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Super Virasoro Algebra and Solvable Supersymmetric Quantum Field Theories

Itaru YAMANAKA and Ryu SASAKI
山中 到 佐々木 隆

Research Institute for Theoretical Physics
Hiroshima University, Takehara, Hiroshima 725, Japan

Abstract

Interesting and deep relationships between super Virasoro algebras and super soliton systems (super KdV, super mKdV and super sine-Gordon equations) are investigated at both classical and quantum levels. An infinite set of conserved quantities responsible for solvability is characterized by super Virasoro algebras only. Several members of the infinite set of conserved quantities are derived explicitly.

1. Introduction

Solving quantum nonlinear field theories (,or field theories with nontrivial interactions) exactly is one of the dreams of many physicists and mathematicians. Usually the word "solving" is used to mean the exact calculation of S-matrix or Green's functions. We try to propose another definition of solvability. And it is expected that this approach will shed a light on the research for solving quantum nonlinear field theory.

In classical particle systems, the criterion of complete integrability is the existence of as many independent and mutually involutive (vanishing Poisson bracket) conserved quantities as degrees of freedom possessed by the system (Liouville)[1)]. Integrable classical nonlinear field theories in 1+1 dimensions, or classical soliton systems have an infinite set of polynomial conserved quantities in involution.[2)] When we try to construct the quantum version of integrable (solvable) nonlinear field theories, by analogy to the above classical cases, we may regard the existence of an infinite set of mutually commutable conserved quantities as one guiding principle. From this point of view, the present authors have investigated 1+1 dimensional quantum soliton systems that have an infinite set of conserved quantities.[3,4)]

It is widely recognized that the so called "quantum inverse scattering method"[5)] is a powerful method to solve the nonlinear systems. Let's briefly summarize the difference between the quantum inverse scattering method and our approach. The generating function of the polynomial conserved quantities is the scattering data $S(\lambda)$ defined in the inverse scattering method

$$\ln S(\lambda) = \sum_{j=1}^{\infty} M_j (i\lambda)^{-j} \quad , \qquad (1.1)$$

where M_j stands for each member of the infinite conserved quantities. At the classical level the following two equations hold simultaneously,

$$\{ \quad S(\lambda) \ , \quad S(\gamma) \} = 0 \quad , \qquad (1.2)$$

$$\{ \ln S(\lambda) \ , \ln S(\gamma) \} = 0 \quad , \qquad (1.3)$$

where { , } denotes the Poisson bracket. These two equations are equivalent. However we can conceive two different schemes of quantum field theories corresponding to these two equations(1.2-3). They are :

(A) a quantum field theory in which the quantum scattering data operators $S_q(\lambda)$ are well defined and commute with each other

$$[\quad S_q(\lambda) \ , \quad S_q(\gamma)] = 0 \quad , \qquad (1.4)$$

where the subscript q means quantum version.

(B) a quantum field theory in which an infinite set of polynomial quantum operators M_{qj}, $j=1,2,\cdots\infty$,or $(\ln S(\lambda))_q$ is well defined and they commute with each other

$$[\quad M_{qj} \quad , \quad M_{qk} \quad] = 0 \quad , \quad j,k=1,2,\cdots,\infty \quad ,\text{or} \qquad (1.5)$$

$$[(\ln S(\lambda))_q, (\ln S(\gamma))_q] = 0 \quad . \qquad (1.6)$$

Due to the problems of operator ordering and divergences, in the scheme (A), ln $S(\lambda)_q$ is in general not well defined even if $S(\lambda)_q$ is and, in the scheme (B), $\exp[(\ln S(\lambda))_q]$ is not well defined even if $(\ln S(\lambda))_q$ is. Therefore there is no a priori reason why these two scheme should be equivalent. Most of the known quantum systems solvable by the quantum inverse scattering method belong to the scheme (A). Our approach is the scheme (B). The present authors showed that such quantization is possible for some models[3,4]. The techniques developed in conformal field theory to evaluate commutators are quite useful. Thanks to these techniques, the quantum conserved quantities are obtained by adding quantum corrections to the classical conserved quantities. These quantum conserved operators are Heisenberg operators. In other words, our results are non perturbative and exact in the sense that the quantum corrections stop at finite order of the Planck constant $\hbar$. So far quantum conserved quantities have been calculated only in relativistic field theories using perturbation theory[6].

Recently conformal field theory[7-10] has attracted much attention. The unitarizable conformal field theories with the discrete series[8] of central charges $c<1$ are especially interesting since they are examples in which Green's functions are calculable[7,10]. From the previous unified point of view on the solvability, we expect that conformal field theories also have an infinite set of mutually commutative conserved quantities. At the classical level, there is an indication that this is so. Gervais pointed out that an infinite set of mutually involutive polynomial functions of the Virasoro generators exists provided that the Virasoro commutation relations are regarded as the Poisson brackets[11]. These are nothing but the well known infinite set of conserved quantities of the Korteweg de Vries (KdV) eq., a solvable classical field theory. We can extend this relation between the conformal field theory (Virasoro algebra) and a nonlinear soliton system (KdV equation). As Gervais pointed out the second Poisson bracket of the KdV eq. is the classical Virasoro algebra. In addition, the Miura transformation[12] that connects the KdV and the modified KdV (mKdV) is just the oscillator representation[13] of the Virasoro algebra. If we take the first Poisson bracket of the mKdV then the second Poisson bracket of the KdV is simply the consequence of the Miura transformation. A half of the sine-Gordon Hamiltonian (i.e. $e^{i\beta\phi}$ or $e^{-i\beta\phi}$ part of $\cos\beta\phi$) can be considered as a Primary field, a familiar concept in the area of conformal field theory, with respect to the Virasoro algebra. On the other hand, it is known that KdV, mKdV and sine-Gordon (sG) essentially share common conserved quantities. This knowledge give us a new characterization of the conserved quantities of the KdV-mKdV-sine-Gordon equations[4].

