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HIGHER CODIMENSIONAL BOUNDARY VALUE PROBLEM

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Introduction.

Let M be an n-dimensional real analytic manifold and N be its d-codimensional closed real analytic submanifold. Let X and Y be complexifications of M and N respectively such that Y is a closed complex submanifold of X. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module (i.e. a system of linear partial differential equations with analytic coefficients) for which Y is non-characteristic. Our purpose is to define the boundary value of a hyperfunction solution of $\mathcal{M}$ defined on a 'wedge-like' domain with edge N. In fact, the boundary value of such a solution can be defined as a hyperfunction solution on N of the tangential system $\mathcal{M}_Y$ of $\mathcal{M}$ to Y. Moreover we can prove local and microlocal uniqueness theorem of the Holmgren type. By introducing the notion of F-mild hyperfunctions, we can also give an explicit meaning to the boundary value of a hyperfunction solution of $\mathcal{M}$ through its defining functions. We remark that Schapira [7] has recently defined the boundary value of a solution of $\mathcal{M}$ defined on a domain not necessarily wedge-like by using the theory of microlocalization of sheaves.

In Sections 1 and 2 we assume the existence of a 'partial complexification' $\tilde{M}$ of M. However, it will turns out in Section 3 that this assumption is unnecessary by resorting to the theory of higher-codimensional F-mild hyperfunctions (generalization to higher codimensional case of the 1-codimensional theory ([4])). This theory also enables us to express the boundary value explicitly.

§1. Several sheaves attached to the boundary.

Let M be an n-dimensional real analytic manifold and N be its d-codimensional closed real analytic submanifold. We denote by

$$\mathrm{Mon}_N(M) = (M \setminus N) \cup S_N M$$

the real monoidal transform of M with center N, here $S_N M$ is the sphere bundle of N in M. Then $\mathrm{Mon}_N(M)$ becomes a real analytic manifold with real analytic boundary $S_N M$ (see [6, Chapter 1] for real monoidal transform). Roughly speaking, $S_N M$ in $\mathrm{Mon}_N(M)$ will represents the direction along which the boundary value is taken. We denote by $\mathcal{B}_M$ the sheaf of hyperfunctions on M and by $\iota : M\setminus N \longrightarrow \mathrm{Mon}_N(M)$ the inclusion map. Then we put

Definition 1.1.

$$\mathcal{B}_{N,M} := \iota_*(\mathcal{B}_M|_{M\setminus N}), \qquad \mathcal{B}_{N|M} := \mathcal{B}_{N,M}|_{S_N M} .$$

In the sequel we shall define a sheaf $\tilde{\mathcal{B}}_{N|M}$ on $S_N M$ and an injective sheaf homomorphism $\alpha : \mathcal{B}_{N|M} \longrightarrow \tilde{\mathcal{B}}_{N|M}$. For this purpose we assume, for the moment, that there exists a real analytic submanifold $\tilde{M}$ of M containing N such that the triplet $(N, M, \tilde{M})$ are locally isomorphic (by a complex analytic local coordinate system $z = (z_1,\cdots,z_n)$ of X) to the triplet $(\{0\}\times R^{n-d}, R^n, R^d\times C^{n-d})$. We call such a local coordinate system <u>admissible</u>. Hence $\tilde{M}$ is a 'partial complexification' of M. Then we can take a complexification Y of N so that it is a closed real analytic submanifold of $\tilde{M}$ (then $Y \cap \tilde{M} = N$ holds). Let us consider the real monoidal transform $\mathrm{Mon}_Y(\tilde{M}) = (\tilde{M}\setminus Y) \cup S_Y\tilde{M}$ and denote by $\tilde{\imath} : \tilde{M} \setminus Y \longrightarrow \mathrm{Mon}_Y(\tilde{M})$ the inclusion map. There is a sheaf $\mathcal{B}\mathcal{O}$ of hyperfunctions with holomorphic parameters on $\tilde{M}$. Note that $\mathrm{Mon}_Y(\tilde{M})$ (resp. $S_Y\tilde{M}$) can be viewed as a partial complexification of $\mathrm{Mon}_N(M)$ (resp. S_NM). In the same way as Definition 1 we define

