数理解析研究所講究録
第638巻 1988年 185-202

非線型偏微分方程式の解の準斉次超局所特異性の伝播

東大・理　山崎昌男(Masao Yamazaki)

ここでは，Meyerによる非線型偏微分方程式の超局所準楕円性定理を準斉次の場合に拡張し，同時に解の滑らかさに対して予め課される条件をなるべく弱くしたものを示すと共に，結果として得られる超局所正則性をより精密にする．また，条件の精密さを示す例を与える．この結果の詳細は[12]に報告される予定である．

また，Bonyによる一般の非線型偏微分方程式の解の超局所特異性の伝播についての定理を準斉次の場合に拡張する．この結果は，半線型方程式の解の準斉次超局所特異性の伝播に関するGodin及び桜井氏の結果の拡張にもなっている．また，特異性の伝播が成立つために十分な，解の滑らかさについての条件を緩め，より精密な超局所正則性を示す．更に，特異性伝播のための条件を超局所準楕円性のための条件と比較し，これらの条件の意味を論ずる．この結果の詳細は[13]に報告される予定である．

PROPAGATION OF QUASI-HOMOGENEOUS MICROLOCAL SINGULARITIES OF SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Masao Yamazaki (Department of Mathematics, University of Tokyo)

Abstract. In this article we give a quasi-homogeneous version of Meyer's microlocal elliptic theorem for general fully nonlinear partial differential equations. We also relax the condition originally posed on the smoothness of the solutions, and give a sharper estimate of the microlocal smoothness. We also discuss some examples which shows the sharpness of the condition. Details on this result will be published in [12].

Secondly, we generalize Bony's theorem on propagation of microlocal singularities of solutions to general fully nonlinear partial differential equations to the quasi-homogeneous case. This result also generalizes the theorems given by Godin and Sakurai on propagation of quasi-homogeneous microlocal singularities of solutions of semilinear equations. We also relax the condition on the smoothness of the solutions sufficient for the propagation of singularities, and obtain sharper microlocal regularities. Then we compare the conditions for the propagation of singularities with that for the microlocal ellipticity, and discuss the meanings of the conditions. Details on this result will be published in [13].

§0. **Introduction.** Among a number of studies on microlocal analysis for nonlinear partial differential equations, Bony [1] was the first where general nonlinear equations were treated. He showed the microlocal hypoellipticity at non-characteristic points and the propagation of singularities along simple bicharacteristic strips, by making use of paradifferential operators as the main tool. Along this line, Meyer [7] improved Bony's microlocal hypoellipticity theorem.

On the other hand, Lascar [6] introduced the notion of quasi-homogeneous wave front set, and obtained the propagation of singularities of solutions to the equations of Schrödinger type. Here by the quasi-homogeneous treatement we mean the following: we give a weight m_l to each coordinate variable x_l , regard the differential operator $\partial/\partial x_l$ as an operator of order m_l , and consider the function spaces, principal symbols and bicharacteristic strips suitable to this setting. This treatment is natural for a number of important nonlinear equations; for example, nonlinear parabolic equations, nonlinear equations of Schrödinger type, and the KdV equation.

The first purpose of this article is to obtain a quasi-homogeneous version of the microlocal hypoellipticity theorem with little assumption posed *a priori* on the smoothness of the solutions. It is verified by Kobayashi-Nakamura [5] that the microlocal hypoellipticity theorem is not applicable for solutions with strong singularity, by constructing solutions of semilinear hyperbolic equations with singular spectra on the non-characteristic points. Hence it is of interest to relax the condition on the smoothness of solutions for which the microlocal hypoellipticity theorem is applicable. The condition obtained in this article depends on the non-linearity of the equations. We also give some examples which shows the sharpness of our condition for the hypoellipticity theorem. At the same time we give a sharper estimate of the microlocal regularity.

The second purpose of this article is to give a generalization of the result of Bony [1] on the propagation of singularities to the quasi-homogeneous case. In this direction, Yamazaki [10], [11] showed the microlocal hypoellipticity in general function spaces of Besov type. Also, Godin [3] and Sakurai [8] showed the propagation of singularties for a class of semilinear equations. Then Sakurai [9] obtained the propagation theorem along simple bicharacteristic strips for general semilinear equations. Here we aim to generalize his result to general nonlinear equations. Also, we do not assume that the principal symbol is real, as in Hörmander [4]. Here we also aim to relax the condition posed originally on the regularity of the solutions, and to obtain sharper estimates.

