数理解析研究所講究録
第 636 巻 1987 年 1-15

Minimal genus Seifert surfaces for unknotting number 1 knots

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1. Introduction

Let K be a knot in the 3-sphere S^3, D a disk which intersects K in precisely two points, of opposite orientation, and $L = \partial D$. Then K_L denotes the knot in S^3 obtained as the image of K after doing + or −1 surgery to L. L (D resp.) is called a *crossing link* (*crossing disk* resp.). We say that K_L is obtained from K by a single *crossing change* along the crossing link L. Then the *unknotting number* $u(K)$ of K is the minimal number of times of crossing changes which are needed to transform K into the trivial knot.

We say that the oriented surface F $(\subset S^3)$ is obtained from two oriented surfaces F_1 and F_2 in S^3 by a *plumbing* if:

1) $F = F_1 \underset{R}{\cup} F_2$, where R is a rectangle with four edges A_1, B_1, A_2, B_2 such that A_i (B_i resp.) is an arc properly embedded in F_1 (F_2 resp.), and $\cup A_i \subset \partial F_2$, $\cup B_i \subset \partial F_1$,

2) There exist 3-balls D_1, D_2 in S^3 such that:

i) $D_1 \cup D_2 = S^3$, $D_1 \cap D_2 = \partial D_1 = \partial D_2 = S$ a 2-sphere,

ii) $F_1 \subset D_1$, $F_2 \subset D_2$ and $F_1 \cap S = F_2 \cap S = R$.

The *Hopf band* is a ±1 twisted unknotted annulus (Figure 1.1). In this paper we will show that there is a special kind of minimal genus Seifert surfaces for the unknotting number 1 knots.

Theorem. *Let* K *be an unknotting number 1 knot of genus*

$g(>0)$. Then there is a minimal genus Seifert surface T *for* K *such that* T *is obtained from a Hopf band and a genus $g-1$ surface by a plumbing along the disk* D *in Figure 1.1. Moreover, the crossing link for* K *is ambient isotopic in* S^3-K *to the image of* ℓ *in Figure 1.1.*

Remark. We note that the genus g-1 surface of Theorem has two boundary components.

As consequences of Theorem, we have:

Corollary 1. *The unknotting number 1 genus 1 knots are precisely non-trivial doubled knots ([7.112p.]). Moreover the crossing links are found in the obvious positions (Figure 1.2).*

Corollary 2(cf. [1,section 2]). *A non-trivial pretzel knot* K(2p+1,2q+1,2r+1) *has unknotting number 1 if and only if* {2p+1,2q+1,2r+1} *contains* either {1,1}, {-1,-1}, {3,-1}, or {-3,1}.

We note that K(2p+1,2q+1,2r+1) is a trivial kont if and only if (2p+1,2q+1,2r+1) contains {1,-1}. In section 4, by using Theorem, we show that the unknotting number of 9_{25} ([7]) is 2, so that, among the knots of ≤9 crossings, the unknotting numbers of 8_{10}, 8_{16}, 9_{10}, 9_{13}, 9_{32}, 9_{35}, 9_{38}, 9_{49} are still unknown.

I would like to express my hearty thanks to Yasutaka Nakanishi, and Hitoshi Murakami for useful and helpful conversations.

2. Proof of Theorem

Let D be the crossing disk, i.e. $\partial D = L$, $D \cap K = 2$ points. Let S be an orientable surface of minimal genus in $S^3 - \mathrm{Int}\, N(L)$ such that $\partial S = K$. Then, if necessary by moving S by an ambient isotopy of $S^3 - \mathrm{Int}\, N(K)$, we may suppose that $D \cap S$ consists of an arc joining the 2 points $D \cap K$. Let $P = \partial N(L)$. Then, by the proof of [2,Theorem], we see that $M = S^3 - \mathrm{Int}\, N(K \cup L)$ is S_P-atoroidal (see [3,Definition 1.6]). Then, by [3, Corollary 2.4] or [8,Theorem 5.1], we see that for all but at most one framing, the manifold N obtained from M by attaching a solid torus to P is irreducible and S remains norm minimizing in N.