It is known that a quantum version of this characterization exists[4]. We can calculate several quantum conserved quantities explicitly as "polynomials" of quantum Virasoro generators. These quantum conserved quantities exhibits remarkable features at the known special values of the coupling constant in Coleman's theory[14] of quantum sine-Gordon equation. A recursion formula for the quantum conserved quantities at a special value of the coupling constant is given.(eq.(5.17) of Ref.4)

On the other hand, recently, the super version[15] of the Virasoro algebra has attracted much attention[9] in the area of superstring theory and critical phenomena theory. Therefore a supersymmetric extension of the previous story is very interesting. So far some authors have studied the relation[16] between the super Virasoro algebra and "super KdV (s-KdV) and super mKdV (s-mKdV) equations"[17]. However, their equations actually do not have the supersymmetry. In the field theoretical point of view, supersymmetric systems should have not only a Grassmann odd field but also a super invariant Hamiltonian or an equation of motion. In this sense, the supersymmetric KdV equation proposed by Manin and Radul[18]has the desired symmetry. We adopt this s-KdV equation. In order to display the supersymmetry manifestly, we use the superspace formulation[19]throughout this paper. We will discuss the relation between the super Virasoro algebra and the super soliton systems (super KdV, super mKdV, super sine-Gordon) at classical and quantum levels.

This paper is organized as follows. The section 2 is for the super soliton equations at the classical level. We present the recursion formulas of the conserved quantities of super KdV,mKdV and sine-Gordon[20] systems based on the superspace formulation. In section 3, we introduce the Poisson bracket or the Hamiltonian structure. As in the non-super case, we consider the super Miura transformation as the generator of the super Virasoro algebra and a part of the super sine-Gordon Hamiltonian as a super primary field, and we derive the new characterization of s-(m)KdV-sG hierarchy. In section 4, we quantize the super soliton theories and derive the super version of the previous characterization. With it we derive the explicit forms of some quantum conservation quantities.

2. Conserved Quantities of Classical super (m)KdV-sine-Gordon

In this section, we introduce a special type of super soliton theories and study their integrability. To hold supersymmetry explicit, we use the superspace formulation. The superspace coordinates $\hat{\sigma}$ consist of a space coordinate σ $(0\leq\sigma<2\pi)$ and a corresponding Grassmann odd super coordinate θ. Superfields are defined as Grassmann odd or even functions of the superspace coordinates $\hat{\sigma}$. For notations in the superspace formulation, we generally follow Ref.19.

2-1 Super Korteweg de Vries (s-KdV) Equation

The s-KdV eq. reads[18)]

$$\partial_t W = \partial_\sigma^3 W + \frac{3}{\kappa^2}\partial_\sigma (WDW) \tag{2.1.1}$$

in which $W=W(t,\hat{\sigma};\kappa)$ is a Grassmann odd superfield depending on time(t) and superspace($\hat{\sigma}$) coordinates, $D(=\partial_\theta - i\theta\partial_\sigma)$ is the super derivative and κ is a coupling constant. The s-KdV field W has two component fields φ(Grassmann odd) and ν(even) when expanded in powers of θ,

$$W(t,\hat{\sigma};\kappa) = \varphi(t,\sigma;\kappa)/2 + \theta\ \nu(t,\sigma;\kappa) . \tag{2.1.2}$$

We adopt the periodic boundary condition,

$$\varphi(t,\sigma+2\pi;\kappa) = \pm\ \varphi(t,\sigma;\kappa) \quad , \tag{2.1.3}$$

$$\nu(t,\sigma+2\pi;\kappa) = \nu(t,\sigma;\kappa) \quad , \tag{2.1.4}$$

where the double sign is important only at the quantum level.

The associated linear problem of the s-KdV eq. is given as[18)]

$$(\partial_\sigma^2 + \frac{1}{\kappa^2}WD)\Psi(t,\hat{\sigma};\kappa) = \lambda^2\Psi(t,\hat{\sigma};\kappa) \quad , \tag{2.1.5}$$

$$\partial_t\Psi(t,\hat{\sigma};\kappa) = (4\partial_\sigma^3 + \frac{6}{\kappa^2}WD\partial_\sigma + \frac{3}{\kappa^2}\partial_\sigma WD)\ \Psi(t,\hat{\sigma};\kappa) \ , \tag{2.1.6}$$

in which Ψ is a Grassmann odd field depending on $(t,\hat{\sigma};\kappa)$ and on the spectral parameter λ. In other words, the compatibility condition of eqs.(2.1.5) and (2.1.6) reduces to the s-KdV eq.(2.1.1). We can find an odd superfield Z that is a functional of Ψ and satisfies a conservation law and a Riccati equation;

$$\partial_t Z = DF \quad , \tag{2.1.7}$$

$$\partial_\sigma Z = 2\lambda Z - \frac{1}{\kappa^2}W - ZDZ \quad , \qquad (2.1.8)$$

where F is a rational function of Ψ,$D\Psi$,W and DW. Eq.(2.1.8) admits the following asymptotic solution,

$$Z = \sum_{n=1}^{\infty} \frac{Z_n}{(2\lambda)^n} \quad . \qquad (2.1.9)$$

By substituting this expansion into eq.(2.1.8), we find the recursion formula for the conserved quantities $K_n[W]$ of the s-KdV equation,

$$Z_{n+1} = \partial_\sigma Z_n + \sum_{k=1}^{n-1} Z_k\, DZ_{n-k} \;,\; Z_1 = \frac{1}{\kappa^2}W \;, \qquad (2.1.10)$$

$$K_n[W] = \frac{1}{2\pi}\,\kappa^{2n}\int_0^{2\pi} d\hat{\sigma}\; Z_{2n+1}(W) \quad , \; n=0,1,2,\cdots, \qquad (2.1.11)$$

in which

$$\int_0^{2\pi} d\hat{\sigma} \equiv \int_0^{2\pi} d\sigma \int d\theta \quad . \qquad (2.1.12)$$

Explicit forms of some lower members of the conserved quantities are given in Appendix I. They are Grassmann even conserved quantities.

