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A uniqueness set for the linear partial differential operators of the second order

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dを2以上の自然数とする。$W \subset \mathbb{C}^{d+1}$に対して$\mathcal{O}(W)$でW上の正則函数のなす空間を表わす．任意の$\lambda \in \mathbb{C}$に対して$\mathcal{O}_\lambda(W) = \{ f \in \mathcal{O}(W) ; (\Delta_z + \lambda^2) f = 0 \}$，$\Delta_z = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial z_j^2}$と定義する．$M = \{ z \in \mathbb{C}^{d+1} ; \sum_{j=1}^{d+1} z_j^2 = 0 \}$とおく．[13]では制限写像$\alpha_\lambda : f \to f|_M$が$\mathcal{O}_\lambda(\mathbb{C}^{d+1})$から$\mathcal{O}(\mathbb{C}^{d+1})|_M$への全単射であることを示したが，$\mathcal{O}_\lambda(\mathbb{C}^{d+1})$の元はM上の値のみで一意的に決定できるという意味でMを微分作用素$\Delta_z + \lambda^2$のuniqueness setと呼ぶことにする．

ここでは$\Delta_z + \lambda^2$を一般の定数係数2階偏微分作用素にかえても類似の現象が起こることを示す（定理4.1）．また$\widetilde{B}(r)$を$\mathbb{C}^{d+1}$上の半径rのLie球とすると前述の制限写像α_λは$\mathcal{O}_\lambda(\widetilde{B}(r))$から$\mathcal{O}(\widetilde{B}(r))|_M$への線型位相同型になり（定理2.1），さらに$\mathcal{O}_\lambda(\widetilde{B}(r))$は複素球面$\widetilde{S}$上の汎函数の空間のFourier-Borel変換$\mathcal{P}_\lambda$の像として表わされる（定理3.1）．

Introduction.

Let d be a positive integer and $d \geqslant 2$. Let us denote by $\mathcal{O}(W)$ the space of holomorphic functions on $W \subset C^{d+1}$. For any $\lambda \in C$ we define $\mathcal{O}_\lambda(W) = \{f \in \mathcal{O}(W);\ (\Delta_z + \lambda^2)f = 0\}$, where $\Delta_z = \sum_{j=1}^{d+1}(\partial/\partial z_j)^2$. Let $M = \{z \in C^{d+1};\ z_1^2 + z_2^2 + \cdots + z_{d+1}^2 = 0\}$. In [13] it was shown that the restriction mapping $\alpha_\lambda : f \longrightarrow f|_M$ is a one-to-one linear mapping of $\mathcal{O}_\lambda(C^{d+1})$ onto $\mathcal{O}(C^{d+1})|_M$. In this sense we call M a uniqueness set of the differential operator $\Delta_z + \lambda^2$.

Here we will show that the same kind of phenomenon holds more generally for a linear partial differential operator of the second order with constant coefficients (Theorem 4.1) and that α_λ is a linear topological isomorphism of $\mathcal{O}_\lambda(\widetilde{B}(r))$ onto $\mathcal{O}(\widetilde{B}(r))|_M$, where $\widetilde{B}(r)$ is the Lie ball of radius r (Theorem 2.1).

The Fourier-Borel transformation P_λ has been studied in [2], [6], [10], [12], [13] etc. We will determine tne inverse image of $\mathcal{O}_\lambda(\widetilde{B}(r))$ by the transformation P_λ (Theorem 3.1).

§ 1. Preliminaries.

Let d be a positive integer and assume $d \geqslant 2$. $S = \{x \in R^{d+1};\ \|x\| = 1\}$ denotes the unit sphere in R^{d+1}, where $\|x\|^2 = \sum_{j=1}^{d+1} x_j^2$. ds denotes the unique $O(d+1)$ invariant measure on S with $\int_S 1\, ds$

= 1, where $O(k)$ is the orthogonal group of degree k. $\| \|_\infty$ is the sup norm on S. $H_{n,d}$ is the space of spherical harmonics of degree n in dimension d + 1. For spherical **harmonics**, see Müller [11].

For $S_n \in H_{n,d}$ $\widetilde{S}_n$ denotes the unique homogeneous harmonic polynomial of degree n on C^{d+1} such that $\widetilde{S}_n|_S = S_n$.