We note that the image of S after the + or -1 surgery on ∂M is not a minimal genus Seifert surface for $K_L = O$. Hence if N' is obtained from M by the ∞ surgery, then S remains norm minimizing in $N' = S^3$. Hence the image of S after the ∞ surgery on P is a minimal genus Seifert surface for K.

Let S' be the image of S after the + or -1 surgery on P D' the image of the crossing disk D for L, and $a' = S' \cap D'$. Then a' is an arc properly embedded in S'. Since $\partial S'$ is the trivial knot, S' has a compressing disk d ,i.e. $d \cap S' = \partial d$ is an essential loop on S'. Since S is a minimal genus Seifert surface for K, we see that $\partial d \cap a' \neq \phi$. Let S'' be the component of the surface obtained from S' by doing the surgery along d such that $\partial S'' = K_L$. Then, by moving S'' by a tiny isotopy rel K_L, we may suppose that $\partial D'$ intersects S'' in two points, of opposite orientation. Moreover we may suppose that $D' \cap S''$ consists of two arcs. We get a genus g surface $\bar{S}$ from S by piping along a component of $\partial D'-S$. Then we deform $\bar{S}$ by an ambient isotopy as in Figure 2.1. Then Figure 2.1 (ii) shows that the conclusion of Theorem holds.

This completes the proof of Theorem.

3. Proof of Corollaries

Proof of Corollary 1. It is easy to see that every non-trivial doubled knot is an unknotting number 1 genus 1 knot. Let K be an unknotting number 1, genus 1 knot. Then, by Theorem, there is a minimal genus Seifert surface for K obtained by plumbing a genus 0 surface S' and a Hopf band. Since S' has two boundary components, S' is an annulus. Hence it is easily observed that the conclusion of Corollary 1 holds.

This completes the proof of Corollary 1.

Proof of Corollary 2. It is easily observed that $K(2p+1,2q+1,2r+1)$ has a genus 1 Seifert surface. Hence, by Corollary 1, $K(2p+1,2q+1,2r+1)$ has unknotting number 1 if and only if it is a doubled knot. Suppose that $K(2p+1,2q+1,2r+1)$ is a doubled knot. Since pretzel knots are simple ([5,Theorem II]), it is a twist knot([7,112p.]). We note that if $\{2p+1,2q+1,2r+1\} \supset \{1,1\}$, or $\{-1,-1\}$, then $K(2p+1,2q+1,2r+1)$ is a twist knot, and that every twist knot can be expressed in this form. Hence, by [5,Theorem I], $K(2p+1,2q+1,2r+1)$ is not a twist knot if $|2p+1|$, $|2q+1|$, $|2r+1| \geq 3$. Hence we need to consider the case when precisely one of $|2p+1|$, $|2q+1|$, $|2r+1|$ is equal to 1. Since $K(\alpha,\beta,\gamma) = K(\gamma,\alpha,\beta)$ and

$K(-\alpha,-\beta,-\gamma)$ is the mirror image of $K(\alpha,\beta,\gamma)$, we may suppose that $2q+1 = 1$. Then we deform the diagram of $K(2p+1,1,2r+1)$ as in Figure 3.1. The picture shows that $K(2p+1,1,2r+1)$ is a 2-bridge knot. Since $K(2p+1,1,2r+1)$ is a twist knot, by [4,Corollary], we see that $K(2p+1,1,2r+1)$ has a unique incompressible Seifert surface. On the other hand, by [4,Theorem 1(e)], we see that if $|2p+2| > 2$, $|2r+2| > 2$, then $K(2p+1,1,2r+1)$ has two mutually non-isotopic incompressible Seifert surface. Hence $|2p+2| = 2$ or $|2r+2| = 2$. Since $K(2p+1,1,2r+1)$ is a non trivial knot, we have $p = -2$ or $r = -2$, so that $2p+1 = -3$ or $2r+1 = -3$.

4. $u(9_{25}) = 2$.