The outline of this article is as follows. In Section 1 we list our notations and make some preliminary remarks and definitions. Then we state our microlocal hypoellipticity theorem in Section 2, and we give some examples in Section 3. Finally, in Section 4, we state our theorem on the propagation of singularities.

§1. Notations and preliminary remarks. We start with some notations. For a multi-index $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{N}^n$ and two vectors $x = (x_1,\cdots,x_n)$, $\xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$, we put $|\alpha| = \alpha_1+\cdots+\alpha_n$, $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $x\cdot\xi = x_1\xi_1+\cdots+x_n\xi_n$. Here $\mathbb{N}$ denotes the set of natural numbers (= nonnegative integers).

Next, for the coordinate variable $x = (x_1,\cdots,x_n) \in \mathbb{R}^n$, let dx denote the Lebesgue measure on $\mathbb{R}^n$, and put $\bar{d}x = (2\pi)^{-n}dx$. We omit the domain of integration if it is the whole space $\mathbb{R}^n$. Next we put $i = \sqrt{-1}$, $\partial_{x_\ell} = \partial/\partial x_\ell$ and $\partial_x^\alpha = (\partial_{x_1})^{\alpha_1}\cdots(\partial_{x_n})^{\alpha_n}$.

Further, let $\mathcal{S}'$ denote the space of tempered distributions on $\mathbb{R}^n$, and for an open set Ω in $\mathbb{R}^n$, let $\mathcal{D}'(\Omega)$ denote the spaces of all distributions on Ω. For $u(x) \in \mathcal{S}'$, let $\hat{u}(\xi) = \mathcal{F}[u](\xi)$ denote the Fourier transform of $u(x)$; that is, we put $\hat{u}(\xi) = \int \exp(-ix\cdot\xi)u(x)dx$.

Now we state our general assumptions. We give a weight $M = (m_1,\cdots,m_n) \in (\mathbb{R}^+)^n$ satisfying $\min_{\ell=1,\cdots,n} m_\ell = 1$ to the coordinate variable $x \in \mathbb{R}^n$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. Then we put $|M| = m_1+\cdots+m_n$ and $\Theta = \{ \ell = 1,\cdots,n;\ m_\ell = 1 \}$. The dimension n and the weight M are fixed throughout this article. Next, let $\langle\xi\rangle$ denote the unique positive root of $t^{-2}+\sum_{\ell=1}^{n} t^{-2m_\ell}\xi_\ell^2 = 1$, which is regarded as an equation with respect to t. For an open set $\Omega \subset \mathbb{R}^n$, a subset $V \subset T^*\Omega\backslash 0 = \Omega\times(\mathbb{R}^n\backslash\{0\})$ is called *M-conic* if $(x,\xi) \in V$ and $t > 1$ imply $(x,t^M\xi) \in V$.

Now we give the definition of our function spaces.

Definition. For $s \in \mathbb{R}$, we define the *anisotropic Sobolev space* $H^{M,s}$ by

$$H^{M,s} = \{ u(x) \in \mathcal{S}';\ \|u(x)\|_{M,s} = \left(\int \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2 \bar{d}\xi\right)^{1/2} < \infty \}.$$

At the end of this section we microlocalize this space in the same manner as in Lascar [6].

Definition. Let Ω be an open set in $\mathbb{R}^n$, and let $u(x) \in \mathcal{D}'(\Omega)$ and $(\mathring{x},\mathring{\xi}) \in T^*\Omega\backslash 0$.

Then $u(x)$ is said to be *microlocally in* $H^{M,s}$ *at* $(\mathring{x},\mathring{\xi})$ if there exist a function $\phi(x) \in C_0^\infty(\Omega)$ and an M-conic neighborhood V of $(\mathring{x},\mathring{\xi})$ satisfying $\phi(\mathring{x}) \neq 0$ and $\int_U \langle\xi\rangle^{2s} |\mathcal{F}[\phi u](\xi)|^2 d\xi < \infty$, where $U = \{ \xi \in \mathbb{R}^n;\ (x,\xi) \in V$ for some $x \in \Omega \}$.

And $u(x)$ is said to be *locally in* $H^{M,s}$ *at* $\mathring{x}$ if there exists a function $\phi(x) \in C_0^\infty(\Omega)$ satisfying $\phi(\mathring{x}) \neq 0$ and $\phi(x)u(x) \in H^{M,s}$.