2-2 Super Modified Korteweg de Vries (s-mKdV) Equation

The super Miura transformation has the form (U is an odd superfield)

$$W(t,\hat{\sigma};\kappa) = U(t,\hat{\sigma})DU(t,\hat{\sigma}) + i\kappa\partial_\sigma U(t,\hat{\sigma}) \quad . \qquad (2.2.1)$$

With s-KdV eqs.(2.1.1) and (2.2.1), we get the following super modified KdV equation,

$$\partial_t U = \partial_\sigma^3 U + \frac{3}{\kappa^2}(\partial_\sigma U)(DU)^2 + \frac{3}{\kappa^2}U(DU)\partial_\sigma DU \quad . \tag{2.2.2}$$

Namely, W defined by eq.(2.2.1) satisfies the s-KdV eq.(2.1.1) if U satisfies the above super mKdV eq,

$$\partial_t W - \left[\partial_\sigma^3 W + \frac{3}{\kappa^2}\partial_\sigma (WDW) \right]$$

$$= (D\frac{U}{\kappa} + \frac{U}{\kappa}D + i\partial_\sigma) \left[\partial_t U - \left[\partial_\sigma^3 U + \frac{3}{\kappa^2}(\partial_\sigma U)(DU)^2 + \frac{3}{\kappa^2}U(DU)\partial_\sigma DU \right]\right] . \tag{2.2.3}$$

So far as we know the explicit form of the super mKdV eq. had not yet been given before. The super Miura transformation factorizes the linear problem (2.1.2),

$$(D-\frac{U}{\kappa})D(D+\frac{U}{\kappa})D\Psi = (i\lambda)^2\Psi \quad . \tag{2.2.4}$$

It should be noticed that this factorization property of the Miura transformation is inherited from the non-supersymmetric mKdV system. Although we can derive the recursion formula of the s-mKdV conserved quantities with eq.(2.1.8) and the super Miura transformation (2.2.1), we will show that the recursion formula can also be derived from the factorized linear problem (2.2.4). Initially we define a superfield Y that is a functional of Ψ and satisfies the Riccati equation (2.2.6),

$$Y = \frac{\Psi - (D+U/\kappa)D\Psi\ /i\lambda}{D\Psi + D(D+U/\kappa)D\Psi/i\lambda} \quad , \tag{2.2.5}$$

$$\partial_\sigma Y = 2\lambda Y - i\frac{U}{\kappa} + i(D\frac{U}{\kappa})YDY \quad . \tag{2.2.6}$$

With the asymptotic expansion,

$$Y = \sum_{n=1}^{\infty} \frac{Y_n}{(2\lambda)^n} \quad , \tag{2.2.7}$$

we derive the recursion formula of the s-mKdV conserved quantities I_n ;

$$Y_{n+1} = \partial_\sigma Y_n - i(D\frac{U}{\kappa}) \sum_{k=1}^{n-1} Y_k \, DY_{n-k} \, , \quad Y_1 = i\frac{U}{\kappa} \, , \tag{2.2.8}$$

$$I_n = \frac{1}{2\pi} \kappa^{2n-3} \int_0^{2\pi} d\hat{\sigma} \, (DU) \, Y_{2n-1} \quad . \tag{2.2.9}$$

They are Grassmann even conserved quantities.

2-3 Super Sine-Gordon (s-sG) Equation

For later discussion, we introduce a super coordinate θ_t, namely, the Grassmann odd variable associated with the time coordinate t. The super time derivative is of the form

$$D_t = \frac{\partial}{\partial \theta_t} - i\theta_t \frac{\partial}{\partial t} \quad \text{and} \quad D_t^2 = -i\partial_t \quad . \tag{2.3.1}$$

We define the super sine-Gordon equation as follows

$$D_t D \, \Phi = -i\frac{2}{\beta} \sin\frac{\beta\Phi}{2} \quad , \tag{2.3.2}$$

where Φ is a fundamental superfield (Grassmann even) of the system. The time (t) evolution of Φ is obtained by applying the D_t operation to eq.(2.3.2). According to Girardello and Sciuto[20], the linear equations are

$$D \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} i\lambda\theta, & -\frac{\beta}{2}D\Phi, & 0 \\ \frac{\beta}{2}D\Phi, & i\lambda\theta, & 2\lambda \\ 0, & -i, & i\lambda\theta \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad \text{and} \tag{2.3.3}$$

$$D_t\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 & , & 0 & , & -\sin(\beta\Phi/2) \\ 0 & , & 0 & , & \cos(\beta\Phi/2) \\ \frac{-i\sin(\beta\Phi/2)}{2\lambda} & , & \frac{i\cos(\beta\Phi/2)}{2\lambda} & , & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}, \qquad (2.3.4)$$

where V_1 and V_2 are Grassmann even superfields and V_3 is an odd superfield. The recursion formula for the s-sG conserved quantities J_n is derived from the linear problem[20],(X is an odd superfield)

$$X_{n+1} = \partial_\sigma X_n + \frac{\beta}{2}(D^2\Phi) \sum_{k=1}^{n-1} X_k\, DX_{n-k}\ , \quad X_1 = -\frac{\beta}{2}D\Phi\ , \qquad (2.3.5)$$

$$J_n = \frac{1}{2\pi}\int_0^{2\pi} d\hat{\sigma}\quad i(D^2\Phi)\ X_{2n-1} \qquad . \qquad (2.3.6)$$

This recursion formula (2.3.5) coincides with the s-mKdV formula (2.2.8) provided that we identify s-mKdV superfield $\frac{U}{\kappa}$ with $\frac{\beta}{2}iD\Phi$. Therefore we can say that the s-KdV, s-mKdV and s-sG systems share essentially the common conserved quantities. We call the set of these conserved quantities the super (m)KdV-sine-Gordon hierarchy.

3. Classical s-(m)KdV-sG Hierarchy and Super Virasoro Algebras

In this section, we introduce the Poisson bracket structure with which the conserved quantities are involutive to each other and we present another characterization of the s-(m)KdV-sG hierarchy, which closely follows the non-super case[4].

3-1 Poisson Brackets

Hereafter we identify U with $iD\Phi$ throughout this paper,

$$U(t,\hat{\sigma}) = iD\Phi(t,\hat{\sigma}) \quad . \tag{3.1.1}$$

The fundamental Poisson brackets are of the forms

$$\{ U(t,\hat{\sigma}) , \Phi(t,\hat{\sigma}') \}_{P.B.} = -2\pi \hat{\delta}(\hat{\sigma},\hat{\sigma}') \quad , \tag{3.1.2}$$

$$\{ U(t,\hat{\sigma}) , U(t,\hat{\sigma}') \}_{P.B.} = -2\pi iD\hat{\delta}(\hat{\sigma},\hat{\sigma}') \quad , \tag{3.1.3}$$

where $\hat{\delta}(\hat{\sigma},\hat{\sigma}')\equiv(\theta-\theta')\delta(\sigma-\sigma')$ is the delta function in the superspace. By means of these Poisson brackets, we can show that any pair of members of s-(m)KdV-sG hierarchy are involutive (,or have vanishing Poisson bracket). The Poisson bracket of the s-KdV field W are derived easily in terms of the super Miura transformation (2.2.1),

$$\{ W(t,\hat{\sigma};\kappa) , W(t,\hat{\sigma}';\kappa) \}_{P.B.}$$
$$=2\pi\Big[-i\kappa^2\partial_\sigma^2 D\hat{\delta}(\hat{\sigma},\hat{\sigma}')-3W(t,\hat{\sigma};\kappa)\partial_\sigma\hat{\delta}(\hat{\sigma},\hat{\sigma}')-iDW(t,\hat{\sigma};\kappa)D\hat{\delta}(\hat{\sigma},\hat{\sigma}')$$
$$-2\partial_\sigma W(t,\hat{\sigma};\kappa)\hat{\delta}(\hat{\sigma},\hat{\sigma}')\Big] \quad . \tag{3.1.4}$$