Let 9_{25} be the knot as in Figure 4.1. In this section we show that the unknotting number of 9_{25} is 2. For the proof of the next lemma, see [6].

Lemma 4.1. *The minimal genus Seifert surfaces for* 9_{25} *are unique up to ambient isotopies of* S^3 *which fix* 9_{25}. *Moreover the surface* S *of Figure 4.1 is the minimal genus Siefert surface.*

Lemma 4.2. *Let* V *be a genus 2 handlebody, and* A $(\subset \partial V)$ *a union of three simple closed curves as in Figure.4.2. Let* D $(\subset V)$ *be a properly embedded disk such that* ∂D *intersects* A *in at most two points. Then* D *is boundary parallel.*

Proof. Assume that D is not boundary parallel. Let D_1, D_2, D_3 be disks properly embedded in V as in Figure 4.2. We suppose that the number of the components of $D\cap(D_1\cup D_2\cup D_3)$ is minimal among all the disks which are not parallel to ∂V. We note that $\partial(D_1\cup D_2\cup D_3)$ cuts ∂V into two pants P_1, P_2 such that for each pair of boundary components of P_i $(i=1,2)$ there are two subarcs of A properly embedded in P_i which joins the boundary components. By using standard cut and paste arguments, we may suppose that no component of $D\cap(D_1\cup D_2\cup D_3)$ is a simple closed curve.

Suppose that $D\cap(D_1\cup D_2\cup D_3) = \phi$. Then D is parallel to some D_i. Hence ∂D intersects A in at least four points, a contradiction.

Suppose that $D\cap(D_1\cup D_2\cup D_3) \neq \phi$. Then let Δ_1, Δ_2 $(\subset D)$ be innermost disks, i.e. $\Delta_i\cap\partial H = \alpha_i$ an arc, $\Delta_i\cap(D_1\cup D_2\cup D_3) = \beta_i$ an arc such that $\alpha_i\cup\beta_i = \partial\Delta_i$. By the minimality assumption on D, we see that α_i is an essential arc in P_1 or P_2. Since $\partial\alpha_i = \partial\beta_i$, $\partial\alpha_i$ is contained in a component of $\partial(D_1\cup D_2\cup D_3)$. Hence we see that each α_i intersects A in at least two points. Hence ∂D intersects A at least four points, a contradiction.

This completes the proof of Lemma 4.2.

Let V' be a genus 4 handlebody, ℓ $(\subset\partial V')$ a simple closed curve, D_1, D_2 disks properly embedded in V' as in Figure 4.3. Then the frontier of a regular neighborhood of $D_1\cup D_2\cup\ell$ consists of three annuli A_1, A_2, A_3 as in Figure 4.3. Let V_1 be the closure of the component of $V'-(A_1\cup A_2\cup A_3)$ which contains ℓ, and $V_2 = c\ell(V'-V_1)$. Then we have:

Lemma 4.3. *Let* D *be a disk properly embedded in* V' *such* *that* ∂D *intersects* ℓ *in two points. Then* D *is isotopic to a disk in* V_1 *by an ambient isotopy which preserves* ℓ.

Proof. It is easily observed that $(V_1;N(\ell)\cup A_1\cup A_2\cup A_3)$ is homeomorphic to $(P\times I,\partial P\times I)$, where P is a fourth punctured sphere, and $(V_2;A_1\cup A_2\cup A_3)$ is homeomorphic to $(V;N(A))$ in Lemma 4.2, where $N(.)$ denotes a regular neighborhood in ∂V_i. By cut and paste arguments, we may suppose that the number of the components of $D\cap(A_1\cup A_2\cup A_3)$ is minimal among all disks which are rel ℓ isotopic to D. Assume that $D\cap(A_1\cup A_2\cup A_3) \neq \phi$. Let Δ $(\subset D)$ be an innermost disk, i.e. $D\cap\partial V' = \alpha$ an arc, $D\cap(A_1\cup A_2\cup A_3) = \beta$ an arc such that $\alpha\cup\beta = \partial\Delta$.