Then, as usual, it is easy to see the following

Proposition 1.1. *Let* Ω, $\mathring{x}$ *and* $u(x)$ *be the same as in the previous definition. Then* $u(x)$ *is locally in* $H^{M,s}$ *at* $\mathring{x}$ *if and only if* $u(x)$ *is locally in* $H^{M,s}$ *at* $(\mathring{x},\mathring{\xi})$ *for every* $\mathring{\xi} \neq 0$.

§2. Microlocal hypoellipticity theorem. We consider nonlinear partial differential equations on Ω of the following form:

$$(2.1)\quad \sum_{k=1}^{K} \partial_x^{\beta(k)}\Big(F_k(x;\ u(x),\cdots,\partial_x^{\alpha}u(x),\cdots)\Big) = f(x),$$

where Ω is an open set of $\mathbb{R}^n$, $K \in \mathbb{N}$, $\beta(k) \in \mathbb{N}^n$ ($k = 1,\cdots,K$), $u(x)$, $f(x) \in \mathcal{D}'(\Omega)$ and the function $F_k(x;\ X,\cdots,X_\alpha,\cdots)$ is of the form $X_{\alpha(k,1)}\cdots X_{\alpha(k,L_k)}G_k(x;\ X,\cdots,X_\alpha,\cdots)$. (We put $L_k = 0$ and $G_k = F_k$ if F_k is not divisible by any X_α.)

Here the number L_k is determined and the multi-indices $\alpha(k,1),\cdots,\alpha(k,L_k)$ are arranged in such a way that the relation $\lambda_{1k} \geq \lambda_{2k} \geq \cdots \geq \lambda_{0k}$ should hold, where

$$\lambda_{0k} = \max\ \{\ M\cdot\alpha;\ G_k \text{ depends on } X_\alpha\ \}$$

$$(\ \lambda_{0k} = -\infty \text{ if } G_k \text{ does not depend on any } X_\alpha\ )$$

and

$$\lambda_{jk} = \begin{cases} M\cdot\alpha(k,j) & (\text{ if } 1 \leq j \leq L_k\ ) \\ \lambda_{0k} & (\text{ if } j > L_k\ ). \end{cases}$$

Next we suppose that $u(x)$ is a solution of (2.1) such that one of the following two conditions holds.

(HR) *The functions* $G_k(x;\ X,\cdots,X_\alpha,\cdots)$ *are* C^∞ *with respect to* $x \in \Omega$ *and every* $X_\alpha \in \mathbb{R}$, *and* $u(x)$ *is a real-valued* *distribution.*

(HC) *The functions* $G_k(x; X, \cdots, X_\alpha, \cdots)$ *are* C^∞ *with respect to* $x \in \Omega$, *and entire with respect to each* X_α, *and* $u(x)$ *is a complex-valued distribution.*

We write the formal development of (2.1) as

$$(2.2) \quad A(x;\, u(x), \cdots, \partial_x^\alpha u(x), \cdots) = f(x),$$

and we put $a_\beta(x) = \dfrac{\partial A}{\partial(\partial_x^\beta u)}\left(x;\, u(x), \cdots, \partial_x^\alpha u(x), \cdots\right)$ for $\beta \in \mathbb{N}^n$.

Now we define the *weighted order* of the equation (2.1) by

$$m = \max \{\, M\cdot\alpha;\ \partial_x^\alpha u(x) \text{ appears in } (2.2) \,\},$$

and the *weighted principal symbol* of (2.1) by $P_m(x,\xi) = \sum_{M\cdot\beta=m} a_\beta(x)(i\xi)^\beta$, and we say that a point $(x,\xi) \in T^*\Omega\backslash 0$ is *non-characteristic with respect to* M *and* $u(x)$ if $P_m(x,\xi) \neq 0$.

Remark 2.1. If the equation (2.1) is not semilinear, the weighted principal symbol may depend on the choice of the solution $u(x)$, and it may not be smooth. We shall return to this problem in Remark 2.5.