We can call $W(t,\hat{\sigma};\kappa)$ the super Virasoro field ,for the components φ and ν in eq.(2.1.2) satisfy the super Virasoro algebras at the Poisson bracket level,

$$\{\varphi(t,\sigma;\kappa),\varphi(t,\sigma';\kappa)\}_{P.B.} = -8\pi i(\nu(t,\sigma;\kappa)+\kappa^2\partial_\sigma^2)\delta(\sigma-\sigma'), \tag{3.1.5}$$

$$\{\varphi(t,\sigma;\kappa),\nu(t,\sigma';\kappa)\}_{P.B.}=2\pi\Big[3\varphi(t,\sigma;\kappa)\partial_\sigma\delta(\sigma-\sigma')+\partial_\sigma\varphi(t,\sigma;\kappa)\delta(\sigma-\sigma')\Big], \tag{3.1.6}$$

$$\{\nu(t,\sigma;\kappa),\nu(t,\sigma';\kappa)\}_{P.B.} = 2\pi\Big[\kappa^2\partial_\sigma^3\delta(\sigma-\sigma')+4\nu(t,\sigma;\kappa)\partial_\sigma\delta(\sigma-\sigma')$$
$$+2\partial_\sigma\nu(t,\sigma;\kappa)\delta(\sigma-\sigma')\Big]. \qquad (3.1.7)$$

The fundamental Poisson brackets (3.1.2) and (3.1.3) give the Hamiltonian structure. The s-KdV equation (2.1.1), s-mKdV eq. (2.2.2) and s-sG eq. (2.3.2) are written as the canonical equation of motion with respective Hamiltonians $\frac{K_1}{2}, \frac{I_2}{2}, H_{sG}$;

$$\partial_t W = \{ W , \frac{K_1}{2} \}_{P.B.} , \qquad \text{s-KdV equation} \qquad (3.1.8)$$

$$\partial_t U = \{ U , \frac{I_2}{2} \}_{P.B.} , \qquad \text{s-mKdV equation} \qquad (3.1.9)$$

$$D_t U = \{ U , H_{sG} \}_{P.B.} , \qquad \text{s-sG equation} \qquad (3.1.10)$$

where

$$H_{s\text{-}KdV}(\kappa) \equiv \frac{K_1}{2} = \frac{1}{4\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \quad (DW)W \quad , \qquad (3.1.11)$$

$$H_{s\text{-}mKdV}(\kappa) \equiv -i\frac{I_2}{2} = \frac{1}{4\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[\kappa^2(DU)\partial_\sigma^2 U + U(DU)^3 \right], \qquad (3.1.12)$$

$$H_{s\text{-}sG}(\beta) \equiv \frac{4}{2\pi\beta^2}\int_0^{2\pi} d\hat{\sigma} \left[1-\cos(\beta\Phi/2) \right] \quad . \qquad (3.1.13)$$

It should be noted that the Hamiltonian of the super sG eq., i.e. (3.1.13) is Grassmann odd, whereas the other Hamiltonians (3.1.11) and (3.1.12) are even. Hereafter the time variable (t) is fixed and suppressed since we are concerned only with equal time Poisson brackets and their quantum versions.

3-2 The Special Feature of The Classical s-(m)KdV-sG Hierarchy

The members of the s-(m)KdV-sG hierarchy are differential polynomials of W by the definition of the hierarchy and W consists of the super Virasoro generators as is shown in the previous subsection. On the other hand, it is well known that the integral of a primary field of unit conformal dimension commutes with all the Virasoro generators. These facts motivate us to verify the following relation[4],

$$\{ W(\hat{\sigma};\kappa_c) \, , \int d\hat{\sigma}' \, e^{i\beta\Phi(\hat{\sigma}')/2} \}_{P.B.} = 0 \quad . \tag{3.2.1}$$

In this equation,

$$\kappa_c \equiv 2/\beta \tag{3.2.2}$$

and

$$V(\hat{\sigma};\beta) \equiv \exp(i\beta\Phi(\hat{\sigma})/2) \tag{3.2.3}$$

is a conformal dimension 1/2 super primary field with respect to the super Virasoro field W. It should be remarked that the integration $\int d\theta$ picks up the desired primary field with unit conformal dimension. Note that $\exp(i\beta\Phi/2)$ is a part of the s-sG Hamiltonian density. We arrive at the following theorem.

<u>Theorem 1</u> Let F[W] be a $\int d\hat{\sigma}$ integration of a differential polynomial of W. If it is even with respect to κ as a function of U then F is a conserved quantity of s-sG system. Namely we have

$$F[W(\hat{\sigma};\kappa)] = F[W(\hat{\sigma};-\kappa)] \Rightarrow \{ F[W(\hat{\sigma};\kappa_c)] \, , H_{sG}(\beta) \}_{P.B.} = 0 \; . \tag{3.2.4}$$

This is proved with eq.(3.2.1) and

$$H_{s-sG}(\beta) = \frac{4}{2\pi\beta^2} \int d\hat{\sigma} \left[1-(V(\hat{\sigma};\beta)+V(\hat{\sigma};-\beta))/2 \right] . \qquad (3.2.5)$$

This theorem means that s-(m)KdV-sG hierarchy is characterized by the left hand side of statement (3.2.4). In other words, we can use this condition to calculate the s-(m)KdV-sG hierarchy explicitly. In section 4, we shall consider the quantum version of Theorem 1.

4. Quantum s-(m)KdV-sG Hierarchy and Super Virasoro Algebras

In this section, we proceed to the solvable quantum field theories. The quantum version of Theorem 1 will be established and we will show the explicit forms of the lower members of the quantum s-(m)KdV-sG hierarchy.