Then we claim that $\Delta\cap\ell \neq \phi$. Assume that $\Delta\cap\ell = \phi$. Then Δ is properly embedded in V_1 or V_2. Suppose that $\Delta\subset V_1$. Since $\partial\Delta\cap P\times\{\varepsilon\} = \phi$ $(\varepsilon=0$ or $1)$, we see that $\partial\Delta$ is contractible in ∂V_1. Hence Δ is parallel to ∂V_1, and D can be moved by a rel ℓ isotopy to delete Δ, contradicting the minimality assumption. Suppose that $\Delta\subset V_2$. Then, by Lemma 4.2, D can be moved by a rel ℓ isotopy to delete Δ, a contradiction.

By the above claim, we see that D contains exactly two innermost disks, which intersect ℓ in 1 point. Hence each component of $D\cap V_2$ is a rectangle such that the two edges are contained in $\partial V'$ and the rest edges are contained in $A_1\cup A_2\cup A_3$. Then, by Lemma 4.2, we see that every component of $D\cap V_2$ can be pushed into V_1, a contradiction.

This completes the proof of Lemma 4.3.

Proof of "$u(9_{25}) = 2$" It is easily observed that 9_{25} is transformed into a tivial knot by using two croosing changes. Hence we show that $u(9_{25}) \neq 1$.

Assume that $u(9_{25}) = 1$. Let D be the crossing link for 9_{25} and E a handlebody obtained by thickening the minimal genus Seifert surface S (Figure 4.1), such that $9_{25} \subset \partial E$. Then let $N = c\ell(S^3 - E)$. Then, by Theorem, there is a disk Δ properly embedded in N, which intersects 9_{25} in two points (Figure 4.4). We move S by a rel 9_{25} ambient isotopy so that S is the closure of a component of $\partial N - 9_{25}$. Then we note that $\Delta \cap S$ is rel 9_{25} ambient isotopic to the arc $D \cap S$. It is directly observed that $(N, 9_{25})$ is homeomorphic to (V', ℓ) of Lemma 4.3, where Δ_1, Δ_2 in Figure 4.5 correspond to D_1, D_2 of Figure 4.3. Let γ_1, γ_2 be arcs properly embedded in S as in Figure 4.1 (ii). Let V_1' be the submanifold of N which corresponds to V_1 of Lemm 4.3. Then, by the above observation, we see that the subsurface $\lambda = V_1' \cap S$ of S is a regular neighborhood of $\partial S \cup \gamma_1 \cup \gamma_2$ in S. Hence, by Lemma 4.3, we may suppose that $\Delta \cap S = D \cap S$ is contined in λ. Figure 4.6 is a schematic picture of S obtained from Figure 4.1 (ii) by ignoring the twists. Then let a_1, a_2, b_1, b_2 be the oriented simple closed curves on S as in Figure 4.6, which represent a generator of $H_1(S;\mathbb{Z})$. Then a Seifert matrix of S is given by:

	a_1	a_2	b_1	b_2
a_1^+	-1	0	0	0
a_2^+	0	1	0	0
b_1^+	1	0	2	1
b_2^+	0	1	1	2

9_{25} is transformed into the trivial knot O by a crossing change along D. Then let S' be the Seifert surface for O which is the image of S, and α_1, α_2, β_1, β_2 ($\subset S'$) be the images of a_1, a_2, b_1, b_2 respectively. Since $D \cap S \subset \lambda$, we see that $\ell k(\alpha_i, \beta_j^+) = \ell k(a_i, b_j^+)$, $\ell k(\beta_i, \alpha_j^+) = \ell k(b_i, a_j^+)$, $\ell k(\beta_i, \beta_j^+) = \ell k(b_i, b_j^+)$. Hence a Seifert matrix of S' is:

$$A = \begin{array}{c} \\ \alpha_1^+ \\ \alpha_2^+ \\ \beta_1^+ \\ \beta_2^+ \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \end{array} \\ \begin{pmatrix} p & r & 0 & 0 \\ r & q & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