Definition 1.2.

$$\mathcal{B}\mathcal{O}_{Y,\tilde{M}} := \tilde{\imath}_*(\mathcal{B}\mathcal{O}|_{\tilde{M}\setminus Y}), \qquad \mathcal{B}\mathcal{O}_{Y|\tilde{M}} := \mathcal{B}\mathcal{O}_{Y,\tilde{M}}|_{S_Y\tilde{M}} .$$

Lemma 1.1. $\quad R^{\nu}\tilde{\imath}_*(\mathcal{B}\mathcal{O}|_{\tilde{M}\setminus Y}) = 0 \quad \text{for} \quad \nu \neq 0.$

Lemma 1.2. Suppose $X = C^n$, $\tilde{M} = R^d \times C^{n-d}$, $M = R^n$, $Y = \{0\} \times C^{n-d}$. *Then for any open proper convex set* U *of* S^{d-1} *and for any Stein open set* Ω *of* C^{n-d},

$$H^{\nu}(U\times\Omega;\ \mathcal{B}\mathcal{O}_{Y|\tilde{M}}) = 0 \quad \text{for any} \quad \nu \neq 0,$$

where $S^{d-1} \times C^{n-d}$ *is identified with* $S_Y\tilde{M}$.

Lemma 1.3. Under the same conditions as Lemma 1.1

$$\mathcal{H}^{\nu}_{S^{d-1}\times\,(R^{n-d}+\sqrt{-1}G)}(\mathcal{B}\mathcal{O}_{Y|\tilde{M}})\big|_{S^{d-1}\times\, R^{n-d}} = 0$$

for any proper convex closed cone G *of* R^{n-d} *and* $\nu \neq n-d$.

Lemma 1.4. There is a sheaf isomorphism

$$\mathcal{B}_{N,M} \cong \mathcal{H}^{n-d}_{\mathrm{Mon}_N(M)}(\mathcal{B}\mathcal{O}_{Y,\tilde{M}})\otimes\omega_{M/\tilde{M}},$$

where $\omega_{M/\tilde{M}}$ *denotes the sheaf of relative orientation of* M *in* $\tilde{M}$.

Now we define a sheaf $\tilde{\mathcal{B}}_{N|M}$ as follows:

Definition 1.3. $\quad \tilde{\mathcal{B}}_{N|M} := \mathcal{H}^{n-d}_{S_NM}(\mathcal{B}\mathcal{O}_{Y|\tilde{M}})\otimes\omega_N,$

where ω_N denotes the sheaf of orientation of N.

By definition it is easy to see that there is a natural sheaf homomorphism $\alpha : \mathcal{B}_{N|M} \longrightarrow \tilde{\mathcal{B}}_{N|M}$. By Lemma 1.3 and the fact that the flabby dimension of $\mathcal{B}\mathcal{O}_{Y|\tilde{M}}$ is n-d, it follows that $\tilde{\mathcal{B}}_{N|M}$ is a flabby sheaf on $S_N M$.

Now let us microlocalize these sheaves. For the sake of simplicity of the notation we put

$$L = \mathrm{Mon}_N(M), \quad \tilde{L} = \mathrm{Mon}_Y(\tilde{M}), \quad L_0 = S_N M, \quad \tilde{L}_0 = S_Y\tilde{M}$$

and denote by

$$\mathrm{Mon}^*_L(\tilde{L}) = (\tilde{L}\setminus L) \cup S^*_L\tilde{L}, \qquad \mathrm{Mon}^*_{L_0}(\tilde{L}_0) = (\tilde{L}_0\setminus L_0) \cup S^*_{L_0}\tilde{L}_0$$

the comonoidal transforms of $\tilde{L}$ (resp. $\tilde{L}_0$) with center L (resp. L_0), here $S^*_L\tilde{L}$ denotes the conormal sphere bundle of L in $\tilde{L}$ (cf. [6, Chapter 1]). Let

$$\pi' : \mathrm{Mon}^*_L(\tilde{L}) \longrightarrow \tilde{L} \quad \text{and} \quad \pi_0 : \mathrm{Mon}^*_{L_0}(\tilde{L}_0) \longrightarrow \tilde{L}_0$$

be the canonical projections.

