem 1(Dold). Let $f : U \to M$ be a C^1 map defined on an open subset U of a manifold M, and let m be a positive integer. Suppose that f^m has only finitely many fixed points and that every fixed point of f^m is simple and contained in the interior Int M of M. Then

$$\begin{aligned} I_m(f) &= N_E(m) - N_O(m) && \text{if } m \text{ is odd,} \\ &= N_E(m) - N_O(m) + 2(N_{OO}(m/2) - N_{EO}(m/2)) && \text{if } m \text{ is even.} \end{aligned}$$

In some special cases, this theorem was proved and was applied to the study of periodic solutions of periodic differential systems by Levinson [12, Section 6], Massera[13], Shiraiwa[16]. In [4] Dold used this theorem to prove that m divides the number $I_m(f)$ for any continuous map f defined on an open set of an ENR with $\mathrm{Fix}(f^m)$ compact. (This result was also obtained by Zabreiko and Krasnosel'skii[17], [11, Theorem 31.4].) In Sections 3 and 4, we shall give two other applications of Theorem 1.

3. The first application

Throughout this section, we assume that M is a compact manifold.

Let $f : M \to M$ be a continuous map.

Definition 2. A positive integer n is an L_2-<u>period</u> of f if

$$L_2(f^{i+n}) = L_2(f^i) \quad \text{for any} \quad i \geqq 1,$$

where $L_2(f^i)$ denotes the mod 2 reduction of the Lefschetz number $L(f^i)$ of f^i. We denote the minimal L_2-period by $\alpha(f)$.

The following proposition clearly implies that $\alpha(f)$ always exists and is an odd number:

Proposition 1. Let p be a prime number and A a square matrix with entries in Z/pZ. Then there exists a positive integer n such that for any $i \geqq 1$,

$$\text{tr } A^{i+n} = \text{tr } A^i$$

and that n is not divisible by p.

Proof. Since $\{A^i\}_{i \geqq 1}$ is a finite set, there exist positive integers I and k such that $A^I = A^{I+k}$. Then for any $i \geqq I$,

$$A^i = A^I A^{i-I} = A^{I+k} A^{i-I} = A^{i+k}.$$

Let i be a positive integer. Then, $qi \geqq I$ for some power q of p. Since

$$\text{tr } B^{p^j} = \text{tr } B \tag{3.1}$$

for any $j \geqq 0$ and any square matrix B with entries in Z/pZ ([2, Proposition 5],[15, Lemma]), we have

$$\operatorname{tr} A^i = \operatorname{tr} A^{iq} = \operatorname{tr} A^{iq + qk} = \operatorname{tr} A^{(i+k)q} = \operatorname{tr} A^{i+k}.$$

Hence k is a period for $\{ \operatorname{tr} A^i \}_{i \geqq 1}$. Decompose k as nr, where n is not divisible by p and r is a power of p. Then, by (3.1), for any i,

$$\operatorname{tr} A^{i+n} = \operatorname{tr} A^{(i+n)r} = \operatorname{tr} A^{ir} = \operatorname{tr} A^i.$$

Thus n is also a period, and the proof is completed. q.e.d.

Given an m-periodic point x of f, we call the set $\{ f^i(x) \mid i \geqq 1 \}$ an m-<u>periodic</u> <u>orbit</u> of f. As an application of Theorem 1, we have :

Theorem 2. Let $f : M \to M$ be a C^1 map and m an odd number with $m \geqq 3$. Suppose that every fixed point of f^m is simple and contained in Int M. Suppose also that m is divisible by $\alpha(f)^2$ or by a prime number p which is congruent to 2^i modulo $\alpha(f)$ for some $i \geqq 0$. Then, the number $\bar{N}(m)$ of m-periodic orbits of f is even.

Proof. By the assumption, $\# \operatorname{Fix}(f^m) < \infty$. Hence by Theorem 1,

$$\bar{N}(m) = \# P(m)/m = 2\, N_O(m)/m + I_m(f)/m.$$

Hence, it is sufficient for the proof to show that the mod 2 reduction $I_{m,2}(f)$ of $I_m(f)$ is equal to zero. Let $\alpha = \alpha(f)$. Suppose first $\alpha^2 \mid m$. Then $\mu(d) = 0$ unless $d \mid m/\alpha$. Therefore, since $L_2(f^k) = L_2(f^\alpha)$ for any multiple k of α,

$$I_{m,2}(f) = \sum_{d \mid m/\alpha} \mu(d)\, L_2(f^{m/d}) = \Big(\sum_{d \mid m/\alpha} \mu(d) \Big)\, L_2(f^{\alpha}) = 0.$$