, where p,q,r are some integers. Then the Alexander polynomial of O is given by $\det(tA-{}^tA) = 3(pq-r^2)(t^4+1) - (12pq-12r^2+2p+2q+2r)(t^3+t) + (18pq-18r^2-4p-4q-4r+1)t^2$. Hence we have $pq-r^2 = 0$, and $12pq-12r^2+2p+2q+2r = 0$, so that $p = q = r = 0$. Since $p = 0$, we see that a_1 intersects $D \cap S$ algebraically once, and we do +1 surgery on ∂D to get O. On the other hand, $q = 0$ shows that we do -1 surgery on ∂D to get O, a contradiction.

Hence we have $u(9_{25}) = 2$.

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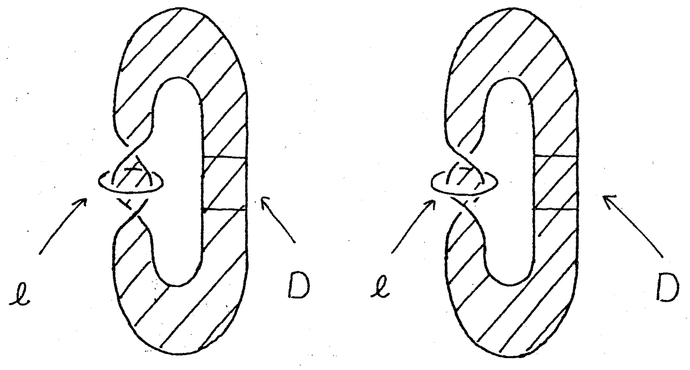