Definition 1.4.

$$\mathcal{C}_{N,M} := \mathcal{H}^{n-d}_{S^*_L\tilde{L}}(\pi'^{-1}\mathcal{B}\mathcal{O}_{Y,\tilde{M}})\otimes\omega_{M|\tilde{M}}, \quad \mathcal{C}_{N|M} := \mathcal{C}_{N,M}|_{\pi'^{-1}(L_0)},$$

$$\tilde{\mathcal{C}}_{N|M} := \mathcal{H}^{n-d}_{S^*_{L_0}\tilde{L}_0}(\pi_0^{-1}\mathcal{B}\mathcal{O}_{Y|\tilde{M}})\otimes\omega_N.$$

Note that we can identify $S^*_{L_0}\tilde{L}_0$ with the subset $\pi'^{-1}(L_0)$ of $S^*_L\tilde{L}$ by the map $T^*\tilde{L} \underset{\tilde{L}}{\times} \tilde{L}_0 \longrightarrow T^*\tilde{L}_0$. Now let

$$\tau' : \mathrm{Mon}_L(\tilde{L}) \longrightarrow \tilde{L}, \qquad \tau_0 : \mathrm{Mon}_{L_0}(\tilde{L}_0) \longrightarrow \tilde{L}_0$$

be canonical projections associated with real monoidal transforms.

Definition 1.5. $\tilde{\mathcal{A}}_{N,M} := (\varepsilon'_*(\mathcal{B}\mathcal{O}_{Y,\tilde{M}}|_{\tilde{L}\setminus L}))|_{S_L\tilde{L}}$,

$$\tilde{\mathcal{A}}_{N|M} := ((\varepsilon_0)_*(\mathcal{B}\mathcal{O}_{Y|\tilde{M}}|_{\tilde{L}_0\setminus L_0}))|_{S_{L_0}\tilde{L}_0},$$

where $\varepsilon': \tilde{L}\setminus L \longrightarrow \mathrm{Mon}_L(\tilde{L})$ and $\varepsilon_0: \tilde{L}_0\setminus L_0 \longrightarrow \mathrm{Mon}_{L_0}(\tilde{L}_0)$ are inclusion maps.

Note that $S_{L_0}\tilde{L}_0$ can be regarded as a subset of $S_L\tilde{L}$. By the same arguments as Sato-Kawai-Kashiwara [6, Chapter I] or Morimoto [3], we can show that there exist injective sheaf homomorphisms (boundary value maps)

$$b': \tilde{\mathcal{A}}_{N,M} \longrightarrow \tau'^{-1}\mathcal{B}_{N,M}, \qquad \tilde{b}_0 : \tilde{\mathcal{A}}_{N|M} \longrightarrow \tau_0^{-1}\tilde{\mathcal{B}}_{N|M}$$

and surjective sheaf homomorphisms (spectral maps)

$$sp': \pi'^{-1}\mathcal{B}_{N,M} \longrightarrow \mathcal{C}_{N,M}, \qquad sp_0 : \pi_0^{-1}\mathcal{B}_{N|M} \longrightarrow \tilde{\mathcal{C}}_{N|M}.$$

Note that the usual spectral map is denoted by $sp : \pi^{-1}\mathcal{B}_M \longrightarrow \mathcal{C}_M$, where $\pi : S_M^*X \longrightarrow M$ is the canonical projection.