The Lie **norm** $L(z)$ and dual Lie norm $L^*(z)$ on C^{d+1} are defined as follows:

$$L(z) = L(x + iy) = [\|x\|^2 + \|y\|^2 + 2\{\|x\|^2\|y\|^2 - (x\cdot y)^2\}^{\frac{1}{2}}]^{\frac{1}{2}},$$

$$L^*(z) = \sup\{|z\cdot\zeta| \,;\, L(\zeta) \leq 1\},$$

where $z, \zeta \in C^{d+1}$, and $z\cdot\zeta = z_1\zeta_1 + z_2\zeta_2 + \ldots + z_{d+1}\zeta_{d+1}$, $x, y \in R^{d+1}$ (see Drużkowski [1]). We put

$$\widetilde{B}(r) = \{z \in C^{d+1}; L(z) < r\} \quad \text{for } 0 < r \leq \infty$$

and

$$\widetilde{B}[r] = \{z \in C^{d+1}; L(z) \leq r\} \quad \text{for } 0 \leq r < \infty.$$

Let us denote by $\mathcal{O}(\widetilde{B}(r))$ the space of holomorphic functions on $\widetilde{B}(r)$. Then $\mathcal{O}(\widetilde{B}(r))$ is an FS space. $\mathcal{O}(\widetilde{B}(\infty)) = \mathcal{O}(C^{d+1})$ is the space of entire functions on C^{d+1}. Let us define

$$\mathcal{O}(\widetilde{B}[r]) = \operatorname*{ind\,lim}_{r' > r} \mathcal{O}(\widetilde{B}(r')).$$

Then $\mathcal{O}(\widetilde{B}[r])$ is a DFS space. For $\lambda \in C$, we put $\mathcal{O}_\lambda(\widetilde{B}(r)) = \{f \in \mathcal{O}(\widetilde{B}(r));\ (\Delta_z + \lambda^2)f = 0\}$ and $\mathcal{O}_\lambda(\widetilde{B}[r]) = \{f \in \mathcal{O}(\widetilde{B}[r]);\ (\Delta_z + \lambda^2)f = 0\}$, where $\Delta_z = (\partial/\partial z_1)^2 + (\partial/\partial z_2)^2 + \dots + (\partial/\partial z_{d+1})^2$. $P_n(C^{d+1})$ denotes the space of homogeneous polynomials of degree n on C^{d+1}.

For $r > 0$ we put

$$X_{r,L} = \{f \in \mathcal{O}(C^{d+1});\ \sup_{z \in C^{d+1}} |f(z)| \exp(-rL(z)) < \infty\}.$$

Then $X_{r,L}$ is a Banach space with respect to the norm

$$\|f\|_{r,L} = \sup_{z \in C^{d+1}} |f(z)| \exp(-rL(z)).$$

Define

$$\mathrm{Exp}(C^{d+1}: (r:L)) = \operatorname{proj\,lim}_{r' > r} X_{r',L} \qquad \text{for } 0 \leq r < \infty,$$

$$\mathrm{Exp}(C^{d+1}: [r:L]) = \operatorname{ind\,lim}_{r' < r} X_{r',L} \qquad \text{for } 0 < r \leq \infty.$$

$\mathrm{Exp}(C^{d+1}: (r:L))$ is an FS space and $\mathrm{Exp}(C^{d+1}: [r:L])$ is a DFS space. $\mathrm{Exp}(C^{d+1}) = \mathrm{Exp}(C^{d+1}: [\infty:L])$ is called the space of entire functions of exponential type. $\mathrm{Exp}'(C^{d+1}: (r:L))$ and $\mathrm{Exp}'(C^{d+1}: [r:L])$ denote the spaces dual to $\mathrm{Exp}(C^{d+1}: (r:L))$ and $\mathrm{Exp}(C^{d+1}: [r:L])$ respectively.