Figure 1.1

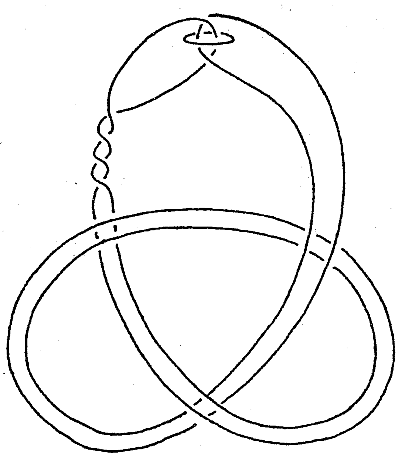

Figure 1.2

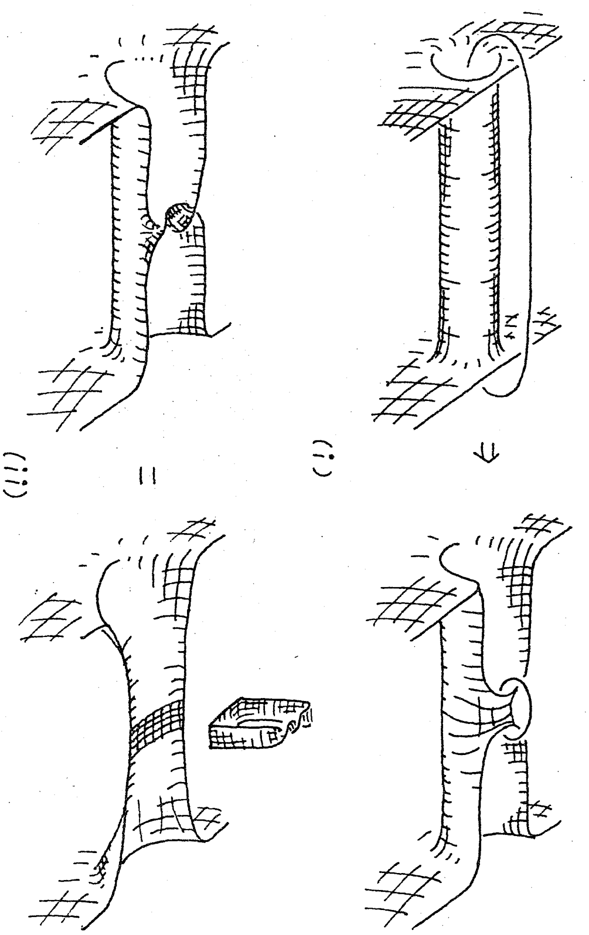

Figure 2.1

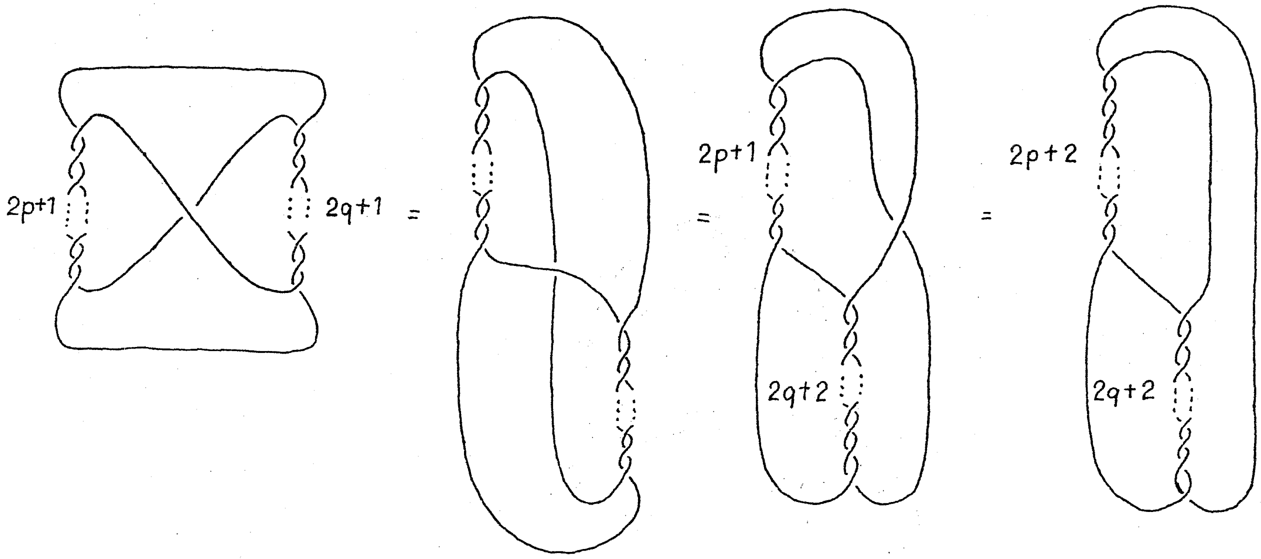