Proposition 1.1. *(i) There exists an exact sequence*

$$0 \longrightarrow \mathcal{B}\mathcal{O}_{Y,\tilde{M}}|_L \longrightarrow \mathcal{B}_{N,M} \longrightarrow \pi'_*\mathcal{C}_{N,M} \longrightarrow 0.$$

(ii) Let U be an open convex subset of $S_L\tilde{L}$ (i.e. each fiber of U with respect to τ' is a convex set of S^{n-d-1}) and let f be a section of $\mathcal{B}_{N,M}$ over $\tau'(U)$. Then there exists a section F of $\tilde{\mathcal{A}}_{N,M}$ over U such that $b'(F) = f$ if and only if $\mathrm{supp}(sp'(f)) \subset U^\circ$, where U° is the polar set of U.

Proposition 1.2. *(i) There exists an exact sequence*

$$0 \longrightarrow \mathcal{B}\mathcal{O}_{Y|\tilde{M}}|_{L_0} \longrightarrow \tilde{\mathcal{B}}_{N|M} \longrightarrow (\pi_0)_*\tilde{\mathcal{C}}_{N|M} \longrightarrow 0.$$

(ii) Let U be an open convex subset of $S_{L_0}\tilde{L}_0$ (i.e. each fiber

of U with respect to τ_0 is convex) and f be a section of $\tilde{\mathcal{B}}_{N|M}$ over $\tau_0(U)$. Then there exists a section F of $\tilde{\mathcal{A}}_{N|M}$ over U such that $b_0(F) = f$ if and only if $supp(sp_0(f)) \subset U^{\circ}$.

There is a sheaf homomorphism $\alpha : \mathcal{C}_{N|M} \longrightarrow \tilde{\mathcal{C}}_{N|M}$ compatible with the homomorphism $\alpha : \mathcal{B}_{N|M} \longrightarrow \tilde{\mathcal{B}}_{N|M}$, i.e. there is a commutative diagram

$$\begin{array}{ccc} \pi_0^{-1}\mathcal{B}_{N|M} & \xrightarrow{sp'} & \mathcal{C}_{N|M} \\ \downarrow \alpha & & \downarrow \alpha \\ \pi_0^{-1}\tilde{\mathcal{B}}_{N|M} & \xrightarrow{sp_0} & \tilde{\mathcal{C}}_{N|M} \end{array} .$$

Theorem 1.1. $\alpha : \mathcal{B}_{N|M} \longrightarrow \tilde{\mathcal{B}}_{N|M}$ and $\alpha : \mathcal{C}_{N|M} \longrightarrow \tilde{\mathcal{C}}_{N|M}$ are injective homomorphisms.

Proof. We can prove this theorem in the same way as the case $d = 1$ (see [5, §1]) by using the curvilinear wave expansion for hyperfunctions with holomorphic parameters.

§2. Formulation of boundary value problem.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module (i.e. a system of linear partial differential equations) defined on an neighborhood of N in X. Here $\mathcal{D}_X$ denotes the sheaf on X of rings of linear partial differential operators (of finite order) with holomorphic coefficients. We assume that Y is non-characteristic with respect to $\mathcal{M}$, i.e. the characteristic variety $SS(\mathcal{M}) \subset T^*X$ of $\mathcal{M}$ does not intersect $\mathring{T}^*_Y X = T^*_Y X \setminus 0$. Then the tangential system $\mathcal{M}_Y$ of $\mathcal{M}$ to Y is a coherent $\mathcal{D}_Y$-module (cf. [1]). We denote by $\tau : L_0 = S_N M \longrightarrow N$, $\tau^* : S^*_{L_0}\tilde{L}_0 \longrightarrow S^*_N Y$, and $\pi^* : S^*_{L_0}\tilde{L}_0 \longrightarrow L_0$ the canonical projections.