$\widetilde{S} = \{z \in C^{d+1};\ z_1^2 + z_2^2 + \dots + z_{d+1}^2 = 1\}$ is the complex sphere. We define $\operatorname{Exp}(\widetilde{S}: [r:L]) = \operatorname{Exp}(C^{d+1}: [r:L])|_{\widetilde{S}}$ and $\operatorname{Exp}(\widetilde{S}: (r:L)) = \operatorname{Exp}(C^{d+1}: (r:L))|_{\widetilde{S}}$. The topology of $\operatorname{Exp}(\widetilde{S}: [r:L])$ is defined to be the quotient topology $\operatorname{Exp}(C^{d+1}: [r:L])/I_{\exp[r:L]}(C^{d+1})$, where we put $I_{\exp[r:L]}(C^{d+1}) = \{f \in \operatorname{Exp}(C^{d+1}: [r:L]);\ f = 0 \text{ on } \widetilde{S}\}$. We also define the topology of $\operatorname{Exp}(\widetilde{S}: (r:L))$ similarly. $\operatorname{Exp}'(\widetilde{S}: [r:L])$ and $\operatorname{Exp}'(\widetilde{S}: (r:L))$ denote the spaces dual to $\operatorname{Exp}(\widetilde{S}: [r:L])$ and $\operatorname{Exp}(\widetilde{S}: (r:L))$ respectively.

If f is a function or a functional on S, we denote by $S_n(f;\ )$ the n-th spherical harmonic component of f:

$$(1.1)\qquad S_n(f; s) = N(n,d) \langle f, P_{n,d}(\cdot s)\rangle \quad \text{for } s \in S,$$

where

$$(1.2)\qquad N(n,d) = \dim H_{n,d} = \frac{(2n+d-1)(n+d-2)!}{n!\ (d-1)!}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and of dimension d + 1. We put $L_n(x) = \|x\|^n P_{n,d}(\alpha\cdot x/\|x\|)$ for fixed $\alpha \in S$. Then L_n is the unique homogeneous harmonic polynomial of degree n with the following properties:

$$L_n(Ax) = L_n(x) \quad \text{for all } A \in O(d+1) \text{ such that } A\alpha = \alpha.$$

$$L_n(\alpha) = 1.$$

We see that $S_n(f;\ )$ belongs to $H_{n,d}$ for $n = 0,1,\ \dots$.

Put $\Lambda_+ = \{(n,k) \in Z_+^2\ ;\ \ n \equiv k \pmod 2 \text{ and } n \geq k\}$, where $Z_+ = \{0,\ 1,\ 2,\ \dots\}$. For any $F \in \mathcal{O}(\widetilde{B}(r))$ we can determine uniquely $S_{n,k}(F;\ ) \in H_{k,d}$ for every $(n,k) \in \Lambda_+$ in such a way that

$$(1.3) \qquad F(z) = \sum_{(n,k)\in\Lambda_+} (\sqrt{z^2}\ )^{n-k}\, \widetilde{S}_{n,k}(F;\ z),$$

where $z^2 = z_1^2 + z_2^2 + \dots + z_{d+1}^2$, and the right hand side of (1.3) converges uniformly on every compact set of $\widetilde{B}(r)$. The $S_{n,k}(F;\ )$ is called the (n,k)-component of F (see [8] [9]).

Next we consider a complex cone M as follows:

$$M = \{z \in C^{d+1},\ z^2 = 0\}.$$

M is identified with the cotangent bundle on S minus its zero section. $P_n(M)$ denotes the restriction to M of $P_n(C^{d+1})$. We put the subset N of M as follows:

$$N = \{z = x + iy \in M;\ \|x\| = \|y\| = 1\},$$

where $x,y \in R^{d+1}$. The unit cotangent bundle to S is identified with N and we have $N \cong O(d+1)/O(d-1)$. dN denotes the unique $O(d+1)$-invariant measure on N with $\int_N 1\, dN(z) = 1$.

It is known that for any $f_n \in H_{n,d}$ and any $g_m \in H_{m,d}$

$$(1.4) \qquad \int_S f_n(s)\, \overline{g_m(s)}\, ds$$

$$= \frac{n!\, N(n,d)\, \Gamma((d+1)/2)}{2^{2n}\, \Gamma(n + \frac{d+1}{2})} \int_N \widetilde{f}_n(z)\, \overline{\widetilde{g}_m(z)}\, dN(z),$$

(see, for example [4][5][13]).

§2. A uniqueness set for the differential operator $\Delta_z + \lambda^2$.