Further, we introduce some numbers. First we put

$$\rho = \max_{1\le k\le K} \left(\lambda_{0k} + |M|/2,\ \max_{h\ge 2} \{|M|/2 + (\textstyle\sum_{j=1}^{h} \lambda_{jk} - |M|)/h\}\right)$$

and

$$\sigma = \max \left\{ \rho \ , \ \max_{1 \leq k \leq K, \ h \geq 2} \{ |M|/2 + (M \cdot \beta(k) + \sum_{j=1}^{h} \lambda_{jk} - m)/(h-1) \} \right\}.$$

Next, for every $k = 1, \cdots, K$ and $s > \sigma$, put

$$(2.3) \quad \mu_k(s) = \min_{h \geq 2} \left\{ \sum_{j=1}^{h} (s - \lambda_{jk} - |M|/2) + |M|/2 \right\} - M \cdot \beta(k).$$

Now let $\mu(s)$ be a real number such that $\mu(s) \leq \mu_k(s)$ holds for every $k = 1, \cdots, N$ and that $\mu(s) < \mu_k(s)$ holds for every k satisfying $\lambda_{3k} \geq s - |M|/2$ and

(2.4) There exists an integer $j \in \{1, \cdots, L_k\}$ satisfying $\lambda_{jk} = s - |M|/2$ such that $\partial_x^{\alpha(k,j)} u(x)$ is not essentially bounded.

Example 2.3. If the equation (2.1) is fully nonlinear of weighted order m, we have $\sigma = \rho = m + |M|/2$ and $\mu(s) = 2s - 2m - |M|/2$. If the equation has better linearity, then σ is smaller and $\mu(s)$ is greater. In particular, if (2.1) is linear, we can take σ arbitrarily small and $\mu(s)$ arbitrarily large. Some other examples are given in Yamazaki [12].

Then our microlocal hypoellipticity theorem is the following.

Theorem 2.4. *Let* $(\mathring{x}, \mathring{\xi}) \in T^*\Omega \backslash 0$ *be non-characteristic with respect to* M *and* $u(x)$. *Suppose that* $u(x)$ *is locally in* $H^{M,s}$ *at* $\mathring{x}$ *for some* $s > \sigma$, *and that* $f(x)$ *is microlocally in*

$H^{M,t}$ at $(\mathring{x},\xi)$ for some $t \leq \mu(s)$. Then $u(x)$ is microlocally in $H^{M,t+m}$ at $(\mathring{x},\xi)$.

Remark 2.5. The condition $s > \rho$ guarantees that the nonlinear terms are well-defined and can be linearized by means of the paradifferential operators. This condition depends, at least formally, on the expression (2.1).

On the other hand, the condition

$$(2.5) \quad s > \max_{1 \leq k \leq K,\ h \geq 2} \{|M|/2 + (M \cdot \beta(k) + \sum_{j=1}^{h} \lambda_{jk} - m)/(h-1)\}$$

implies that we can take

$$(2.6) \quad \mu(s) + m > s,$$

which is necessary for the main theorem to be meaningful. Besides, (2.6) implies that $P_m(x,\xi)$ is continuous. We also remark that the condition (2.5) depends only on the formal development (2.2), not on the expression (2.1).

§3. Some examples.

Example 3.1. Consider the following semilinear elliptic equation:

(3.1) $\Delta u+au^{\ell}+a'u = f$ ($\ell,n \in \mathbb{N}$, $\ell,n \geq 2$).

Put $M = (1,\cdots,1)$. Then we have $\rho = n/2-n/\ell$ and $\sigma = \max\{ \rho, n/2-2/(\ell-1) \}$. We also have $\mu(s) = \ell s-(\ell-1)n/2-2$ for $s < n/2$, and all points are non-characteristic with respect to M. Hence, if f is C^{∞} and if u is locally in H^{s} for some $s > \sigma$, we can see easily that u is C^{∞} near $\mathring{x}$, using Theorem 2.4 and Proposition 1.1 repeatedly.

Suppose further that $n \geq 3$. Then we have $\sigma = n/2-2/(\ell-1)$ for $\ell \geq n/(n-2)$, and we see that $\sigma < 1$ holds if and only if ℓ is less than the *critical Sobolev exponent* $(n+2)/(n-2)$. In this case we can apply the argument above to the variational solutions belonging to the space H^{1}. (See, for example, Brézis-Nirenberg [2].)

Remark 3.2. The condition (2.5) is essential. In fact, let ℓ be a natural number greater than $n/(n-2)$, and put $b = -2/(\ell-1)$ and $M = (1,\cdots,1)$. Then we have $\ell b+n = b-2+n > 0$ and *a fortiori* $2b+n > 0$. Hence, for the equation (3.1), we have $\sigma = b+n/2 > \rho = n/2-n/\ell$ in this case, and hence (2.5) is equivalent to $s > \sigma$. On the other hand, it follows that $u(x) = |x|^{b}$ is a solution of (3.1) with $a = b(2-b-n)$ and

$a' = 0$. Furthermore, Assertion 1) of Proposition 2.8 of [12] implies that $u(x)$ is locally in $H^{b-\varepsilon}$ at $x = 0$ for every $\varepsilon > 0$. However, $u(x)$ is not C^∞ near $x = 0$, which implies that Theorem 2.4 is not applicable in this case.