Suppose m is divisible by a prime p with $p \equiv 2^i \pmod{\alpha}$ for some i. Decompose m as $m = p^j k$, where $j, k \geqq 1$ and k is not divisible by p. Then

$$(3.2) \qquad I_m(f) = I_k(f^{p^j}) - I_k(f^{p^{j-1}}),$$

Since α is a period for the sequence $\{I_{k,2}(f^i)\}_{i \geqq 1}$,

$$I_{k,2}(f^{p^s}) = I_{k,2}(f^{2^{is}}) \qquad \text{for } s \geqq 0.$$

Hence by (3.1), $I_{k,2}(f^{p^s}) = I_{k,2}(f)$ for $s \geqq 0$.

Therefore by (3.2), $I_{m,2}(f) = 2\, I_{k,2}(f) = 0$. q.e.d.

Let $b(M)$ be the maximum of the Betti numbers of M. It is easy to show that if $b(M) = 1$ (resp. 2) then $\alpha(f) = 1$ (resp. 1 or 3). Hence, by Theorem 2, we have:

Corollary. Let m be an odd number. Assume that $b(M) = 1$, $m \geqq 3$ or that $b(M) = 2$, $m \geqq 5$. Let $f : M \to M$ be a C^1 map and suppose that every fixed point of f^m is simple and contained in Int M. Then $\bar{N}(m)$ is even.

In the case where f is a diffeomorphism of a disk, this corollary has been obtained by Levinson[12], Massera[13]. Theorem 2 follows from Shiraiwa[16, Theorem 3] in the case where f is a diffeomorphism with $L(f) = L(f^i)$ for any i.

When $b(M) \geqq 3$, the condition for $\bar{N}(m)$ to be even does not seem simple in many cases. For example, let M be a disk with 3 holes. Suppose a 3×3 matrix (a_{ij}) representing the homomorphism $f_* : H_1(M ; Z) \to H_1(M ; Z)$ is given by $a_{ij} = 0$ or 1 according as $i + j \leqq 3$ or $i + j > 3$ respectively. Then $\alpha(f) = 7$ and the computation of $I_m(f)$ shows that, for m odd, $\bar{N}(m)$ is odd if and only if $m = r^i$ or $7r^i$, where r is a prime number satisfying $r \equiv 3,5,6 \pmod 7$.

4. The existence of infinitely many periodic points

In this section, we shall generalize some theorems of Franks in [7],[8]. Here we assume the following conditions:

i) M is a compact manifold and $f : M \to M$ is a C^1 map.

ii) f^m has only simple fixed points for any $m \geqq 1$, and f has no periodic points on the boundary of M.

iii) All non-zero eigenvalues of $f_* : H_*(M ; Q) \to H_*(M ; Q)$ are n-th roots of unity for some $n > 0$.

For a positive integer r and $i \geqq 0$, let $\gamma(i,r)$ be the number of eigenvalues λ of $f_{*i} : H_i(M ; Q) \to H_i(M ; Q)$ which are r-th primitive roots of unity (counting multiplicity) and set

$$\gamma_r = \sum_{i \geqq 0} (-1)^i \gamma(i,r)/\phi(r),$$

where $\phi(r)$ denotes the Euler function. For a positive integer a, let

$$\gamma(f, a) = \sum_r \mu(r/a) \gamma_r ,$$

where the sum is taken over all odd numbers r dividing n and $\mu(r/a)$ means 0 if r is not divisible by a. Let

$$\bar{N}_{ab}(m) = N_{ab}(m)/m \quad \text{for} \quad a,b = E,\ O.$$

Theorem 3. Let a be an odd number and let $i(a)$ denote the smallest $i \geqq 0$ such that $2^i a$ does not divide n. Then for any $i \geqq i(a) - 1$, we have

$$\bar{N}_{EO}(2^i a) - \bar{N}_{OO}(2^i a) = \gamma(f,a) + \sum_{j=0}^{i} (\bar{N}_{OE}(2^j a) - \bar{N}_{EE}(2^j a))$$

As immediate consequences of Theorem 3, we have:

Corollary 1. Let a be an odd number. Suppose that

$$\gamma(f,a) > \sum_{j=0}^{\infty} \bar{N}_{EE}(2^j a) \quad \text{or} \quad \gamma(f,a) < -\sum_{j=0}^{\infty} \bar{N}_{OE}(2^j a).$$

Then f has infinitely many periodic points.