Our main theorem in this section is the following

Theorem 2.1 ([14]). (i) The restriction mapping $F \longrightarrow F|_M$ defines the following bijections:

$$(2.1) \qquad \alpha_\lambda: \mathcal{O}_\lambda(\widetilde{B}(r)) \longrightarrow \mathcal{O}(\widetilde{B}(r))|_M \quad \text{for any } \lambda \in C.$$

(ii) If $f \in \mathcal{O}(\widetilde{B}(r))|_M$ then $\alpha_\lambda^{-1} f$ can be expressed as follows:

$$(2.2) \qquad \alpha_\lambda^{-1} f(z) = \int_N f(\rho z'/2)\, K_\lambda(z, \frac{\overline{z'}}{\rho})\, dN(z') \quad \text{for } z \in \widetilde{B}(r),$$

where $L(z) < \rho < r$ and

$$(2.3) \qquad K_\lambda(z, \zeta) = \sum_{n=0}^{\infty} \Big\{ N(n,d)\, \Gamma(n + \frac{d+1}{2})\, (\lambda\sqrt{z^2}/2)^{-n-\frac{d-1}{2}}$$

$$J_{n+\frac{d-1}{2}}(\lambda\sqrt{z^2})\, (z\cdot\zeta)^n \Big\}.$$

In particular, if $\lambda = 0$ we have a "Poisson" formula:

$$(2.4) \qquad \alpha_0^{-1} f(z) = \int_N f(\rho z'/2) \frac{(1 + \bar{z}' \cdot \frac{z}{\rho})}{(1 - \bar{z}' \cdot \frac{z}{\rho})^d} dN(z').$$

(iii) α_λ is a linear topological isomorphism of $\mathcal{O}_\lambda(\widetilde{B}(r))$ onto $\mathcal{O}(\widetilde{B}(r))|_M$ if we equip $\mathcal{O}(\widetilde{B}(r))|_M$ with the topology of uniform convergence on every compact set of $\widetilde{B}(r) \cap M$.

We need the following lemmas in order to prove the theorem.

Lemma 2.2. Let $F \in \mathcal{O}_\lambda(\widetilde{B}(r))$ and $S_{n,k}$ be the (n,k)-component of F. Then we have

$$(2.5) \qquad S_{n,k} = \frac{(i\lambda/2)^{n-k} \, \Gamma(k + \frac{d+1}{2})}{\Gamma(\frac{n-k}{2} + 1) \, \Gamma(\frac{n+k+d+1}{2})} S_{k,k}$$

for $(n,k) \in \Lambda_+$ and

$$(2.6) \qquad \limsup_{n \to \infty} \| S_{n,n} \|_\infty^{1/n} \leq 1/r.$$

Conversely if we are given a sequence of spherical harmonics $\{S_{n,k}\}_{(n,k) \in \Lambda_+}$ satisfying (2.5) and (2.6) and if we put for $z \in \widetilde{B}(r)$

$$(2.7) \qquad F(z) = \sum_{(n,k) \in \Lambda_+} (\sqrt{z^2})^{n-k} \, \widetilde{S}_{n,k}(z)$$

then the right hand side of (2.7) converges uniformly and absolutely on every compact set of $\widetilde{B}(r)$ and F belongs to $\mathcal{O}_\lambda(\widetilde{B}(r))$. Furthermore we have

$$\widetilde{S}_{n,k}(z) = \widehat{S}_{n,k}(F;\ z) \qquad \text{for } (n,k) \in \Lambda_+ .$$

Remark the case $\lambda = 0$ is known (see [9]).

<u>Sketch of the proof.</u> By [8] Theorem 3.2 we have

$$(2.8) \qquad \Delta_z F(z) = \sum_{(n,k)\in\Lambda_+} \Delta_z((\sqrt{z^2}\ )^{n-k}\, \widetilde{S}_{n,k}(z))$$

$$= \sum_{\substack{(n,k)\in\Lambda_+ \\ n>k}} (n-k)(n+k+d-1)(\sqrt{z^2}\ )^{n-k-2}\, \widetilde{S}_{n,k}(z).$$

(2.8) gives us, for $0 \leqslant k \leqslant n-2$ with $n \equiv k \pmod 2$,

$$(2.9) \qquad (n-k)(n+k+d-1)S_{n,k} = -\lambda^2 S_{n-2,k} ,$$

because $F \in \mathcal{O}_\lambda(\widehat{B}(r))$ and $H_{n,d} \perp H_{m,d}$ if $n \neq m$. (2.5) follows from (2.9). (2.6) follows from [8] Theorem 3.2 (3.33).