4.1 Quantization

The canonical quantization is achieved by replacing the Poisson bracket by the following equal time commutation relations

$$[\ U(\hat{\sigma})\ ,\ \Phi(\hat{\sigma}')\ ]_- = -\hbar 2\pi i\hat{\delta}(\hat{\sigma},\hat{\sigma}') \quad , \qquad (4.1.1)$$

$$[\ U(\hat{\sigma})\ ,\ U(\hat{\sigma}')\ ]_+ = \hbar 2\pi D\hat{\delta}(\hat{\sigma},\hat{\sigma}') \quad , \qquad (4.1.2)$$

where subscripts + and - stand for anti-commutation and commutation relation, respectively. For the quantum calculation, we need the mode expansion of Φ and U and the definition of the Fock space. The superfields are expanded into the component fields,

$$\Phi(\hat{\sigma}) = \phi(\sigma) + i\theta\psi(\sigma) \quad , \qquad (4.1.3)$$

$$U(\hat{\sigma}) = iD\Phi(\hat{\sigma}) = -\psi(\sigma) + \theta u(\sigma) \quad , \qquad (4.1.4)$$

where $u \equiv \partial_\sigma \phi$. The Fourier expansion of the component fields and hermiticity of oscillators are

$$\phi(\sigma) = q + \alpha_0 \sigma + i \sum_{n\in Z} \frac{\alpha_n}{n} e^{-in\sigma} \quad , \quad \alpha_n^\dagger = \alpha_{-n} \quad , \qquad (4.1.5)$$

$$\psi(\sigma) = \begin{cases} \sum_{r\in Z+1/2} b_r e^{-ir\sigma} \quad , \quad b_r^\dagger = b_{-r} \quad , \qquad (NS) \qquad (4.1.6) \\ \sum_{r\in Z} d_n e^{-in\sigma} \quad , \quad d_n^\dagger = d_{-n} \quad , \qquad (R) \qquad (4.1.7) \end{cases}$$

$$u(\sigma) = \sum_{n\in Z} \alpha_n e^{-in\sigma} \quad , \qquad (4.1.8)$$

where (4.1.6) and (4.1.7) are Neveu-Schwarz (NS) and Ramond (R) sectors[15], respectively. The commutators of oscillators are derived as follows,

$$[\, \alpha_n \, , \, \alpha_m \,]_- = \hbar n \delta_{n+m,0} \quad , \quad [\, q \, , \, \alpha_n \,]_- = i\hbar \delta_{n,0} \quad , \qquad (4.1.9)$$

$$[\, b_r \, , \, b_s \,]_+ = \hbar \delta_{r+s,0} \qquad (4.1.10)$$

$$[\, d_n \, , \, d_m \,]_+ = \hbar \delta_{n+m,0} \qquad (4.1.11)$$

We interpret α_n ,n>0 (n<0), b_r ,r>0 (r<0) and d_n ,n>0 (n<0) as annihilation (creation) operators. In order to take a well defined product of field operators, we define the normal ordering : : as reordering all the annihilation operators to the right of the creation operators. The vacuum $|0\rangle$ is defined by

$$\alpha_n |0\rangle = 0 \quad (n \geq 0) \quad , \qquad (4.1.12)$$

$$b_r |0\rangle = 0 \quad (r \geq 1/2) \quad , \quad d_n |0\rangle = 0 \quad (n \geq 1) \quad , \qquad (4.1.13)$$

and the Fock space is built by repeated application of creation operators. We introduce a fictitious time τ and a complex variable $\xi=\tau+i\sigma$, since we need to calculate contour integrals to evaluate the commutators. The space coordinate σ is on a circle and then the complex coordinate ξ is on a cylinder. Hereafter we shall consider the quantum field theory on the cylinder.

Next we review the method of evaluating commutators[21]. Here we consider the commutator of two integrated quantities P and Q,

$$P = \oint_0 d\hat{\xi}\ p(U, DU, D^2U, \cdots) \quad , \tag{4.1.14}$$

$$Q = \oint_0 d\hat{\eta}\ q(U, DU, D^2U, \cdots) \quad , \tag{4.1.15}$$

where p and q are differential polynomials of U, and

$$\oint_0 d\hat{\xi} \equiv \frac{1}{2\pi i}\int_\tau^{\tau+2\pi i} d\xi \int d\theta \qquad \text{and} \qquad D = \partial_\theta + \theta\partial_\xi \ . \tag{4.1.16}$$

The commutator of :P: and :Q: are rewritten by changing the contour of integration,

$$[\ :P:\ ,\ :Q:\ ] = \oint_0 d\hat{\eta} \oint_{c_\eta} d\hat{\xi} \quad T\ :p(\hat{\xi}):\ :q(\hat{\eta}): \tag{4.1.17}$$

where c_η stands for a small contour around η and T means the τ-ordering. The τ-ordering is defined as follows

$$T\ :p(\hat{\xi}):\ :q(\hat{\eta}): = \begin{cases} :p(\hat{\xi}):\ :q(\hat{\eta}):\ , & (\mathrm{Re}\ \xi > \mathrm{Re}\ \eta) \\ \pm:q(\hat{\eta}):\ :p(\hat{\xi}):\ , & (\mathrm{Re}\ \eta > \mathrm{Re}\ \xi) \end{cases} \tag{4.1.18}$$

where the sign factor takes minus if both q and p are Grassmann odd. In order to calculate the commutator (4.1.17), we expand the

τ-ordered product $T{:}p(\hat{\xi}){::}q(\hat{\eta}){:}$ by Wick's theorem. Namely we expand a τ-ordered product into a sum of a normal ordered product times propagators. The simplest example of Wick's theorem is

$$T\,U(\hat{\xi}_1)\,U(\hat{\xi}_2) = :U(\hat{\xi}_1)U(\hat{\xi}_2): + \hbar\,\Delta(\hat{\xi}_1,\hat{\xi}_2) \quad , \qquad (4.1.19)$$

in which the propagator Δ is defined by

$$\hbar\,\Delta(\hat{\xi}_1,\hat{\xi}_2) \equiv \langle 0|T\,U(\hat{\xi}_1)U(\hat{\xi}_2)|0\rangle$$

$$= \begin{cases} \hbar\left[\frac{1}{2}\mathrm{cosech}\frac{1}{2}(\xi_1-\xi_2) + \theta_1\theta_2\,\frac{1}{4}\,\mathrm{cosech}^2\,\frac{1}{2}(\xi_1-\xi_2)\right] & ,(\mathrm{NS}) \qquad (4.1.20) \\ \hbar\left[\frac{1}{2}\,\coth\,\frac{1}{2}(\xi_1-\xi_2) + \theta_1\theta_2\,\frac{1}{4}\,\mathrm{cosech}^2\,\frac{1}{2}(\xi_1-\xi_2)\right] & \\ \quad = \hbar\,\frac{1}{2}\,\coth\,\frac{1}{2}\xi_{12} & ,\ (\mathrm{R}) \qquad (4.1.21) \end{cases}$$

where $\xi_{12} \equiv \xi_1-\xi_2-\theta_1\theta_2$ is the distance between $\hat{\xi}_1=(\xi_1,\theta_1)$ and $\hat{\xi}_2=(\xi_2,\theta_2)$ in the (ξ,θ) superspace.