Corollary 2. Let a be an odd number satisfying $\gamma(f,a) = 0$, and let $i(a)$ be as in Theorem 3. Suppose that

i) $N_{EE}(2^i a) = N_{OO}(2^i a) = 0$ for any $i \geqq 0$, or

ii) $N_{EO}(2^i a) = N_{OE}(2^i a) = 0$ for any $i \geqq 0$.

Then we have

$$\bar{N}_{EO}(2^i a) + \bar{N}_{OO}(2^i a) = \sum_{j=0}^{i} (\bar{N}_{EE}(2^j a) + \bar{N}_{OE}(2^j a))$$

for any $i \geqq i(a) - 1$. In particular, if f has a $2^i a$-periodic point for some $i \geqq i(a) - 1$, then it has infinitely many periodic points.

Corollary 1 has been proved by Franks [7] in the case where $a = 1$, $N_{OO}(2^i) = 0$ for any i, and M is a circle, a sphere, a closed interval, or a disk. Clearly $\gamma(f,a) = 0$ if $i(a) = 0$. Hence, Corollary 2 has been proved by Franks [8, Theorem C] in case of $i(a) = 0$.

Proof of Theorem 3. Using Theorem 1, we can prove the following equalities by induction on i:

(4.1) $$\bar{N}_{EO}(2^i a) - \bar{N}_{OO}(2^i a)$$
$$= \sum_{j=0}^{i} (\bar{N}_{OE}(2^j a) - \bar{N}_{EE}(2^j a) + \bar{I}_{2^j a}(f)),$$

where $\bar{I}_k(f) = I_k(f)/k$. Hence it is sufficient to show that if $i \geqq i(a) - 1$, then $\sum_{j=0}^{i} \bar{I}_{2^j a}(f) = \gamma(f,a)$. For $r \geqq 1$, let A_r be the set of all r-th primitive roots of unity. For $j \geqq 1$, let

$$\psi(r, j) = \sum_{\lambda \in A_r} \lambda^j,$$

$$M_r(m) = \frac{1}{m} \sum_{d|m} \mu(d)\, \psi(r, m/d).$$

Clearly we have:

(4.2) $$\bar{I}_m(f) = \sum_{r|n} \gamma_r M_r(m).$$

Also we have:

Lemma. For $m, r \geqq 1$, $M_r(m) = \mu(r/m)$.

Proof. We prove this lemma by induction on the number $s(m)$ of primes dividing m. If $s(m) = 0$, then $m = 1$ and the lemma is trivial. Assume the lemma holds for any r and any m such that $s(m)$ is less than an integer s. Let $r, m \geqq 1$ and suppose $s(m) = s$. Decompose m and r as $m = p^i m'$, $r = p^j r'$, where m', r', $i \geqq 1$, $j \geqq 0$, and p is a prime which does not divide m', r'. Let $v = \min\{i, j\}$ and $w = \min\{i - 1, j\}$. For any k, we have

$$\psi(r, km) = \psi(r'', m)\,\phi(r)/\phi(r''),$$

where $r'' = r/(r,k)$, because the k times power of an r-th primitive root of unity is an $r/(r,k)$-th primitive root of unity. Hence, by the induction hypothesis, we have

$$(4.3)\qquad M_r(m) = \frac{1}{m}\sum_{d'|m'} \mu(d')\{\psi(r, p^i m'/d') - \psi(r, p^{i-1}m'/d')\}$$

$$= \frac{m'}{m}\{M_{rp^{-v}}(m')\,\phi_v - M_{rp^{-w}}(m')\,\phi_w\}$$

$$= p^{-i}\mu(r'/m')\{\mu(p^{j-v})\,\phi_v - \mu(p^{j-w})\,\phi_w\}$$

where $\phi_v = \phi(p^j)/\phi(p^{j-v})$, $\phi_w = \phi(p^j)/\phi(p^{j-w})$.

If m does not divide r, then we can assume p^i does not divide r. Therefore, $v = w$ and by (4.3) $M_r(m) = 0$. If m divides r, then $v = i$ and $w = i - 1$. Hence by (4.3), $M_r(m) = \mu(r/m)$. Thus the proof is completed. q.e.d.

Now let $i \geqq i(a) - 1$. By (4.2) and Lemma, since $\sum_{j=0}^{i} \mu(r/2^j a) = 0$ for r even, we have

$$\sum_{j=0}^{i} \bar{I}_{2^j a}(f) = \sum_{r|n} \gamma_r \sum_{j=0}^{i} \mu(r/2^j a) = \gamma(f, a).$$

Thus the proof is completed.