Conversely, suppose we are given a sequence $\{S_{n,k}\}$ satisfying (2.5) and (2.6). Then we can show that the right hand side of (2.7) converges uniformly and absolutely on every compact set of $\widetilde{B}(r)$ since the Šilov boundary of $\widetilde{B}(\rho)$ is $\{\rho e^{i\theta} s;\ 0 \leqslant \theta < 2\pi,\ s \in S\}$ (see Hua [3]). Therefore F belongs to $\mathcal{O}(\widehat{B}(r))$ and $S_{n,k} = S_{n,k}(F;$

It is easy to show that $\Delta_z F = -\lambda^2 F$. q.e.d.

Lemma 2.3. For $F \in \mathcal{O}(\widetilde{B}(r))$ we have for any $z \in C^{d+1}$

$$(2.10) \qquad \widetilde{S}_{n,n}(F;\ z) = N(n,d) \int_N F(\rho z'/2)\ (\bar{z}' \cdot \tfrac{z}{\rho})^n\ dN(z'),$$

where ρ is any real number such that $0 < \rho < r$ and the right hand side of (2.10) is independent of ρ.

Sketch of the proof. Since $S_{n,n}(F;\ ) \in H_{n,d}$ it is valid for any $s \in S$

$$(2.11) \qquad S_{n,n}(F;\ s) = N(n,d) \int_S S_{n,n}(F;\ s')\ P_{n,d}(s \cdot s')\ ds'.$$

By (2.11) and (1.4) we have (2.10) because $P_n(M) \perp P_m(M)$ $(n \neq m)$ and $\widetilde{P}_{n,d}(z \cdot s) = \{2^n \Gamma(n + \frac{d+1}{2})\ (z \cdot s)^n / \{n!\ N(n,d)\ \Gamma(\frac{d+1}{2})\}$ on N.

Sketch of the proof of Theorem 2.1. (i) For any $\lambda \in C$ it is clear that $\mathcal{O}_\lambda(\widetilde{B}(r))|_M \subset \mathcal{O}(\widetilde{B}(r))|_M$. Let $F \in \mathcal{O}_\lambda(\widetilde{B}(r))$. Then for any $z \in M \cap \widetilde{B}(r)$ we have

$$(2.12) \qquad F(z) = \sum_{n=0}^{\infty} \widetilde{S}_{n,n}(F;\ z).$$

By (2.12) we have for any $z' \in N$

$$(2.13) \qquad F(rz'/4) = \sum_{n=0}^{\infty} (r/4)^n \widetilde{S}_{n,n}(F;\, z'),$$

because $(r/4)N \subset \widetilde{B}(r) \cap M$. If $\alpha_\lambda(F) = 0$, $\widetilde{S}_{n,n} = 0$ on N by (2.13) and the orthogonality of homogeneous polynomials on N. So the spherical harmonic function $S_{n,n} = 0$ by (1.4) and $F = 0$ by (2.5). Therefore α_λ is injective.

Next for $f \in \mathcal{O}(\widetilde{B}(r))$ we define the function F as follows:

$$F(z) = \sum_{(n,k)\in\Lambda_+} (\sqrt{z^2}\,)^{n-k}\, \widetilde{S}_{n,k}(z),$$

where

$$\widetilde{S}_{n,k}(z) = \frac{\Gamma(k + \frac{d+1}{2}) \quad (i\lambda/2)^{n-k}}{\Gamma(\frac{n-k}{2} + 1)\, \Gamma(\frac{n+k+d+1}{2})} \quad \widetilde{S}_{k,k}(f;\, z).$$

As $f \in \mathcal{O}(\widetilde{B}(r))$, $\limsup_{n\to\infty} \|S_{n,n}\|_\infty^{1/n} = \limsup_{n\to\infty} \|S_{n,n}(f;\ )\|_\infty^{1/n} \leq 1/r$ by [8] Theorem 3.2. Hence by Lemma 2.2 $F \in \mathcal{O}_\lambda(\widetilde{B}(r))$ and $F|_M = f|_M$. Therefore α_λ is surjective.

(ii) By the proof of surjectivity of α_λ in (i) and (2.10) we get (2.2). In particular, we obtain (2.4) since $K_0(z, \frac{\overline{z'}}{\rho}) = \sum_{k=0}^{\infty} N(k,d)\, (\overline{z}' \cdot \frac{z}{\rho})^k$ and $\sum_{k=0}^{\infty} N(k,d)x^k = (1 + x)(1 - x)^{-d}$ for $x \in C$, $|x| < 1$ (see, for example Müller [11] Lemma 3).