4.2 Super Virasoro Algebras and Super Primary Fields

In this subsection, we derive the condition that characterizes the quantum s-(m)KdV-sG hierarchy. In other words, we establish the quantum version of Theorem 1, statement (3.2.4). For this purpose, we introduce the quantum super Virasoro field and a quantum super primary field. The procession to the quantum theorem is parallel with the classical case in section 3.

The classical s-KdV field W is the super Virasoro field at the Poisson bracket level. By analogy, we take

$$W_q(\hat{\xi};\kappa)\equiv\ :UDU:\ -\ \kappa\partial_\xi U\ -\hbar\omega\theta\quad ,\qquad \omega=\begin{cases}\frac{1}{8} & (NS)\\ 0 & (R)\end{cases}\quad , \tag{4.2.1}$$

for the quantum super Virasoro field. In the classical limit ($\hbar\to 0$), W_q reproduces the classical field W. The constant $\hbar\omega\theta$ in eq.(4.2.1) is determined so that W_q satisfies the super Virasoro algebras in the following sense.

The operator product expansion of W_q fields is calculated by Wick's theorem,

$$T\left[\frac{1}{2\hbar}W_q(\hat{\xi}_1)\right]\left[\frac{1}{2\hbar}W_q(\hat{\xi}_2)\right]$$

$$=\frac{\hat{c}}{4}\xi_{12}^{-3}+\left(\frac{3}{2}\theta_{12}\xi_{12}^{-2}+\frac{1}{2}\xi_{12}^{-1}D+\theta_{12}\xi_{12}^{-1}\partial_\xi\right)\left[\frac{1}{2\hbar}W_q(\hat{\xi}_2)\right]+O(\xi_{12}^{0}), \tag{4.2.2}$$

where the central charge is given by

$$\hat{c}=1-2\kappa^2/\hbar\qquad , \tag{4.2.3}$$

and $\theta_{12}\equiv\theta_1-\theta_2$. This form is known as the operator product expansion of the super Virasoro field. We define the integration operator type of the super Virasoro generators;

$$L_nO(\hat{\xi}_2)\equiv\oint_{c_2}d\hat{\xi}_1T\left[\frac{1}{2\hbar}W_q(\hat{\xi}_1)\right]\xi_{12}^{n+1}\ O(\hat{\xi}_2)\quad ,\qquad n\in Z\ , \tag{4.2.4}$$

$$G_rO(\hat{\xi}_2)\equiv\oint_{c_2}d\hat{\xi}_1T\left[-\frac{1}{\hbar}W_q(\hat{\xi}_1)\right]\xi_{12}^{r+1/2}\theta_{12}\ O(\hat{\xi}_2),\qquad r\in Z+\frac{1}{2}\ ,\quad (R\text{ or }NS) \tag{4.2.5}$$

where the contour c_2 is a small circle around $\hat{\xi}_2$ and it should be noted that L_n and G_r operate on an arbitrary local field operator $O(\hat{\xi})$. L_n and G_r obey the super Virasoro algebras[15),]

$$[\,L_n, L_m\,]_- = (n-m)L_{n+m} + \frac{\hat{c}}{8}n(n^2-1)\delta_{n+m,0} \quad , \qquad (4.2.6)$$

$$[\,L_n, G_r\,]_- = (\frac{n}{2} - r)G_{n+r,0} \quad , \qquad (4.2.7)$$

$$[\,G_r, G_s\,]_+ = 2L_{r+s} + \frac{\hat{c}}{2}(r^2 - \frac{1}{4})\delta_{r+s,0} \quad . \qquad (4.2.8)$$

Next we define a quantum super primary field,

$$V_q(\hat{\xi};\beta) \equiv e^{\frac{\beta^2}{8}\hbar\xi} :e^{i\beta\Phi/2}: \quad , \qquad (4.2.9)$$

that reproduces the classical one V eq.(3.2.3) in the classical limit ($\hbar \to 0$). The operator product expansion of W_q and V_q is

$$T\left(\frac{1}{2\hbar}W_q(\hat{\xi}_1;\kappa)\right)\left(V_q(\hat{\xi}_2;\beta)\right)$$

$$=\Delta_q\theta_{12}\frac{1}{\xi_{12}^2}V_q(\hat{\xi}_2;\beta) + \frac{1}{2\xi_{12}}D_2V_q(\hat{\xi}_2;\beta) + \theta_{12}\frac{1}{\xi_{12}}\partial_2V_q(\hat{\xi}_2;\beta) + o(\xi_{12}^0), \qquad (4.2.10)$$

in which $\Delta_q = \frac{\beta^2}{8}\hbar + \frac{\beta\kappa}{4}$ (conformal dimension of V_q). Especially in the case of $\Delta_q = 1/2$, or

$$\kappa = \kappa_q \equiv \frac{2}{\beta}\left[\,1 - \frac{\beta^2}{4}\hbar\,\right] \quad , \qquad (4.2.11)$$

we get the following relation

$$[\,W_q(\hat{\xi}_1;\kappa_q)\,,\,\oint_0 d\hat{\xi}_2\,V_q(\hat{\xi}_2;\beta)\,]_+ = 0 \quad . \qquad (4.2.12)$$

In the classical limit, κ_q, W_q and V_q reproduce κ_c, W and V respectively. This relation eq.(4.2.12) is the quantum version of eq.(3.2.1). Therefore it is natural to define the Hamiltonian H_{qsG} of quantum super sine-Gordon system,

$$H_{qsG}(\beta) \equiv \frac{4}{\beta^2}\oint_0 d\hat{\xi}\left[1-(V_q(\hat{\xi};\beta)+V_q(\hat{\xi};-\beta))/2 \right] \quad , \quad (4.2.13)$$

which is Grassmann odd. Consequently we can prove Theorem 2 with eq.(4.2.12) and the definition of H_{qsG}.