We now compare our result with that of Kobayashi-Nakamura [5].

Remark 3.3. Let $\phi(x)$ be an analytic function defined near $0 \in \mathbb{R}^n$ satisfying $\phi(0) = 0$ and $\operatorname{grad} \phi(0) \neq 0$, and let $P(D)$ and $Q(D)$ be partial differential operators with constant coefficients of order m and $m-h$ respectively. Assume that $P(D)$ is strictly hyperbolic with respect to the time function $\phi(x)$. Next we put $\varkappa = -h/(\ell-1)$.

Then, under some assumptions on the coefficients of $P(D)$ and $Q(D)$, Kobayashi-Nakamura [5] constructed a local solution $u(x)$ of the equation

$$P(D)u+Q(D)(u^{\ell}) = 0$$

and a smooth function $v(x)$, both defined near 0, such that $u(x)$ is the sum of $v(x)(\phi(x)+i0)^{\varkappa}$ and a more regular function.

The wave front set (modulo sufficiently regular terms) of $u(x)$ is contained in the set $V = \{ (x, \operatorname{grad} \phi(x));\ \phi(x) = 0 \}$, which consists of non-characteristic points. On the other hand, Assertion 2) of Proposition 2.8 of [12] implies that $u(x)$ is locally in $H^{\varkappa+1/2-\varepsilon}$ for every $\varepsilon > 0$. On the other hand, the condition (2.5) can be rewritten as $s > \sigma+n/2$. It is to be expected that we will be able to fill in this gap by introducing some function spaces associated with an appropriate second microlocalization.

Since the fiber of V is strictly convex, there is little difficulty in justifying the definition of the nonlinear terms in this case. However, in general, some conditions other than (2.5) are necessary.

§4. **The propagation theorem.** In this section we consider the equation (2.1) again. In order to state our theorem on the propagation of microlocal singularities, we introduce some notions following Lascar [6].

Definition. We call the vector field

$$X(x,\xi) = \sum_{\ell\in\Theta} \left(\frac{\partial}{\partial\xi_\ell}(Re\ P_m(x,\xi))\cdot\frac{\partial}{\partial x_\ell} - \frac{\partial}{\partial x_\ell}(Re\ P_m(x,\xi))\cdot\frac{\partial}{\partial\xi_\ell}\right)$$

the quasi-homogeneous Hamiltonian vector field associated to the solution $u(x)$. Next, if $X(\mathring{x},\mathring{\xi}) \neq 0$ at a point $(\mathring{x},\mathring{\xi}) \in T^*\Omega\backslash 0$ such that $P_m(x,\xi) = 0$, then we say that the equation (2.1) is *of principal type at* $(\mathring{x},\mathring{\xi})$ *with respect to the weight* M *and the solution* $u(x)$. An integral curve of X contained in the set $V_0 = \{\ (x,\xi) \in T^*\Omega\backslash 0;\ Re\ P_m(x,\xi) = 0\ \}$ is called a *null bicharacteristic strip associated to the weight* M *and the solution* $u(x)$. Further, we put

$$\tau = \max\left\{\rho\ ,\ \max_{1\le k\le K,h\ge 2}\{|M|/2+(M\cdot\beta(k)+\sum_{j=1}^{h}\lambda_{jk}+1-m)/(h-1)\}\right\}.$$

Example 4.1. If the equation (2.1) is fully nonlinear of weighted order m, we have $\tau = m+1+|M|/2$.

Then our theorem on the propagation of microlocal singularities is the following.

Theorem 4.2. *Let* $\Gamma = \{\ \gamma(t);\ a \le t \le b\ \} \subset V_0$ *be a null bicharacteristic strip associated to* M *and* $u(x)$, *and suppose* *that the following six conditions are satisfied.*

(4.1) $u(x)$ *is locally in* $H^{M,s}$ *at every point of* Ω, *where* $s > \sigma$.

(4.2) *The equation* (2.1) *is of principal type with respect to the solution* $u(x)$.

(4.3) *For every* $\ell, \ell' \in \Theta$, *the derivative* $\partial_{x_\ell} \mathrm{Re}\, P_m(x,\xi)$ *is Lipshitz continuous with respect to* $x_{\ell'}$.