(iii) It is clear that α_λ is continuous. (2.2) gives that for any ρ and ρ' with $0 < \rho' < \rho < r$

$$(2.14)\qquad \sup_{L(z)\leq \rho'} |\alpha_\lambda^{-1} f(z)| \leq \exp \tfrac{1}{4}(|\lambda|^2 \rho'^2) \cdot \left(1 + \frac{\rho'}{\rho}\right)\left(1 - \frac{\rho'}{\rho}\right)^{-d} \sup_{z \in \frac{\rho}{2}N} |f(z)|.$$

As $(\rho/2)N$ is the compact set of $\widetilde{B}(r) \cap M$ (2.14) showes that α_λ^{-1} is continuous. q.e.d.

<u>Corollary 2.4.</u> If F belongs to $\mathcal{O}_\lambda(\widetilde{B}(\rho))$ for $0 < \rho < r$ and $F|_M$ belongs to $\mathcal{O}(\widetilde{B}(r))|_M$ then F belongs to $\mathcal{O}_\lambda(\widetilde{B}(r))$.

§3. The Fourier-Borel transformation.

The Fourier-Borel transformation P_λ for a functional $T \in \mathrm{Exp}'(C^{d+1}: [r:L])$ is defined by

$$(3.1)\qquad P_\lambda T(z) = \langle T_\zeta, \exp(i\lambda\zeta\cdot z)\rangle \qquad \text{for } z \in C^{d+1},$$

where $\lambda \in C\setminus\{0\}$ is a fixed constant. In this section we will determine the functional space on $\widetilde{S}$ whose image by P_λ coincides with $\mathcal{O}_\lambda(\widetilde{B}(r))$. Our main theorem in this section is the following

<u>Theorem 3.1([14]).</u> The transformation P_λ establishes linear topological isomorphisms

$$(3.2)\qquad P_\lambda : \mathrm{Exp}'(\widetilde{S}: [|\lambda| r/2: L]) \xrightarrow{\sim} \mathcal{O}_\lambda(\widetilde{B}(r)) \qquad (0 < r \leq \infty),$$

$$(3.3)\qquad P_\lambda : \mathrm{Exp}'(\widetilde{S}: (|\lambda| r/2: L)) \xrightarrow{\sim} \mathcal{O}_\lambda(\widetilde{B}[r]) \qquad (0 \leq r < \infty).$$

Remark. The above theorem was proved by Morimoto [10] in the case of $r = \infty$.

We need the following lemma in order to prove the theorem.

Lemma 3.2. If S_n is the spherical harmonic component of f' then

$$f' \in \mathrm{Exp}'(\widetilde{S}: [r:L]) \iff \limsup_{n \to \infty} (\|S_n\|_\infty/n!)^{1/n} \leq 1/r, \tag{3.4}$$

$$f' \in \mathrm{Exp}'(\widetilde{S}: (r:L)) \iff \limsup_{n \to \infty} (\|S_n\|_\infty/n!)^{1/n} < 1/r. \tag{3.5}$$

Lemma 3.2 can be proved in the same way as in the proof of [10] Theorem 6.1.

Sketch of the proof of Theorem 3.1. By using the results of [6] and Theorem 2.1, Corollary 2.4 and Lemma 3.2, Theorem 3.1 can be proved. q.e.d.

§4. A uniqueness set for linear partial differential operators of the second order.

Consider the differential operator

$$P(D) = \sum_{i=1}^{d+1} \sum_{j=1}^{d+1} a_{i,j} \frac{\partial^2}{\partial z_i \partial z_j} + \sum_{j=1}^{d+1} b_j \frac{\partial}{\partial z_j} + c, \tag{4.1}$$

where $a_{i,j}$, b_j , $c \in \mathbf{C}$ and $a_{i,j} = a_{j,i}$ for $i,j = 1,2, \ldots , d+1$, and we put

$$(4.2) \qquad A = \left(a_{i,j} \right) .$$

As A is a symmetric matrix there is some $T \in U(d+1)$ and $\lambda_1, \lambda_2, \dots, \lambda_{d+1} \in R$ such that

$$(4.3) \qquad TA{}^tT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{d+1} \end{pmatrix} ,$$

where U(K) is the unitary group of degree k. We define

$$(4.4) \qquad A_0 = {}^tT \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_{d+1}) \end{pmatrix} T,$$

where we put

$$f(\lambda) = \begin{cases} 1/\lambda & (\lambda \neq 0) \\ 0 & (\lambda = 0). \end{cases}$$

If $TA{}^tT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{d+1} \end{pmatrix}$ and $SA{}^tS = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{d+1} \end{pmatrix}$ for some $T, S \in U(d+1)$ and $\lambda_j, \mu_j \in C$ $(j = 1,2, \dots, d+1)$ then

$${}^tT \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_{d+1}) \end{pmatrix} T = {}^tS \begin{pmatrix} f(\mu_1) & & \\ & \ddots & \\ & & f(\mu_{d+1}) \end{pmatrix} S.$$

When A is regular we have $A_0 = A^{-1}$.