<u>Theorem 2</u> Let $F[W_q]$ be a $\oint d\hat{\xi}$ integration of a differential polynomial of W_q. If it is even with respect to κ as a function of U then F is a conserved quantity of the quantum s-sG system. Namely we have

$$F[W_q(\hat{\xi};\kappa)] = F[W_q(\hat{\xi};-\kappa)] \Rightarrow \left[F[W_q(\hat{\xi};\kappa_q)], H_{qsG}(\beta) \right]_{\pm} = 0 \ . \quad (4.2.14)$$

This Theorem means that the conserved quantities of the quantum s-sG system are characterized by the left hand side condition of the statement (4.2.14).

4-3 Quantum s-(m)KdV-sG Hierarchy

Due to Theorem 2, we can calculate conserved quantities of the quantum s-sG system. However, we should pay attention to the fact that a functional of W_q, such as the $F[W_q]$ in Theorem 2, needs to be carefully defined so as to avoid the singularities of a product of quantum field operators at the same space-time point. Here we try to adopt the integration type of super Virasoro generators, eqs.(4.2.4–5) to define well defined functions of W_q. The ansatz for a conserved quantity of s-sG system has the form

$$A \equiv \oint_0 d\hat{\xi}\ a(L_n, G_r)\ I(\xi) \qquad , \qquad (4.3.1)$$

in which $a(L_n, G_r)$ is a Grassmann odd "polynomial" of L_n,$(n \le -2)$ and G_r,$(r \le -3/2)$, and $I(\xi)$ is the identity operator. By Theorem 2, we arrive at the following theorem.

<u>Theorem 2'</u> If A defined by eq.(4.3.1) satisfies the condition,

$$A(\kappa) = A(-\kappa) \qquad (4.3.2)$$

then A is a conserved quantity of the quantum s-sG system.

We use Theorem 2' to calculate the first six conserved quantities (K_{q0}–K_{q5}) of quantum s-sG system in the Ramond sector. They are listed in Appendix II. The present authors used a formula manipulation computer language, REDUCE. It should be remarked that <u>the coefficients depend only on the central charge of the super Virasoro algebras</u>. We discuss the Neveu-Schwartz sector at the end of this section.

Next we consider the classical limit of the quantum conserved quantities of s-sG system. In the classical limit, the τ-ordered product of operators becomes a mere product of numbers. Therefore, in the definition of L_n and G_r, eqs.(4.2.4) and (4.2.5), the contour integrals are evaluated as follows,

$$L_n\ O(\hat{\xi}_2) \xrightarrow{(\hbar \to 0)} \frac{1}{2(-n-2)!\ \hbar} \left[D^{-2n-3} W(\hat{\xi}_2)\right] O(\hat{\xi}_2) \quad \text{and} \qquad (4.3.3)$$

$$G_r\, O(\hat{\xi}_2) \xrightarrow{(\hbar\to 0)} \frac{-1}{(-r-3/2)!\hbar} \left[D^{-2r-3} W(\hat{\xi}_2)\right] O(\hat{\xi}_2) \quad . \tag{4.3.4}$$

In this limit, the quantum conserved quantities, K_{q0}–K_{q5} in Appendix II, reduce to the classical ones listed in Appendix I up to normalization factors;

$$K_{qn} \xrightarrow{(\hbar\to 0)} \propto K_n \quad . \tag{4.3.5}$$

In section 2, we showed that the s-KdV, s-mKdV and s-sG systems have common conserved quantities that are in involution with each other in the classical case. Therefore the second s-sG conserved quantity J_2 is the s-(m)KdV Hamiltonian $I_2/2$ up to a constant factor. We expect these soliton systems to maintain these relationships after the quantization. By the classical limit argument, it is natural that a quantum s-(m)KdV Hamiltonian is defined as the second quantum s-sG conserved quantity,

$$H_{qs\text{-}(m)KdV} = K_{q1} = \oint_0 d\hat{\xi} \quad L_{-2}\, G_{-3/2}\, I(\xi) \quad . \tag{4.3.6}$$

Here we assume all s-sG conserved quantities are commutable with each other. Therefore we can conclude that s-(m)KdV and s-sG system have common conserved quantities also in the quantum case. We call the set of conserved quantities the quantum s-(m)KdV-sG hierarchy.

Theorem 2' gives us a method to calculate the quantum s-(m)KdV -sG hierarchy. Another method to derive the hierarchy is the calculation of the quantum s-KdV conserved quantities. They are defined by,

$$[\ H_{qs\text{-}(m)KdV}\ ,\ A\ ] = 0 \quad , \tag{4.3.7}$$

where A is the ansatz (4.3.1). We can rewrite the definition (4.3.7) as follows by changing the contour of integral and Taylor expansion,

$$\oint_0 d\hat{\xi} \quad E\, a(L_n,G_r)\, I(\xi) = 0 \quad , \tag{4.3.8}$$

$$E = \sum_{n=0}^{\infty} \left[L_{-n-2}\, L_{n-1} + \frac{(n+1)}{4}\, G_{-n-5/2}\, G_{n-1/2} \right] \quad . \tag{4.3.9}$$

In this case, we can calculate the quantum s-(m)KdV-sG hierarchy by using the super Virasoro algebras eqs.(4.2.6–8) only. In other words, the infinite set of mutually commutable operators responsible for the solvability is uniquely characterized by the super Virasoro algebra. We can summarize this paragraph in the following theorem.

Theorem 3 For a function $a(L_n,G_r)$, if there exists a function $b(L_n,G_r)$ such that

$$E\, a(L_n,G_r)\, I = G_{-1/2}\, b(L_n,G_r)\, I \tag{4.3.10}$$

then $\oint_0 d\hat{\xi}\; a(L_n,G_r)\, I(\xi)$ belongs to the quantum s-(m)KdV-sG hierarchy.

To prove the theorem, it is useful to note that $G_{-1/2}$ is equivalent to the super derivative D in the right hand side of eq.(4.3.10).

A discussion on the hierarchy in the Neveu-Schwarz sector is in order. The super Virasoro algebras (4.2.6–8) hold not only in the Ramond sector but also in the Neveu-Schwarz sector.

Therefore, by Theorem 3, we can have a common argument for the quantum s-(m)KdV-sG hierarchy in both sectors. Consequently K_{q0}–K_{q5} in Appendix II belong to the quantum s-(m)KdV-sG hierarchy also in the NS sector. It should be remarked that the definition of the super Virasoro generators L_n and G_r in the NS sector are different from that in the R sector only by the constant term in eq.(4.2.1).