Proposition 2.1. Assume that Y is non-characteristic with respect to $\mathcal{M}$. Then there are sheaf isomorphisms

$$\tilde{\gamma} : R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M}) \xrightarrow{\sim} \tau^{-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N),$$

$$\tilde{\gamma} : R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) \xrightarrow{\sim} \tau^{*-1}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$

Proof. We shall prove only the first isomorphism (the second one can be proved in the same way as the first one). We denote by $\tilde{\tau}$: $\tilde{L}_0 = S_Y\tilde{M} \longrightarrow Y$ the canonical projection (note that τ is its restriction to S_NM). Let us first prove the isomorphism

(1) $$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\tau}^{-1}(\mathcal{B}\mathcal{O}|_Y)) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}}).$$

Define a sheaf $\mathcal{F}$ on $\tilde{L}_0$ by the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \tilde{\tau}^{-1}(\mathcal{B}\mathcal{O}|_Y) \longrightarrow \mathcal{B}\mathcal{O}_{Y|\tilde{M}} \longrightarrow 0.$$

Then it suffices to prove $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}) = 0$. We may assume that $X = C^n$, $\tilde{M} = R^d \times C^{n-d}$, $M = R^n$, $N = \{0\} \times R^{n-d}$. Put $x^* = (1,0,\cdots,0;0) \in S^{d-1} \times C^{n-d} \cong S_Y\tilde{M}$. By definition we have

$$\mathcal{F}_{x^*} = \varinjlim_{\varepsilon} \Gamma_{U_\varepsilon \setminus \Gamma_\varepsilon}(U_\varepsilon; \mathcal{B}\mathcal{O})$$

with $U_\varepsilon = \{(x',z") \in \tilde{M};\ |x'| < \varepsilon,\ |z"| < \varepsilon\}$ and $\Gamma_\varepsilon = \{(x',z") \in \tilde{M};\ \varepsilon x_1 > |x_2| + \cdots + |x_d|\}$; here we use the notation $x' = (x_1,\cdots x_d)$, $z" = (z_{d+1},\cdots,z_n)$. Since $\mathcal{M}$ is coherent we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F})_{x^*} = R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \varinjlim_{\varepsilon} \Gamma_{U_\varepsilon \setminus \Gamma_\varepsilon}(U_\varepsilon; \mathcal{B}\mathcal{O}))$$

$$= \varinjlim_{\varepsilon} R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{U_\varepsilon \setminus \Gamma_\varepsilon}(U_\varepsilon; \mathcal{B}\mathcal{O}))$$

$$= \varinjlim_{\varepsilon} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_{Z_\varepsilon}(\mathcal{B}\mathcal{O}))_0,$$

where $Z_\varepsilon = \tilde{M} \setminus \Gamma_\varepsilon$. By using Corollary 2.2.2 of [2] we can prove

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_{Z_\varepsilon}(\mathcal{B}\mathcal{O}))_0 = 0.$$

Hence we get the isomorphism (1). Since Y is non-characteristic with respect to $\mathcal{M}$, we have

(2) $$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}|_Y) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X|_Y),$$

where $\mathcal{O}_X$ denotes the sheaf on X of holomorphic functions. On the other hand, the Cauchy-Kowalevsky theorem due to Kashiwara (cf. [1]) gives the isomorphism

(3) $$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X|_Y) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Combining (1)-(3) we finally get an isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}\mathcal{O}_{Y|\tilde{M}}) \xrightarrow{\sim} \tilde{\tau}^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Applying the functor $R\Gamma_{S_N M}(\cdot)\otimes\omega_N$ we get the isomorphism

$$\tilde{\gamma} : R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M}) \xrightarrow{\sim} \tau^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_N).$$

Combining Proposition 2.1 and Theorem 1.1 we get

Theorem 2.1. Assume that Y is non-characteristic with respect to $\mathcal{M}$. Then there are injective homomorphisms

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M}) \longrightarrow \tau^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N),$$

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \longrightarrow \tau^{*-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$

The first homomorphism represents the boundary value (as a hyperfunction solution on N of $\mathcal{M}_Y$) of a hyperfunction solution of $\mathcal{M}$ defined on a wedge-like domain with edge N. This homomorphism means that the boundary value does not depend on the direction along which the boundary value is taken. Its injectivity means that the solution vanishes near N if the boundary value vanishes on N (Holmgren's type theorem).