Figure 3.1

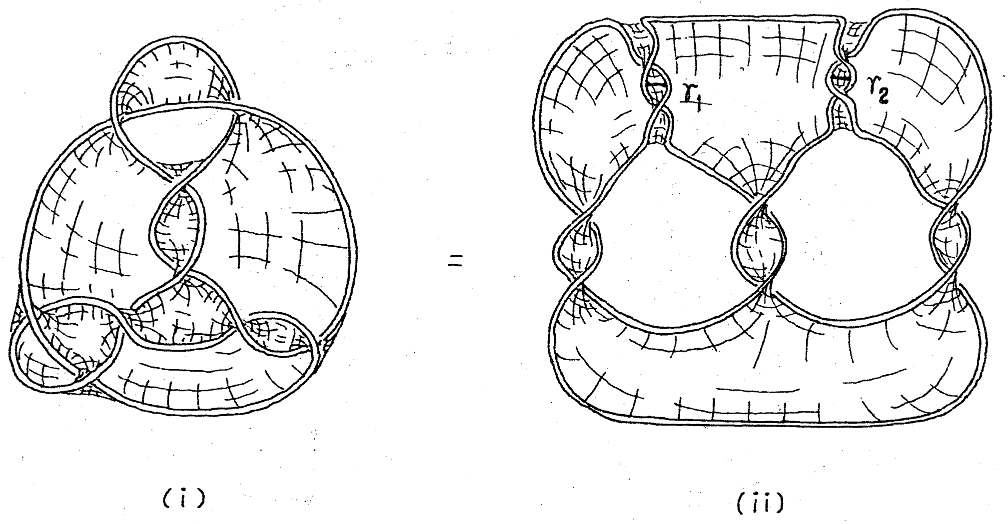

Figure 4.1

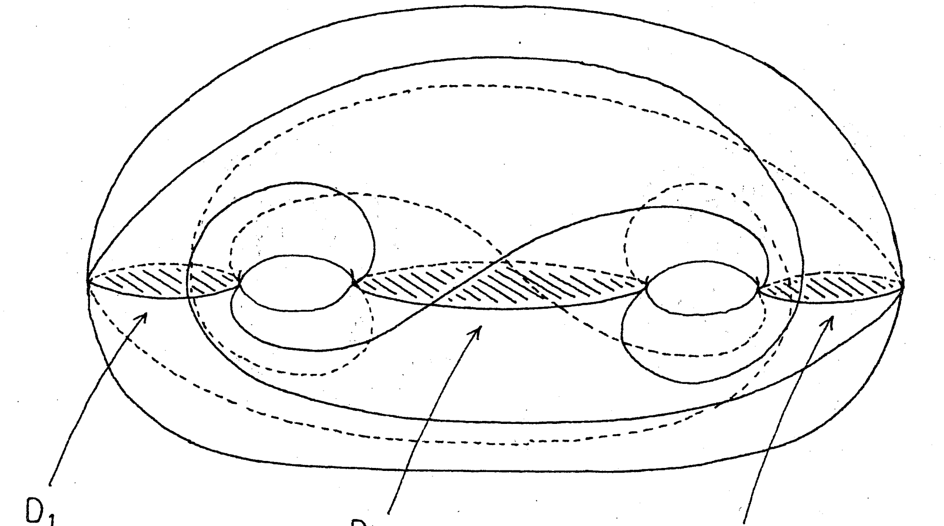

Figure 4.2

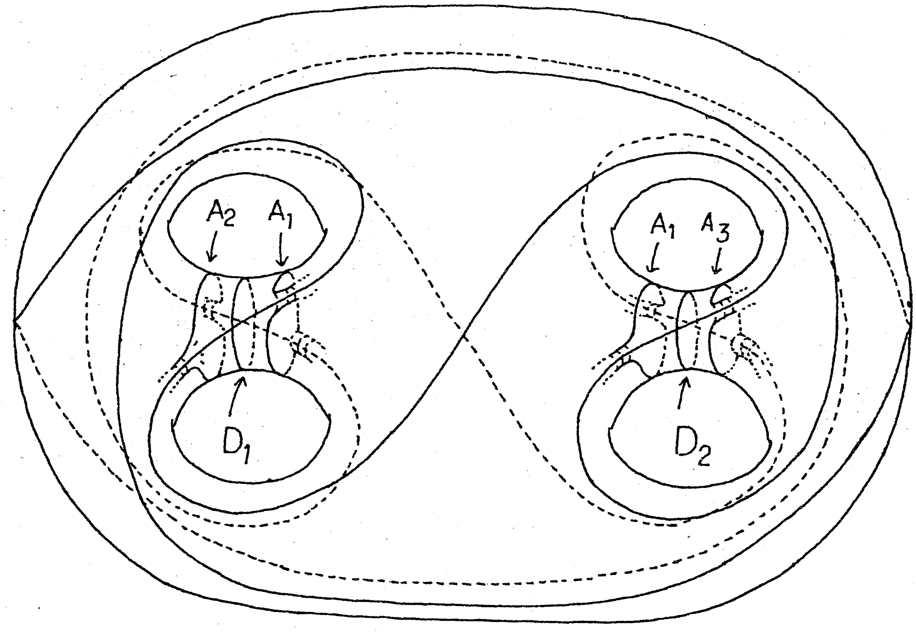

Figure 4.3