(4.4) *We can take an M-conic neighborhood* V *of* Γ *in* $T^*\Omega\backslash 0$ *and a disjoint partition* $I_1, \cdots, I_N$ *of the set* $\{\beta \in \mathbb{N}^n; m-1 < M\cdot\beta \leq m\}$ *such that the conditions* $\mathrm{Im} \sum_{\beta \in I_\nu} a_\beta(x)(i\xi)^\beta \geq 0$ *and*

$$\left| \mathrm{Im}\, i^{|\beta|}\left(a_\beta(x+x') - 2a_\beta(x) + a_\beta(x-x')\right)\right| \leq C \sum_{\ell=1}^{n} |x'_\ell|^{(M\cdot\beta - m+1)\tau(\nu)/m_\ell}$$

$$(\beta \in I_\nu)$$

hold for every $(x,\xi) \in V$ *and every* $\nu = 1, \cdots, N$ *with some constant* C, *where* $\tau(\nu) = 2/(m+1-\max_{\beta\in I_\nu} M\cdot\beta)$.

(4.5) $f(x)$ *is microlocally in* $H^{M,t}$ *at every point of* Γ, *where* $t \leq \mu(s)$.

(4.6) $u(x)$ *is microlocally in* $H^{M,t+m-1}$ *at* $\gamma(b)$.

Then $u(x)$ *is microlocally in* $H^{M,t+m-1}$ *at every point of* Γ.

Remark 4.3. The condition

(4.7) $s > \max_{1 \leq k \leq K,\ h \geq 2} \{|M|/2+(M\cdot\beta(k)+\sum_{j=1}^{h} \lambda_{jk}-m+1)/(h-1)\}$

implies that we can take

(4.8) $\mu(s)+m-1 > s$,

which is necessary for the main theorem to be meaningful. Besides, (4.8) implies that $P_b(x,\xi)$ is continuous for every $b > m-1$, and that $P_m(x,\xi)$ is differentiable with respect to x_ℓ for every $\ell \in \Theta$. However, this is not strong enough to guarantee (4.3) and the latter inequality in (4.4) (The Hölder-type condition). If $\mu(s)+m-2 > s$, then these conditions will be automatically satisfied.

The condition (4.7) depends only on the formal development (2.2), not on the expression (2.1).

Remark 4.4. Theorem 4.2 generalizes Theorem 6.1 of Bony [1], where the case $M = (1,\cdots,1)$ is treated. Moreover, our theorem improves the values of τ and $\mu(s)$.

References

[1] J. M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Scient. Ec. Norm. Sup., 4ème série, 14 (1981), 209-246.

[2] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math., 36 (1983), 437-477.

[3] P. Godin, Lectures at the symposium "Microlocal Analysis and Partial Differential Equations" held at Mathematische Forschungsinstitut Oberwolfach, 1983.

[4] L. Hörmander, On the existence and the regularity of solutions of linear pseudo-differential equations, L'Enseignement Math., 17 (1971), 99-163.

[5] T. Kobayashi and G. Nakamura, Singular solutions for semilinear hyperbolic equations I, Amer. J. of Math., 108 (1986), 1477-1486.

[6] R. Lascar, Propagation des singularités des solutions d'équations pseudo-différentielles quasi homogènes, Ann. Inst. Fourier, Grenoble, 27 (1977), 79-123.

[7] Y. Meyer, Remarques sur un théorème de J. M. Bony, Supplemento ai rendiconti del Circolo matematico di Palermo, serie II, 1 (1981), 1-20.

[8] T. Sakurai, Propagation of singularities of solutions to semilinear Schrödinger equations, Proc. Japan Acad., 61

(1985), Ser. A, 31-34.

[9] ———, Propagation of regularities of solutions to semilinear partial differential equations of quasi-homogeneous type, J. Fac. Sci. Univ. Tokyo, IA., 33 (1986), 347-378.

[10] M. Yamazaki, Régularité microlocale quasi homogène des solutions d'équations aux dérivées partielles non linéaires, C. R. Acad. Sc. Paris, 298 (1984), Série I, 225-228.

[11] ———, Quasi-homogeneous microlocal analysis for nonlinear partial differential equations, Proc. Symp. Pure Math., 45 (1986), Part 2, 573-577.

[12] ———, A quasi-homogeneous version of microlocal analysis for nonlinear partial differential equations, preprint.

[13] ———, Propagation of quasi-homogeneous microlocal singularities of solutions to nonlinear partial differential equations, preprint.