Now let us clarify the meaning of the second homomorphism. First note that there is a natural map

$$p : S_M^*X \setminus S_{\tilde{M}}^*X \longrightarrow S_M^*\tilde{M}.$$

S_N^*Y is regarded as a subset of $S_M^*\tilde{M}$. We denote by $\pi : S_N^*Y \longrightarrow N$ the canonical projection. Since $S_L^*\tilde{L}|_{M\setminus N} = S_M^*\tilde{M}|_{M\setminus N} = S^*_{(M\setminus N)}(\tilde{M}\setminus Y)$, restricted to $S_L^*\tilde{L}|_{M\setminus N}$, $\mathcal{C}_{N,M}|_{M\setminus N}$ can be regarded as a sheaf on $S^*_{(M\setminus N)}(\tilde{M}\setminus Y)$. Then it is easy to see by definition that there is a sheaf homomorphism

$$\psi : p^{-1}(\mathcal{C}_{N,M}|_{S^*_{(M\setminus N)}(\tilde{M}\setminus Y)}) \longrightarrow \mathcal{C}_M|_{S^*_{(M\setminus N)}(\tilde{M}\setminus Y)\cap(S_M^*X\setminus S_{\tilde{M}}^*X)}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} p^{-1}(\mathcal{C}_{N,M}|_{S^*_{(M\setminus N)}(\tilde{M}\setminus Y)}) & \xrightarrow{\psi} & \mathcal{C}_M|_{S^*_{(M\setminus N)}(\tilde{M}\setminus Y)} \\ \downarrow sp' & & \downarrow sp \\ \mathcal{B}_{N,M}|_{M\setminus N} & \xrightarrow{=} & \mathcal{B}_M|_{M\setminus N} \end{array} .$$

Hence the second homomorphism of Theorem 2.2 implies the following microlocal version of Holmgren's type uniqueness theorem:

Theorem 2.2. Let x^* be a point of S_NM and U be an open set of $M \setminus N$ such that $U \cup S_NM$ is a neighborhood of x^* in $Mon_N(M)$. Let y^* be a point of S_N^*Y such that $\pi(y^*) = \tau(x^*)$. Let $u(x)$ be a hyperfunction solution of $\mathcal{M}$ defined on U such that the singular spectrum of its boundary value $\gamma(u)$ does not contain y^*. Then there exists an open set U' of $M\setminus N$ such that $U' \cap S_NM$ is a neighborhood of x^* in $Mon_N(M)$ and that the closure of the singular spectrum of $u|_{U'}$ does not intersect $p^{-1}(y^*)$.

§3. F-mild hyperfunctions.

In the previous sections we assumed that there exists a partial complexification $\tilde{M}$ of M. Now let us introduce the sheaf of F-mild hyperfunctions. By using F-mild hyperfunctons, we can show that the homomorphism Υ of boundary values introduced in Theorem 2.1 can be defined independently of $\tilde{M}$ (hence we don't have to assume the existence of $\tilde{M}$ in the long run). We think, however, that the advantage of F-mild hyperfunctions consists in making possible the explicit and concrete expression of boundary values.

We use the same notation as in the previous sections. In particular, we assume at present that there exists such an $\tilde{M}$ as described in §1. For an admissible local coordinate system z of X, we use the notation $z = (z',z'')$ with $z' = (z_1,\cdots,z_d)$, $z'' = (z_{d+1},\cdots,z_n)$, $z' = x' + \sqrt{-1}y'$, etc.

Definition 3.1. Let x^* be a point of S_NM and $z = (z',z'')$ be an admissible local coordinate system around $\mathring{x} = \tau(x^*)$ such that $z(\mathring{x}) = 0$. Let $u(x)$ be a germ of $\mathcal{B}_{N|M}$ at x^*. Then $u(x)$ is called <u>F-mild</u> at x^* if there exist open convex cones (with vertex at the origin) $\Gamma_1,\cdots,\Gamma_J$ in R^{n-d} with an integer J, and holomorphic functions $F_j(z)$ defined on a neighborhood in X of

$$\{(x',z'') \in R^d \times C^{n-d};\ (x',x'') \in \tau(U),\ |z''| < \varepsilon,\ \mathrm{Im}\, z'' \in \Gamma_j\}$$

with an $\varepsilon > 0$ and a neighborhood U of x^* in $\mathrm{Mon}_N(M)$ such that

$$u(x) = \sum_{j=1}^{J} F_j(x',x''+\sqrt{-1}\Gamma_j 0) \tag{4}$$

as a hyperfunction on $\tau(U) \cap (M\setminus N)$. We denote by $\mathcal{B}^F_{N|M}$ the subsheaf of $\mathcal{B}_{N|M}$ consisting of sections of $\mathcal{B}_{N|M}$ which are F-mild at each point of its defining domain.

