

Microfunctions for boundary values
problems

Pierre SCHAPIRA (Univ. Paris-Nord)

1) Let X be a real manifold of class C^2 , and let $D^+(X)$ denote the derived category of complexes of sheaves on X with cohomology bounded from below. Let $\pi: T^*X \rightarrow X$ denote the cotangent bundle to X .

In [K-S] we associate to $F \in \text{Ob}(D^+(X))$ a closed subset $\text{SS}(F)$ of T^*X , the micro-support of F , and we also introduce the bifunctor μhom from $D^b(X)^0 \times D^+(X)$ to $D^+(T^*X)$ which generalizes the functor μ_M of Sato's microlocalization along M . One has:

$$(1.1) \quad \text{supp}(\mu\text{hom}(F,G)) \subset \text{SS}(F) \cap \text{SS}(G) .$$

2) Let Ω be an open subset of X , $\bar{\Omega}$ its closure, $\partial\Omega = \bar{\Omega} \setminus \Omega$. We say that Ω is lct in X if:

$$(2.1) \quad \forall x \in \partial\Omega, (R\Gamma_{\bar{\Omega}}(\mathbb{Z}_X))_x = 0, (R\Gamma_{\Omega}(\mathbb{Z}_X))_x = \mathbb{Z} .$$

Let $*$ denote the functor $R \underline{\text{Hom}}(\cdot, \mathbb{Z}_X)$. Then if Ω is lct, one has:

$$(2.2) \quad (\mathbb{Z}_{\Omega})^* = \mathbb{Z}_{\bar{\Omega}}, (\mathbb{Z}_{\bar{\Omega}})^* = \mathbb{Z}_{\Omega} .$$

Now let M be a closed submanifold of X . Assume:

$$(2.3) \quad \Omega \text{ is lct in } X \text{ and } \bar{\Omega} \supset M .$$

Then the morphism $\mathbb{Z}_{\bar{\Omega}} \rightarrow \mathbb{Z}_M$ defines by duality:

$$(2.4) \quad b: \mathbb{Z}_M \rightarrow \mathbb{Z}_{\bar{\Omega}} \otimes \omega_{M/X}[d]$$

where $\omega_{M/X}$ is the relative orientation sheaf and d is the codimension of M . We call b the "boundary value" morphism.

- 3) Assume now M is a real analytic manifold of dimension n and X is the complexification of M . Applying the functor $\mu\text{hom}(\cdot, \mathcal{O}_X)$ to (2.4) we get:

Proposition 3.1 Let f be a holomorphic function on Ω . Then $b(f)$ is a well-defined hyperfunction on M , and

$$\text{SS}(b(f)) \subset T_M^*X \cap \text{SS}(\mathbb{Z}_{\bar{\Omega}}).$$

- 4) Let ω be an open subset of M , j the inclusion map $\omega \rightarrow X$. We set:

$$(4.1) \quad C_{\omega|X} = \mu\text{hom}(\mathbb{Z}_{\bar{\omega}}, \mathcal{O}_X) \otimes \omega_{M/X}[n]$$

There is a well-defined morphism:

$$(4.2) \quad \alpha: j_* j^{-1} B_M \rightarrow \pi_* H^0(C_{\omega|X})$$

If M is a coherent \mathcal{D}_X -module, there is a similarly defined morphism:

$$(4.3) \quad \alpha: \text{Ext}_{\mathcal{D}_X}^j(M, j_*j^{-1}B_M) \rightarrow \pi_*H^j(\underline{\text{RHom}}_{\mathcal{D}_X}(M, C_{\omega|X}))$$

Definition 4.1 Let u be a section of $j_*j^{-1}B_M$
 (resp. of $\text{Ext}_{\mathcal{D}_X}^j(M, j_*j^{-1}B_M)$). One sets: $\text{SS}_{\omega}(u) = \text{supp}(\alpha(u))$
 (resp.: $\text{SS}_{\omega}^{M,j}(u) = \text{supp}(\alpha(u))$).

There are similar definitions for microfunctions (using
 the morphism $C_M \rightarrow C_{\omega|X}$).

5) Let N be an analytic submanifold of M , Y the complexification
 of N in X .

Assume:

(5.1) Y is non characteristic for M

(5.2) ω is lct in M and $\bar{\omega} \supset N$.

Using (2.4) one easily obtains a morphism

$$(5.3) \quad b: \underline{\text{RHom}}_{\mathcal{D}_X}(M, j_*j^{-1}B_M) \rightarrow \underline{\text{RHom}}_{\mathcal{D}_Y}(M_Y, B_N)$$

This morphism extends the construction of [Sl] and [Ko],
 (cf. also [\hat{O}]).

Theorem 5.1 Let $u \in \text{Ext}_{\mathcal{D}_X}^j(M, B_M(\omega))$. Then:

i) $\text{SS}_N^{M_Y,j}(b(u)) \subset \mathcal{I}_{\bar{\omega}}^{-1} \text{SS}_{\omega}^{M,j}(u)$

ii) If $j = 0$, $\text{codim}_M N = 1$, this inclusion is an equality.

Here f and $\bar{\omega}$ denote as usual the map from $Y \times T^*X$ to T^*Y and T^*X , respectively. Remark that if $(C_{\omega|X})_{T^*_M X}$ is in degree 0 (e.g.: $\omega = M$ or else $\partial\omega$ is an analytic hypersurface). Then $SS_{\omega}^{M,0}(u) \cap T^*_M X = SS_{\omega}(u)$. In [S-Z] we study a notion of ω -regularity for M which extends that of [S2] and [Ka] and which ensures that

$$SS_{\omega}(u) = \overline{SS(u|_{\omega})}.$$

The results of this paper are detailed in [S3].

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