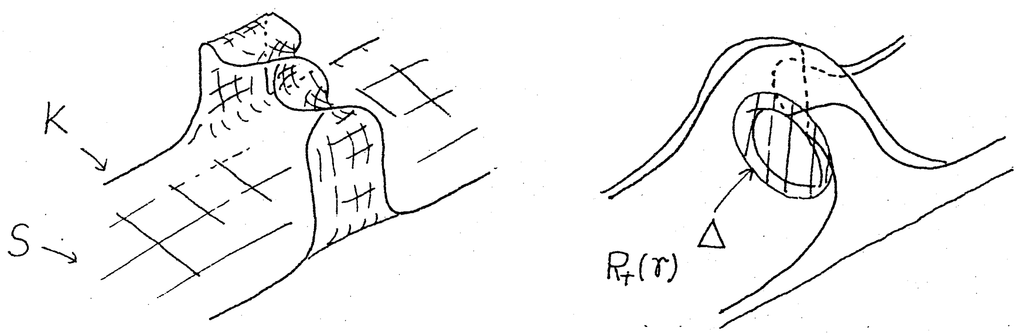

Figure 4.4

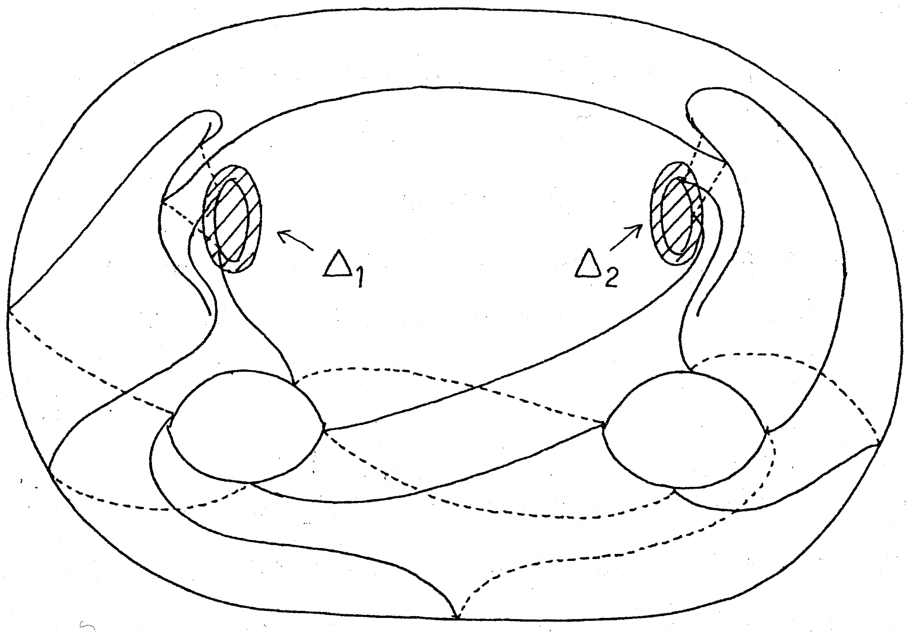

Figure 4.5

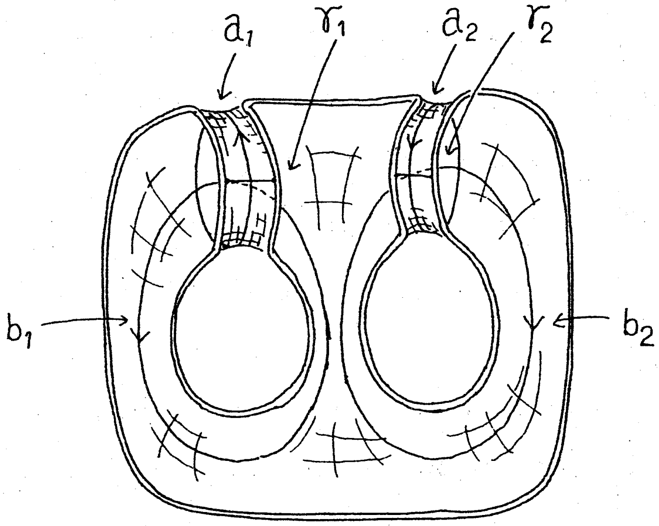

Figure 4.6