5. An extension of Theorem 1

Let M be a connected manifold and $f : M \to M$ a continuous map. Let u be a path in M with $f(u(1)) = u(0)$.

Definition 3. Two elements a and b of $\pi_1(M, u(1))$ are equivalent if there exists an element c of $\pi_1(M, u(1))$ such that $b = ca(u_* f_*(c))^{-1}$, where $u_* : \pi_1(M, u(0)) \to \pi_1(M, u(1))$ and $f_* : \pi_1(M, u(1)) \to \pi_1(M, u(0))$ are the homomorphisms induced by u and f respectively. Denote the set of equivalence classes by $R(f, u)$. An equivalence class is called a <u>Reidemeister class</u> (e.g. [5]).

Define a map $\phi_u : \mathrm{Fix}(f) \to R(f, u)$ by

$$\phi_u(x) = [h(f \circ h)^{-1} u],$$

where h is some path in M from $u(1)$ to x. This map is clearly well defined. For $\alpha \in R(f, u)$, set

$$\mathrm{Fix}_\alpha(f) = \phi_u^{-1}(\alpha).$$

Since M is an ENR, this is an isolated set of fixed points. If $\mathrm{Fix}_\alpha(f)$ is compact, we denote by $I_\alpha(f)$ the fixed point index of $\mathrm{Fix}_\alpha(f)$ for the map f. $I_\alpha(f)$ is a homotopy invariant in the sense that if $f_t : M \to M$ is a homotopy with $\bigcup_t \mathrm{Fix}(f_t)$ relatively compact then $I_\alpha(f)$ is independent of t for each α (Fadell and Husseini[5, Theorem (3.5)]).

For a positive integer m, let $u_m = (f^{m-1} \circ u) \ldots (f \circ u) u$ and denote $R(f^m, u_m)$ simply by $R(f^m)$. For $\alpha \in R(f^m)$ and $a,b =$ E,O, let $P_{ab}(m, \alpha)$ be the set of $x \in P_{ab}(m)$ with $\phi_{u_m}(x)$

$= \alpha$ and let $N_{ab}(m, \alpha) = \# P_{ab}(m, \alpha)$. For a positive integer d, define a map $\Phi_d : R(f^m) \to R(f^{dm})$ by

$$\Phi_d([a]) = [ag^m(a)g^{2m}(a)\dots g^{(d-1)m}(a)],$$

where $g = u_* f_* : \pi_1(M, u(1)) \to \pi_1(M, u(1))$. Then for $\alpha \in R(f^m)$, $\mathrm{Fix}_\alpha(f^m) \subset \mathrm{Fix}_{\Phi_d(\alpha)}(f^{dm})$, since $\phi_{u_{dm}}(x) = \Phi_d(\phi_{u_m}(x))$ for $x \in \mathrm{Fix}(f^m)$. For $\alpha \in R(f^m)$ with $\mathrm{Fix}_\alpha(f^m)$ compact, let

(5.1) $$I_m(f, \alpha) = \sum_{d|m} \mu(d) \sum_\beta I_\beta(f^{m/d}),$$

where the inner sum is taken over all elements β of $R(f^{m/d})$ with $\Phi_d(\beta) = \alpha$. Then we have an improvement of Theorem 1 (in case of $U = M$ and M connected):

Theorem 4. Let M be a connected manifold and $f : M \to M$ a C^1 map. Fix a path u in M with $f(u(1)) = u(0)$. Let m be a positive integer and α an element of $R(f^m)$. Suppose $\mathrm{Fix}_\alpha(f^m)$ consists of a finite number of simple fixed points contained in Int M. Then,

$$\begin{aligned} I_m(f, \alpha) &= N_E(m, \alpha) - N_O(m, \alpha) && \text{if } m \text{ is odd,} \\ &= N_E(m, \alpha) - N_O(m, \alpha) + 2 \sum_\beta (N_{OO}(m/2, \beta) - N_{EO}(m/2, \beta)) && \text{if } m \text{ is even,} \end{aligned}$$

where the sum is taken over all elements β of $R(f^{m/2})$ with $\Phi_2(\beta) = \alpha$.

The proof of this theorem is similar to that of Theorem 1 ([4, pages 431-432]) and is omitted.

In the case where f is a diffeomorphism of an annulus isotopic to the identity, this theorem becomes a result of Kawakami[10], which was applied to the study of periodic systems.