On the other hand, since $S_N M$ is the 1-codimensional boundary of $Mon_N(M)$, there exists the subsheaf $\mathcal{B}^F_{S_N M|Mon_N(M)}$ of $\mathcal{B}_{N|M}$ (the sheaf of F-mild hyprfunctions in the 1-codimensional case introduced by Ôaku [4]). We denote $\mathcal{B}^F_{S_N M|Mon_N(M)}$ by $\mathcal{B}^F_1$ for short. Then it is known that $\mathcal{B}^F_1$ is invariant under local coordinate transformations of M fixing N ([4]).

Lemma 3.1. *$\mathcal{B}^F_{N|M}$ is a subsheaf of $\mathcal{B}^F_1$.*

This lemma follows immediately from the following well-known fact:

Sublemma. *Let $F(z)$ be a holomorphic function on a neighborhood in C^n of $\{z = (x_1, x_2, \cdots, x_{n-1}, x_n+\sqrt{-1}y_n);\ x_1, \cdots . x_n, y_n \in R,\ |x_j| < \varepsilon$ $(j = 1, \cdots, n),\ x_1 \geqq 0,\ 0 < y_n < \varepsilon\}$ with an $\varepsilon > 0$. Then there exists a $\delta > 0$ and an open convex cone Γ of R^{n-1} such that $(0, \cdot\cdot, 0, 1) \in \Gamma$ and that $F(z)$ is holomorphic on a neighborhood in C^n of $\{z = (x_1, z_2, \cdots, z_n) \in R \times C^{n-1};\ |z| < \delta,\ x_1 \geqq 0,\ Im\ z' \in \Gamma\}$.*

By using Lemma 3.1 and the invariance of $\mathcal{B}^F_1$ under local coordinate transformation we get the invariance of $\mathcal{B}^F_{N|M}$:

Proposition 3.1. *For a real analytic manifold M and its closed real analytic submanifold N, the sheaf $\mathcal{B}^F_{N|M}$ of F-mild hyperfunctions is defined and is invariant under real analytic local coordinate transformations of M fixing N.*

Proposition 3.2. *(Edge of the wedge theorem for $\mathcal{B}^F_{N|M}$). Let x^* be a point of $S_N M$ and U be a neighborhood of x^* in $Mon_N(M)$. Let Ω be a neighborhood in X of $\mathring{x} = \tau(x^*)$ and z be an admissible* *local coordinate system of* X *over* Ω. *Let* $F_j(z)$ *be a holomorphic function on a neighborhood in* C^n *of*

$$\{z = (x',z") \in \Omega \cap \tilde{M};\ (x',x") \in \tau(U),\ y" \in \Gamma_j\}$$

such that

$$\sum_{j=1}^{J} F_j(x',x"+\sqrt{-1}\Gamma_j 0) = 0$$

as a hyperfunction on $\tau(U) \cap (M \setminus N)$. *Then for any open convex cones* $\Gamma'_j \subset\subset \Gamma_j$, *there exist a neighborhood* $U' \subset U$ *of* x^* *in* $Mon_N(M)$, *a neighborhood* $\Omega' \subset \Omega$ *of* $\mathring{x}$ *in* X, *and holomorphic functions* $F_{jk}(z)$ *on a neighborhood of*

$$\{z = (x',z") \in \Omega' \cap \tilde{M};\ (x',x") \in \tau(U'),\ y" \in \Gamma'_j + \Gamma'_k\}$$

such that

$$F_j(z) = \sum_{k=1}^{J} F_{jk}(z), \quad F_{jk}(z) + F_{kj}(z) = 0 \quad (1 \leqq j,k \leqq n).$$