We define $\mathcal{O}_P(C^{d+1}) = \{ f \in \mathcal{O}(C^{d+1});\ P(D)f = 0 \}$ and $M_P = \{ z \in C^{d+1};\ A_0 z \cdot z = 0 \}$. $\mathrm{Holo}(M_P) = \mathcal{O}(C^{d+1})|_{M_P}$ denotes the space of holomorphic functions on the complex cone M_P. We define spaces of germs of holomorphic functions as follows:

$$\mathcal{O}_P(\{0\}) = \operatorname*{ind\,lim}_{0 \in W} \mathcal{O}_P(W),$$

$$\mathrm{Holo}_{M_P}(\{0\}) = \operatorname*{ind\,lim}_{0 \in W} \mathcal{O}(W)|_{M_P},$$

where W runs through the neighborhoods of 0. We equip $\mathcal{O}_P(W)$ and $\mathcal{O}(W)|_{M_P}$ with the topologies of uniform convergence on every compact set of W and every compact set of $W \cap M_P$ respectively. It is known $\mathcal{O}_P(W)$ is an FS space. The topologies of $\mathcal{O}_P(\{0\})$ and $\mathrm{Holo}_{M_P}(\{0\})$ are the locally convex inductive limit of the topologies of $\mathcal{O}_P(W)$ and $\mathcal{O}(W)|_{M_P}$ respectively. It is known $\mathcal{O}_P(\{0\})$ is a DFS space.

Our main theorem in this section is the following:

Theorem 4.1 ([15]). Let rank $A \geqslant 2$. The restriction mapping $\alpha_P : F \longrightarrow F|_{M_P}$ defines the following linear topological isomorphisms:

(4.5) $\quad \alpha_P: \mathcal{O}_P(C^{d+1}) \xrightarrow{\sim} \mathrm{Holo}(M_P),$

(4.6) $\quad \alpha_P: \mathcal{O}_P(\{0\}) \xrightarrow{\sim} \mathrm{Holo}_{M_P}(\{0\}).$

Let d' be a positive integer and $1 \leqslant d' \leqslant d$. $B_L(r)$ denotes the Lie ball on $C^{d'+1}$. Let $d'' = d - d'$, $\zeta \in C^{d'+1}$ and $t \in C^{d''}$. Let

W be an open set in $C^{d''}$. For $f \in \mathcal{O}(B_L(r) \times W)$ $(r > 0)$ $S_{n,k}(f:\zeta,t)$ denotes the (n,k)-component of f with respect to ζ $((n,k) \in \Lambda_+)$. Then we can write

$$(4.7) \qquad f(\zeta,t) = \sum_{(n,k)\in\Lambda_+} (\sqrt{\zeta^2}\,)^{n-k}\, \widetilde{S}_{n,k}(f:\zeta,t).$$

When t is fixed, the right hand side of (4.7) converges to $f(\zeta,t)$ uniformly and absolutely on every compact set of $B_L(r)$ and $S_{n,k}(f:\zeta,t)$ is a homogeneous harmonic polynomial of degree k. For fixed $\zeta \in C^{d'+1}$ $\widetilde{S}_{n,k}(f:\zeta,t)$ belongs to $\mathcal{O}(W)$.

In order to prove the theorem we need following lemmas.

Lemma 4.2. Suppose f belongs to $\mathcal{O}^c(C^{d+1})$ and satisfies the differential equation

$$(4.8) \qquad \Delta_\zeta f = \Big(\sum_{j=1}^{d''} b_j \frac{\partial}{\partial t_j} + c\Big)f,$$

where $\Delta_\zeta = (\partial/\partial\zeta_1)^2 + \ldots + (\partial/\partial\zeta_{d'+1})^2$ and b_j, $c \in C$. Then we have

$$(4.9) \qquad \widetilde{S}_{n,k}(f:\zeta,t) = \frac{\Gamma(k + \frac{d'+1}{2})\big(\sum_{j=1}^{d''} b_j \partial/\partial t_j + c\big)^{\frac{n-k}{2}}}{2^{n-k}((n-k)/2)!\,\Gamma((n+k+d'+1)/2)}\, \widetilde{S}_{k,k}(f:\zeta,t)$$

for $(n,k) \in \Lambda_+$.