Now we give two examples of the computation of $I_m(f, \alpha)$.

Example 1. If the Euler characteristic of M is zero, $\pi_1(M)$ is abelian, and f is homotopic to the identity, then clearly $R(f^m) = \pi_1(M)$ and $I_m(f, \alpha) = 0$ for any m, α.

Example 2. We give an example where $\pi_1(M)$ is not abelian and f is not homotopic to the identity. Let M be a disk with n holes and $f : M \to M$ a continuous map. For simplicity, we assume that f has a fixed point x_0. Let u be the constant path at x_0. Identify the fundamental group of M with the free group F_n generated by n generators $x_1, \ldots, x_n$. Let B_n be the braid group on n strings, $\sigma_1, \ldots, \sigma_{n-1}$ the generators of B_n, and $\rho : B_n \to \mathrm{Aut}(F_n)$ the homomorphism defined by (cf. [1, (1.14)])

(5.2)
$$\begin{aligned} \rho(\sigma_i)(x_j) &= x_i x_{i+1} x_i^{-1} && \text{if } j = i, \\ &= x_i && \text{if } j = i+1, \\ &= x_j && \text{otherwise.} \end{aligned}$$

We assume that there exists $\sigma \in B_n$ such that $\rho(\sigma) : F_n \to F_n$ coincides with the homomorphism $f_* : F_n \to F_n$ induced by f. Let $Z[t]$ be the ring of integer polynomials on the variable t and its inverse. For $d \geqq 1$, define a homomorphism $\Psi_d : Z[t] \to Z[t]$ by $\Psi_d(t) = t^d$. For $\sigma \in B_n$, let $B(\sigma)$ be the reduced Burau matrix of σ [1, p. 125]. Then $\mathrm{tr}\, B(\sigma)$ is an element of $Z[t]$. Define a map $e : R(f) \to Z$ by $e(\alpha) = j_1 + \cdots + j_r$, where $\alpha =$

$[x_{i_1}^{j_1} \cdots x_{i_r}^{j_r}]$. Then we have

Proposition 2.

$$\sum_{i \in Z} \sum_{\alpha} I_m(f,\alpha) \cdot t^i = - \sum_{d|m} \mu(d)\, \Psi_d(\mathrm{tr}\, B(\sigma^{m/d})),$$

where the second sum is taken over all elements α of $R(f^m)$ with $e(\alpha) = i$.

Proof. Define an element $\hat{I}(f)$ of the free abelian group $Z[R(f)]$ generated by the set $R(f)$ by

$$\hat{I}(f) = \sum_{\alpha \in R(f)} I_\alpha(f) \cdot \alpha$$

and denote by $\hat{L}(f)$ the generalized Lefschetz number defined by Husseini[9]. Then since M is compact, $\hat{I}(f) = \hat{L}(f)$ [9, Theorem (1.13)]. Also by [6, Theorem 2.3],

$$\text{(5.3)} \qquad \hat{L}(f) = 1 - \sum_{i=1}^{n} \pi\, \{(\partial\rho(\sigma)(x_i))/\partial x_i\},$$

where $\partial/\partial x_i$ denotes the Fox derivative and $\pi : Z[F_n] \to Z[R(f)]$ is the homomorphism induced by the natural projection. Define a homomorphism $\Gamma : Z[R(f)] \to Z[t]$ by $\Gamma(\alpha) = t^{e(\alpha)}$. Then by (5.2),(5.3),

$$\sum_{i \in Z} \sum_{e(\alpha)=i} I_\alpha(f) \cdot t^i = \Gamma(\hat{I}(f)) = - \mathrm{tr}\, B(\sigma).$$

(For another proof of this equality, see [14, Proposition 2].) By this and (5.1), we can easily complete the proof. q.e.d.

For example, assume $n = 3$ and $\sigma = \sigma_1\sigma_2^{-1}$. Then,

$$\mathrm{tr}\, B(\sigma) = 1 - t - t^{-1}, \quad \mathrm{tr}\, B(\sigma^2) = 1 - 2(t + t^{-1}) + t^2 + t^{-2}.$$

Therefore, by Proposition 2,

$$e(\alpha) = \sum_i I_2(f,\alpha) = \begin{cases} 2 & \text{if } i = 1, -1, \\ -2 & \text{if } i = 2, -2, \\ 0 & \text{otherwise.} \end{cases}$$

The maps treated in this example appear naturally in the theory of periodic systems (see [14, Lemma 2]). Hence Theorem 4 and Proposition 2 can be applied to the study of such systems.

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