This proposition is proved by means of curvilinear wave decomposition of hyperfunctions with holomorphic parameters and the edge of the wedge theorem for F-mild hyperfucntions in the 1-codimensional case (cf. [4]). By virtue of this propositon we can define boundary values of F-mild hyperfunctions:

Proposition 3.3. *There exists a sheaf homomorphism*

$$b : \mathcal{B}^F_{N|M} \longrightarrow \tau^{-1}\mathcal{B}_N$$

such that $b(u) = \sum_{j=1}^{J} F_j(0,x"+\sqrt{-1}\Gamma_j 0)$ *if* $u \in \mathcal{B}^F_{N|M}$ *is written as (4). Moreover,* b *is independent of local coordinate system.*

Now we can prove that the $\mathcal{B}_{N|M}$-solutions of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ are F-mild if Y is non-characteristic with respect to $\mathcal{M}$. For this purpose we introduce a sheaf $\tilde{\mathcal{B}}^A$.

Definition 3.2. $\tilde{\mathcal{B}}^A := \mathcal{H}_N^{n-d}(\mathcal{O}_X|_Y)\otimes\omega_N.$

Lemma 3.2. There are injective homomorphisms

$$\tau^{-1}\tilde{\mathcal{B}}^A \longrightarrow \tilde{\mathcal{B}}_{N|M}, \qquad \mathcal{B}_{N|M}/\mathcal{B}^F_{N|M} \longrightarrow \tilde{\mathcal{B}}_{N|M}/\tau^{-1}\tilde{\mathcal{B}}^A.$$

By using this lemma we can prove

Proposition 3.4. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is non-characteristic. Then we have

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M}/\mathcal{B}^F_{N|M}) = 0$$

and in particular,

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^F_{N|M}) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M}).$$

By means of Propositons 3.1, 3.3, 3.4 we can finally prove the invariance of the local boundary value homomorphism γ:

Theorem 3.1. The sheaf homomorphism

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M}) \longrightarrow \tau^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_Y, \mathcal{B}_N)$$

coincides with the one

$$\gamma^F : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^F_{N|M}) \longrightarrow \tau^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_Y, \mathcal{B}_N)$$

induced naturally from $b : \mathcal{B}^F_{N|M} \longrightarrow \tau^{-1}\mathcal{B}_N$. Thus the above γ is defined independently of $\tilde{M}$ (hence without assuming the existence of $\tilde{M}$).

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要約

境界の余次元が２以上の場合に、線型偏微分方程式系に対する（局所的な）境界値問題の定式化をおこなう。Ｍを実解析的多様体、Ｎをその実解析的な閉部分多様体として、Ｘ，ＹをそれぞれＭ，Ｎの複素化とする。$\mathcal{M}$を連接的$\mathcal{D}_X$ー加群、すなわち解析的な係数を持つ線型偏微分方程式系とし、Ｙは$\mathcal{M}$に関して非特性的であると仮定する。このとき、$\mathcal{M}$のＹへの接方程式系$\mathcal{M}_Y$が定義される。

さて、Ｎを刃とするくさび状領域で定義された$\mathcal{M}$の佐藤超関数解に対してその境界値を、Ｎ上の$\mathcal{M}_Y$の佐藤超関数解として定義する。その際、境界に近づく方向を識別するために、Ｎを中心とするＭの実モノイダル変換を考えその境界 S_NM （ＮのＭにおける sphere bundle ）の上で考察する。特に、境界値はＮに近づくときの方向によらないことが示される。また、局所的及び超局所的な意味での Holmgren 型の一意性定理も合わせて示される。

上記の境界値を具体的に表示するために、Ｆーマイルド超関数の理論を展開する。これは余次元１の場合の理論（［４］）の拡張である。なお、最近 Schapira は層の超局所化の理論を用いて、境界値の抽象的な定義を与えている（［７］）。