Lemma 4.3. Let $P(D) = \Delta_\zeta - \sum_{j=1}^{d''} b_j \frac{\partial}{\partial t_j} - c$, $M_{d'} = \{(\zeta,t) \in$

$C^{d+1};\ \zeta^2 = 0\}$, $\mathrm{Holo}(M_{d'}) = \mathcal{O}(C^{d+1})|_{M_{d'}}$ and $\mathrm{Holo}_{M_{d'}}(\{0\}) = \operatorname*{ind\,lim}_{0 \in W} \mathcal{O}(W)|_{M_{d'}}$. The restriction mapping $\alpha : f \longrightarrow f|_{M_{d'}}$ defines the following bijections:

$$(4.10) \qquad \alpha : \mathcal{O}_P(C^{d+1}) \longrightarrow \mathrm{Holo}(M_{d'}),$$

$$(4.11) \qquad \alpha : \mathcal{O}_P(\{0\}) \longrightarrow \mathrm{Holo}_{M_{d'}}(\{0\}).$$

We can prove Lemmas 4.2 and 4.3 in the same way as in the proofs Theorem 2.1 and Lemma 2.2.

Sketch of the proof of Theorem 4.1. There exists some $T \in U(d+1)$ such that

$$TA\,{}^tT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_{d'+1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

where d' is a positive integer with $0 < d' \leqslant d$ and $\lambda_j \neq 0$ ($j = 1, 2, \ldots, d'+1$). Consider

$$\Lambda' = \begin{pmatrix} 1/\sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & 1/\sqrt{\lambda_{d'+1}} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

and $(B_1, B_2, \ldots, B_{d'+1}, C_{d'+2}, \ldots, C_{d+1}) \in C^{d+1}$ which satisfies

$$(2B_1\sqrt{\lambda_1} \ \ldots \ 2B_{d'+1}\sqrt{\lambda_{d'+1}}\ C_{d'+2} \ \ldots \ C_{d+1}) = (b_1\ b_2 \ldots b_{d+1})\,{}^tT.$$

It is clear that α_P is a continuous linear mapping of $\mathcal{O}_P(C^{d+1})$ into $\mathrm{Holo}(M_P)$. For any $g \in \mathcal{O}(C^{d+1})$ we put

$$h(z) = g((\Lambda'T)^{-1}z)\exp\left(\sum_{j=1}^{d'+1} B_j z_j\right).$$

Then it is easy to show that

(4.12) $P(D)g(z)$

$$= \exp\left(-\sum_{j=1}^{d'+1} B_j w_j\right)\left(\sum_{j=1}^{d'+1} (\partial/\partial w_j)^2 + \sum_{j=d'+2}^{d+1} C_j(\partial/\partial w_j) + c - \sum_{j=1}^{d'+1} B_j^2\right)h,$$

where we put $w = \Lambda'Tz$. Since $M_{d'} = (\Lambda'T)M_P$, we can prove that the bijectivity of α_P from (4.12) and Lemmas 4.2 and 4.3.

Furthermore, we have for any $f \in \mathrm{Holo}(M_P)$ and for any ρ and ρ' with $\rho > r$ and $\rho' > r'$

(4.13) $\sup\left\{|\alpha_P^{-1} f(z)|\,;\ z \in (\Lambda'T)^{-1}(\overline{B_L(r)} \times B[r'])\right.$

$$\leq C \sup\left\{|f(z)|;\ z \in (\Lambda'T)^{-1}((\rho/2)N_{d'} \times B[\rho'])\right\},$$

where $N_{d'} = \{\zeta \in C^{d'+1};\ \zeta^2 = 0\} \cap B_L(2)$ and $B[r'] = \{t \in C^{d''};\ \|t\| \leq r'\}$, and C is a constant which depends on r, r', ρ and ρ'. We can see that α_P^{-1} is continuous by (4.13).

We can prove (4.6) similarly. q.e.d.

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