5. Summary and Comments

Following the general philosophy of solvable quantum field theories developed by the present authors[3,4], the supersymmetric solvable quantum field theories together with their relationship with the super Virasoro algebra are discussed in some detail. It is shown that the infinite set of quantum commuting operators characterizing the solvability of the supersymmetric quantum field theories (the super-(m)KdV-sG hierarchy) is expressed as certain "polynomials" in the super Virasoro generators. Explicit forms of some lower members of the infinite set of the commuting "polynomials" of the super Virasoro generators are calculated.

To understand the algebraic meaning of this infinite set of commuting operators for the non-super and the supersymmetric cases is an important and interesting open problem in the theory of infinite dimensional Lie algebras. It is also quite challenging to make a further investigation into the solvability of the super conformal field theory, i.e., calculation of S-matrix elements and Green's functions, from the point of view of conserved quantities.

Another interesting problem is to pursue the relationship between infinite dimensional algebras and solvable equations. Searching the relationship in the generalized KdV-mKdV-Toda-lattice[22)] may be an interesting exercise.

It is well known that the supersymmetric extension of the Virasoro algebra is possible for N=1,2 and 4 in which N denotes the number of supersymmetry generators. The N=1 case is discussed in the present paper. The extended (i.e. N>1) super Virasoro algebras are known to have various interesting properties[23)]. As for the solvable nonlinear theories at the classical and quantum levels, several interesting models having the extended supersymmetry are known[24)]. However, the direct connection between these two notions in the sense of our previous (Ref.4) and the present work is yet to be worked out.

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Appendix I

We list up the classical conserved quantities.

$$K_0 = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[W \right]$$

$$K_1 = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[(DW)W \right]$$

$$K_2 = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[2(DW)^2W + \kappa^2(D^3W)D^2W \right]$$

$$K_3 = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[5(DW)^3W + 6\kappa^2(D^3W)(D^2W)DW + 4\kappa^2(D^3W)^2W + \kappa^4(D^5W)D^4W \right]$$

$$K_4 = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[14(DW)^4W + 28\kappa^2(D^3W)(D^2W)(DW)^2 + 42\kappa^4(D^3W)(DW)W + 8\kappa^4(D^5W)(D^4W)DW + 6\kappa^4(D^5W)^2W + \kappa^6(D^7W)(D^6W) \right]$$

$$K_5 = \frac{1}{2\pi\kappa^2}\int_0^{2\pi} d\hat{\sigma} \left[42\,(DW)^5W + 120\kappa^2(D^3W)(D^2W)(DW)^3 + 300\kappa^2(D^3W)^2(DW)^2W - 35\kappa^4(D^3W)^3D^2W + 45\kappa^4(D^5W)(D^4W)(DW)^2 + 12\kappa^4(D^5W)(D^4W)(D^2W)W + 81\kappa^4(D^5W)^2(DW)W - 20\kappa^6(D^5W)^2D^4W + 10\kappa^6(D^7W)(D^6W)DW + 8\kappa^6(D^7W)^2W + \kappa^8(D^9W)(D^8W) \right]$$

Appendix II

We list up the quantum conserved quantities. For typographical simplicity, we use the following coupling constant k throughout Appendix II,

$$k \equiv \frac{\kappa_q}{\sqrt{\hbar}} .$$

So it should not be identified with κ in Appendix I.

$$K_{q0} = \oint_0 d\hat{\xi} \left[G_{-3/2} \right] I(\xi)$$

$$K_{q1} = \oint_0 d\hat{\xi} \left[L_{-2}G_{-3/2} \right] I(\xi)$$

$$K_{q2} = \oint_0 d\hat{\xi} \left[L_{-2}^2 G_{-3/2} + \frac{(6k^2-7)}{24} L_{-3}G_{-5/2} \right] I(\xi)$$

$$K_{q3} = \oint_0 d\hat{\xi} \left[L_{-2}^3 G_{-3/2} + \frac{(12k^2-31)}{20} L_{-3}G_{-5/2}L_{-2} + \frac{(16k^2-73)}{40} L_{-3}^2 G_{-3/2} + \frac{(40k^4-342k^2+291)}{200} L_{-4}G_{-7/2} \right] I(\xi)$$

$$K_{q4} = \oint_0 d\hat{\xi} \left[L_{-2}^4 G_{-3/2} + m1\, L_{-3}G_{-5/2}L_{-2}^2 + m2\, L_{-3}^2 L_{-2}G_{-3/2} + m3\, L_{-4}G_{-7/2}L_{-2} + m4\, L_{-4}^2 G_{-3/2} + m5\, L_{-5}G_{-9/2} \right] I(\xi)$$

$$m1 = k^2 - 4$$

$$m2 = (2k^2-13)3/4$$

$$m3 = (120k^4-1438k^2+2509)/210$$

$$m4 = (30k^4- 356k^2+ 853)/70$$

$$m5 = (8400k^6-115420k^4+432668k^2-264689)3/78400$$

$$K_{q5} = \oint_0 d\hat{\xi} \Big(L^5_{-2}G_{-3/2} + n1\ L_{-3}G_{-5/2}L^3_{-2} + n2\ L^2_{-3}L^2_{-2}G_{-3/2}$$

$$+ n3\ L^3_{-3}G_{-5/2} + n4\ L_{-4}G_{-7/2}L^2_{-2} + n5\ L_{-4}G_{-7/2}G_{-5/2}G_{-3/2}$$

$$+ n6\ L^2_{-4}G_{-7/2} + n7\ L^2_{-4}L_{-2}\ G_{-3/2} + n8\ L_{-5}G_{-9/2}L_{-2}$$

$$+ n9\ L^2_{-5}G_{-3/2} + n10\ L_{-6}G_{-11/2} \Big)\ I(\xi)$$

$n1 = (12k^2-65)5/42$

$n2 = (20k^2-169)5/28$

$n3 = (-840k^4+16858k^2-65987)/4032$

$n4 = (360k^4-5578k^2+14607)/336$

$n5 = (12k^4-172k^2+523)/168$

$n6 = (-201600k^6+3043512k^4-10871134k^2+4888061)/423360$

$n7 = (648k^4-9854k^2+31981)/336$

$n8 = (100800k^6-1844424k^4+9831890k^2-12590635)/94080$

$n9 = (161280k^6-2940408k^4+15785822k^2-25037413)/188160$

$$n10 = (907200k^8-17387640k^6+105348726k^4-237606943k^2 +109736777)/1058400$$

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